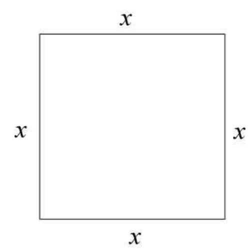
Function and Limits

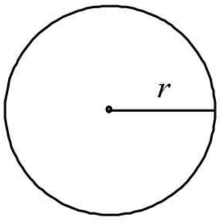
Concept of Functions:

Historically, the term function was first used by German mathematician Leibnitz (1646-1716) in 1673 to denote the dependence of one quantity on another e.g.



1) The area "A" of a square of side "x" is given by the formula A=x^2. As area depends on its side x, so we say that A is a function of x.

2) The area "A" of a circular disc of radius "r" is given by the formula A=πr^2 As area depends on its radius r, so we say that A is a function of r.



3) The volume "V" of a sphere of radius "r" is given by the formula V=$\frac{4}{3}\pi r^3$. As volume V of a sphere depends on its radius r, so we say that V is a function of r.

The Swiss mathematician, Leonard Euler conceived the idea of denoting function written as y=$f(x)$ and read as y is equal to f of x. $f(x)$ is called the value of f at x or image of x under f.

The variable x is called independent variable and the variable y is called dependent variable of f.

If x and y are real numbers then f is called real valued function of real numbers.

Domain of the function:

If the independent variable of a function is restricted to lie in some set, then this set is called the domain of the function e.g.
Dom of $f = \{0 \leq x \leq 5\}$

Range of the function:

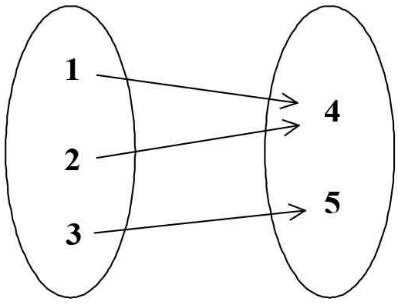
The set of all possible values of $f(x)$ as x varies over the domain of f is called the range of f e.g. $y = 100 - 4x^2$.
As x varies over the domain [0,5] the values of $y = 100 - 4x^2$ vary between y=0 (when x=5) and y = 100 (when x=0)
Range of $f = \{0 \leq y \leq 100\}$

Definition:

A function is a rule by which we relate two sets A and B (say) in such a way that each element of A is assigned with one and only one element of B. For example

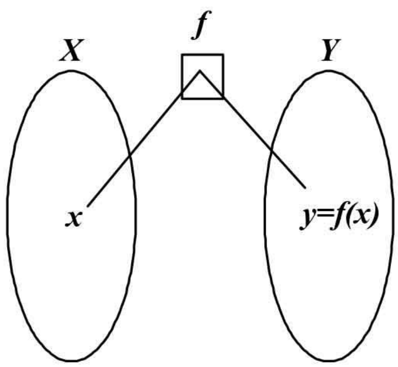
is a function from A to B.

its Domain = {1,2,3} and Range = {4,5}



In general:

A function f from a set 'X' to a set 'Y' is a rule that assigns to each element x in X one and only one element y in Y.(a unique element y in Y)



(f is function from X to Y)

If an element "y, of Y is associated with an element "x, of X, then we write y=f (x) &read as y" is equal to f of x. Here f(x) is called image of f at x or value of f at x .

Or if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y. Then we say that y is a function of x.

The set x is called Domain of f. The set of corresponding elements y in y is called Range of f. we say that y is a function of x.

Exercise 1.1

Q1. (a) Given that $f(x) = x^2 - x$

i. $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

ii. $f(0) = (0)^2 - (0) = 0$

iii. $f(x-1) = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2$

iv. $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 8x^2 + 16 - x^2 - 4 = x^4 + 7x^2 + 12$

(b) Given that $f(x) = \sqrt{x+4}$

$i)\, f(-2) = \sqrt{-2+4} = \sqrt{2}$

$ii)\, f(0) = \sqrt{0+4} = \sqrt{4} = 2$

$iii)\, f(x-1) = \sqrt{x-1+4} = \sqrt{x+3}$

$iv)\, f(x^2+4) = \sqrt{x^2+4+4} = \sqrt{x^2+8}$

Q2. Given that

i) $f(x) = 6x - 9$

$f(a+h) = 6(a+h) - 9 = 6a + 6h - 9$

$f(a) = 6a - 9$

Now $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(6a+6h-9)-(6a-9)}{h}$

$= \dfrac{6a+6h-9-6a+9}{h} = \dfrac{6h}{h} = 6$

ii) $f(x) = \sin x$ *given*

$\because \quad \sin\theta - \sin\varphi = 2\cos\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta+\varphi}{2}\right)$

$f(a+h) = \sin(a+h)$ *and* $f(a) = \sin a$

Now $\dfrac{f(a+h) - f(a)}{h} = \dfrac{\sin(a+h) - \sin a}{h}$

$= \dfrac{1}{h}\left[\sin(a+h) - \sin a\right]$

$= \dfrac{1}{h}\left[2\cos\left(\dfrac{a+h+a}{2}\right)\sin\left(\dfrac{a+h-a}{2}\right)\right] = \dfrac{1}{h}\left[2\cos\left(\dfrac{2a+h}{2}\right)\sin\left(\dfrac{h}{2}\right)\right]$

$= \dfrac{1}{h}\left[2\cos\left(\dfrac{2a}{2}+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)\right] = \dfrac{2}{h}\cos\left(a+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)$

iii) Given that $f(x)=x^3+2x^2-1$

$$f(a+h)=(a+h)^3+2(a+h)^2-1=a^3+h^3+3ah(a+h)+2(a^2+2ah+h^2)-1$$
$$=a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1$$

$f(a)=a^3+2a^2-1$

Now $f(a+h)-f(a)$

$$=\frac{a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-(a^3+2a^2-1)}{h}$$
$$=\frac{1}{h}\left[a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-a^3-2a^2+1\right]$$
$$=\frac{1}{h}\left[h^3+3a^2h+3ah^2+4ah+2h^2\right]=\frac{h}{h}\left[h^2+3a^2+3ah+4a+2h\right]$$
$$=h^2+3a^2+3ah+4a+2h=h^2+3ah+2h+3a^2+4a=h^2+(3a+2)h+3a^2+4a$$

iv) Given that $f(x)=\cos x$

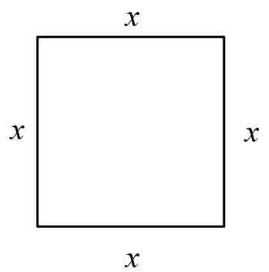
so $f(a+h)=\cos(a+h)$

and $f(a)=\cos a$

Now $\dfrac{f(a+h)-f(a)}{h}$

$$=\frac{\cos(a+h)-\cos a}{h}=\frac{1}{h}\left[-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]=\frac{-2}{h}\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Q3. (a) If 'x' unit be the side of square.



Then its perimeter $P=x+x+x+x=4x$ ………………….. (1)

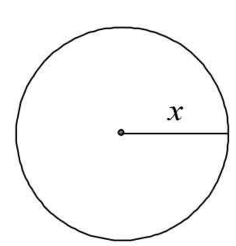
$A=$ Area $=x\,.\,x=x^2$ …………… (2)

From (2) $x=\sqrt{A}$ putting in (1)

$P=4\sqrt{A}$

$\therefore$ P is expressed as Area

(b) Let x units be the radius of circle



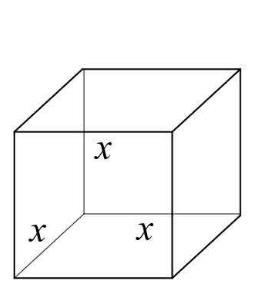
Then Area = A = πx^2 ………………….. (1)

Circumference = C = $2\pi x$ ………………….. (2)

From (2) $x=\dfrac{C}{2\pi}$ Putting in (1)

$$A=\pi\left(\frac{c}{2\pi}\right)^2=\pi\left(\frac{c^2}{4\pi^2}\right)=\frac{c^2}{4\pi}$$

$A=\dfrac{c^2}{4\pi}$ $\because$ *Area is a function of Circumference*



(c) Let x unit be each side of cube.

The Volume of Cube $=x\,.\,x\,.\,x=x^3$ ………………….. (1)

Area of base $=A=x^2$ ………………….. (2)

From (2) $x=\sqrt{A}$ Putting in (1)

$V = \left(\sqrt{A}\right)^3 = (A)^{3/2}$

$Q5. \quad f(x) = x^3 - ax^2 + bx + 1$

$If \quad f(2) = -3$ $\quad and \quad$ $f(-1) = 0$

$(2)^3 - a(2)^2 + b(2) + 1 = -3$

$8 - 4a + 2b + 1 = -3$

$9 - 4a + 2b = -3$

$12 - 4a + 2b = 0$

$Dividing\, by - 2$

$-6 + 2a - b = 0$................. (1)

$(-1)^3 - a(-1)^2 + (-1) + 1 = 0$

$-1 - a - b + 1 = 0$

$-a - b = 0$

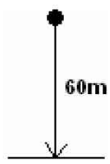
$a + b = 0$ .... (2)

$Solving\,(1)\, and\,(2)$

$$\begin{array}{r} 2a - b - 6 = 0 \\ \underline{a + b \quad = 0} \\ 3a - 6 \quad = 0 \end{array}$$

$a = 2 \quad and \quad (2) \Rightarrow b = -a \quad \Rightarrow \quad b = -2$

$Q6. \quad h(x) = 40 - 10x^2$



$(a) \quad x = 1\sec$

$h(1) = 40 - 10(1)^2$

$= 30m$

$(b) \quad x = 1.5\sec$

$h(1.5) = 40 - 10(1.5)^2$

$= 40 - 10(2.25) = 40 - 22.5 = 17.5m$

$(c) \quad x = 1.7\sec$

$h(1.7) = 40 - 10(1.7)^2$

$= 40 - 10(2.89) = 40 - 28.9 = 11.1m$

ii) Does the stone strike the ground = ?

$h(x) = 0$

$40 - 10x^2 = 0$

$-10x^2 = -40 \quad \Rightarrow x^2 = 4$

$x = \pm 2$

Stone strike the ground after 2 sec.

Graphs of Function

Definition:

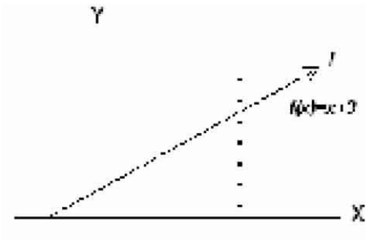
The graph of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane chose co-ordinates (x , y) are input - output pairs for f

.

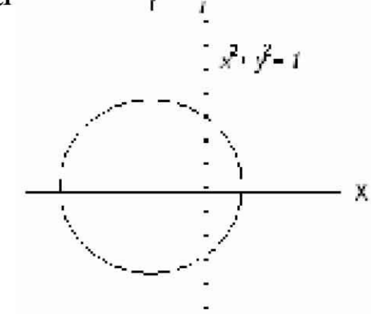
Note that not every curve we draw in the graph of a function. A function f can have only one value $f(x)$ for each x in its domain.

Vertical Line Test

No vertical line can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function. Since some vertical lines intersect the circle Twice. If ‘a’ is the domain of the function f, then the vertical line x = a will intersect th in the single point (a , $f(a)$).



It is graph of function



Is not a graph of function

Types of Function

ALGEBRAIC FUNCTIONS

Those functions which are defined by algebraic expressions.

1) Polynomial Functions:

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots\dots\dots\dots + a_1 x + a_0$ **Is a Polynomial Function for all x where** $a_0 , a_1 , a_2 \dots\dots a_n$ **are real numbers, and exponents are non-negative integer .** a_n **is called leading coefft of p(x) of degree n, Where** $a_n \neq 0$

$\Rightarrow$ *Degree of polynomial function is the* max*imum power of x in equation*

$P(x) = 2x^4 - 3x^3 + 2x - 1$ deg*ree* = 4

2) Linear Function: if the degree of polynomial fn is ‘1, is called linear function .i.e. p(x)=ax+b

or $\Rightarrow$ Degree of polynomial function is one.

$f(x) = ax + b \qquad a \neq 0$

$\because \qquad y = 5x + b$

3) Identity Function: For any set X, a function I: $X \to x$ of the form y = x or $f(x) = x$. Domain and range of I is x. Note. I (x)= ax +b be a linear fn if a=1,b=0 then I(x)=x or y=x is called identity fn

4) Constant Function:

$C : X \to y$ *defined by* called constant fn

$C(x) = a \qquad \forall x \in X$ *and* $a \in y$

e.g. $C : R \to R$

$C(x) = 2$ *or* $y = 2 \quad \forall x \in R$

or

$f : X \to y$ If f (x)=c, (const) then f is

eg y=5

5) Rational Function:

$$R(x) = \frac{P(x)}{Q(x)}$$

Both $P(x)$ *and* $Q(x)$ *are polynomial and* $Q(x) \neq 0$

e.g. $R(x) = \dfrac{3x^2 + 4x + 1}{5x^3 + 2x^2 + 1}$

Domain of rational function is the set of all real numbers for which $Q(x) \neq 0$

6) Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function.

i) $y = a^x \ \therefore x \in R \qquad a > 0$

ii) $y = e^x \ \therefore x \in R$ *and* $e = 2.178$

iii) $y = 2^x$ *or* $y = e^{xh}$

are some exponential functions.

7) Logarithmic Function:

If $x = a^y$ *then* $y = \log_a^{\ x}$ x >0

$\because \quad a > 0 \quad a \neq 1$

'a' is called the base of Logarithemic function

Then $y = \log_a^{\ x}$ *is Logarithmic function of base 'a'*

i) *If base* $= 10$ *then* $y = \log_{10}^{\ x}$

is called common Logarithm of x

ii) *If base* $= e = 2.718$

$y = \log_e^{\ x} = \ln x$ *is called natural* log

8) Hyperbolic Function:

We define as

i) $y = \sinh(x) = \dfrac{e^x - e^{-x}}{2}$ Sine hyperbolic function or hyperbolic sine function

$Dom = \{x / x \in R\}$ *and* $Range = \{y / y \in R\}$

ii) $y = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$ *is called hyperbolic cosine function* $\Rightarrow \quad x \in R, y \in [1, \infty)$

iii) $y = Tanhx = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{\sinh x}{\cosh x}$ *iv)* $y = \coth x = \dfrac{\cosh x}{\sinh x}$

v) $y = \sec hx = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}} \qquad x \in R$

vi) $y = \cos echx = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}} \qquad Dom = \{x \neq 0 : x \in R$

9) Inverse Hyperbolic Function: (Study in B.Sc level)

i) $y=\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$ *for* $\forall x\in R$

ii) $y=\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$ *for* $\forall x\in R$ *and* $x>1$

iii) $y=Tanh^{-1}x=\dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|$ $x\neq 1$ *and* $|x|<1$

iv) $y=\sec h^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1-x^2}}{x}\right)$ $0<x\leq 1$

v) $y=\coth^{-1}x=\dfrac{1}{2}\left|\dfrac{x+1}{x-1}\right|$ $\because$ $|x|>1$

vi) $y=\cos ech^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1+x^2}}{|x|}\right)$ $x\neq 0$

10) Trigonometric Function:

Functions	Domain(x)	Range(y)
i) $y=\sin x$	*All real numbers* $\because -\infty<x+\infty$	$-1\leq y\leq 1$
ii) $y=\cos x$	*All real numbers* $\because -\infty<x<\infty$	$-1\leq y\leq 1$
iii) $y=\tan x$	$x\in R-(2k+1)\dfrac{\pi}{2}$ $k\in Z$	$\because$ 'R' *all real numbers*
iv) $y=\cot x$	$x\in R-k\pi$ $k\in Z$	R
v) $y=\sec x$	$x\in R-(2k+1)\dfrac{\pi}{2}$ $k\in Z$	$R-(-1,1)$ *or* $R-(-1<y<1)$
vi) $y=\cos ecx$	$x\in R-(k\pi)$ $k\in Z$	$R-(-1<y<1)$

11) Inverse Trigonometric Functions:

Function	Dom(x)	Range(y)
$y=\sin^{-1}x\Leftrightarrow x=\sin y$	$-1\leq x\leq 1$	$-\dfrac{\pi}{2}\leq y\leq\dfrac{\pi}{2}$
$y=\cos^{-1}x\Leftrightarrow x=\cos y$	$-1\leq x\leq 1$	$0\leq y\leq\pi$

$y = Tan^{-1}x \Leftrightarrow x = Tany$	$x \in R$ $or\ -\infty < x < \infty$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = Sec^{-1}x \Leftrightarrow x = \sec y$	$x \in R-(-1,1)$	$y \in [0,\pi]-\left\{\dfrac{\pi}{2}\right\}$
$y = Co\sec^{-1}x \Leftrightarrow x = \cos ecy$	$x \in R-(-1,1)$	$y \in \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]-\{0\}$
$y = Cot^{-1}x \Leftrightarrow x = \cot y$	$x \in R$	$0 < y < \pi$

12) <u>Explicit Function:</u>

If y is easily expressed in terms of x, then y is called an explicit function of x.

$\Rightarrow y = f(x)$ *e.g.* $y = x^3 + x + 1$ *etc.*

13) <u>Implicit Function:</u>

If x and y are so mixed up and y cannot be expressed in term of the independent variable x, Then y is called an implicit function of x. It can be written as. $f(x, y) = 0$

e.g. $x^2 + xy + y^2 = 2$ etc.

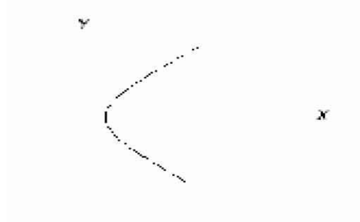
14) <u>Parametric Function:</u>

For a function $y = f(x)$ if both x & y are expressed in another variable say 't' or θ which is called a parameter of the given curve.

Such as:

i) $x = at^2$ Parametric parabola

$y = 2at$



ii) $x = a\cos t$ *Parametric equation of circle* $y^2 = 4a$

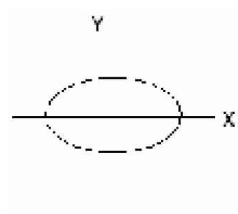
$y = a\sin t$

$x^2 + y^2 = a^2$

iii) $x = a\cos\theta$ *Parametric equation of Ellipse*

$y = b\sin\theta$

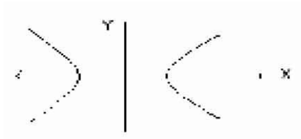
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



vi) $x = a\sec\theta$ *Parametric equation of hyperbola*

$y = b\tan\theta$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Exercise 1.1

$Q7.\quad Parabola \quad \Rightarrow \quad y^2 = 4ax \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

$x = at^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$

$y = 2at \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

$To\,e\lim inating\,'t'\,from\,(ii) \quad t = \dfrac{y}{2a} \qquad putting\,(i)$

$$x = a\left(\frac{y}{2a}\right)^2 \quad \Rightarrow \quad x = a\left(\frac{y^2}{4a^2}\right) \quad \Rightarrow \quad x = \frac{y^2}{4a}$$

$\Rightarrow \quad y^2 = 4ax \qquad which\,is\,same\,as\,(1)$

$which\,is\,equation\,of\;parabola.$

$ii) \quad x = a\cos\theta, \quad y = b\sin\theta$

$\Rightarrow \quad \dfrac{x}{a} = \cos\theta \ldots\ldots\ldots(i) \quad and \quad \dfrac{y}{b} = \sin\theta \ldots\ldots\ldots\ldots(ii) \qquad To\,e\lim inating\,\theta\;from\,(i)\,and\,(ii)$

$Squaring\;and\;adding\;(i)\,and\;(ii)$

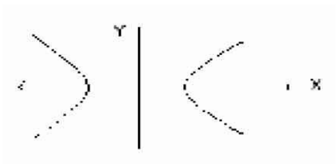
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y^2}{b}\right) = 1 \qquad represent\,a\,Ellipse$$

$iii) \quad x = a\sec\theta, \quad y = b\tan\theta$

$\dfrac{x}{a} = \sec\theta \ldots\ldots\ldots\ldots\ldots(i) \qquad \dfrac{y}{b} = \tan\theta \ldots\ldots\ldots\ldots\ldots\ldots(ii)$

$Squaring\;and\;Subtracting \quad (i)\,and\,(ii)$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2\theta - \tan^2\theta \qquad \Rightarrow \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \tan^2\theta - \tan^2\theta \;\Rightarrow\; \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Which is equation of hyperbola

$Q8. \quad (i) \quad \sinh 2x = 2\sinh x\cosh x$

$$R.H.S = 2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = 2\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x = L.H.S$

$ii) \quad \sec^2 hx = 1 - \tan^2 hx$

$$R.H.S. = 1 - \tan^2 hx = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1 - \left(\frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{e^{2x} + e^{-2x} + 2} = \frac{4}{\left(e^x + e^{-x}\right)^2} = \frac{1}{\left(e^x + e^{-x}\big/2\right)^2}$$

$$= \frac{1}{\cosh^2 x} = \sec h^2 x = L.H.S$$

$iii)$ $\quad \cos eh^2 x = \coth^2 x - 1$

$$R.H.S = \coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2} = \frac{\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x - e^{-x}\right)^2} = \frac{4}{\left(e^x - e^{-x}\right)^2} = \frac{1}{\left(e^x - e^{-x}/2\right)^2} = \frac{1}{\sinh^2 x} = \cos ech 2x = L.H.S$$

$Q9.$ $\quad f(x) = x^3 + x$

replace x by $-x$

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -\left[x^3 + x\right] = -f(x)$$

$\Rightarrow \quad f(x) = x^3 + x$ *is odd function*

$ii)$ $\quad f(x) = (x+2)^2$

replace x by $-x$

$$f(-x) = (-x+2)^2 \neq \pm f(x)$$

$f(x) = (x+2)^2$ *is neither even nor odd*

$iii)$ $\quad f(x) = x\sqrt{x^2+5}$

replace x by $-x$

$$f(-x) = (-x)\sqrt{(-x)^2 + 5} = -\left[x\sqrt{x^2+5}\right] = -f(x)$$ $f(x)$ *is odd function.*

$iv)$ $\quad f(x) = \frac{x-1}{x+1}$

replace x by $-x$

$$f(-x) = \frac{-x-1}{-x+1} = \frac{-(x+1)}{-(x-1)} = \frac{x+1}{x-1} \neq \pm f(x)$$

$f(x)$ *is neither even nor odd function.*

$v)$ $\quad f(x) = x^{\frac{2}{3}} + 6$

replace x by $-x$

$$f(-x) = (-x)^{\frac{2}{3}} + 6 = \left[(-x)^2\right]^{\frac{1}{3}} + 6 = x^{\frac{2}{3}} + 6 = f(x)$$

$f(x)$ *is an even function.*

$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore$ f(x) is neither even nor odd function.

(v) $\mathbf{f(x) = x^{2/3} + 6}$

$$f(-x) = (-x)^{2/3} + 6$$
$$= [(-x)^2]^{1/3} + 6$$
$$= (x^2)^{1/3} + 6$$
$$= x^{2/3} + 6$$
$$= f(x)$$

$\therefore$ f(x) is an even function.

(vi) $\mathbf{f(x) = \dfrac{x^3 - x}{x^2 + 1}}$

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1}$$
$$= \frac{-x^3 + x}{x^2 + 1}$$
$$= \frac{-(x^3 - x)}{x^2 + 1}$$
$$= -f(x)$$

$\therefore$ f(x) is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z. The composition of f and g is a function, denoted by gof, from X to Z and is defined by.

$$(gof)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y. The inverse function of f, denoted by f^{-1}, is a function from Y onto X and is defined by.

$$x = f^{-1}(y) \;,\; \forall\, y \in Y \text{ if and only if } y = f(x), \forall\, x \in X$$

EXERCISE 1.2

Q.1 The real valued functions f and g are defined below. Find

(a) fog (x) (b) gof (x) (c) fof (x) (d) gog (x)

(i) $\mathbf{f(x) = 2x + 1 \;;\; g(x) = \dfrac{3}{x-1} \;,\; x \neq 1}$

(ii) $\quad f(x) = \sqrt{x+1} \quad ; \quad g(x) = \dfrac{1}{x^2} \quad , \quad x \neq 0$

(iii) $\quad f(x) = \dfrac{1}{\sqrt{x-1}} \quad ; \quad x \neq 1 \quad ; \quad g(x) = (x^2+1)^2$

(iv) $\quad f(x) = 3x^4 - 2x^2 \quad ; \quad g(x) = \dfrac{2}{\sqrt{x}} \quad , \quad x \neq 0$

Solution:

(i) $\quad f(x) = 2x + 1 \quad ; \quad g(x) = \dfrac{3}{x-1} \quad , \quad x \neq 1$

(a) $\quad fog\,(x) = f(g(x))$

$$= f\left(\frac{3}{x-1}\right)$$

$$= 2\left(\frac{3}{x-1}\right) + 1$$

$$= \frac{6}{x-1} + 1$$

$$= \frac{6 + x - 1}{x-1}$$

$$= \frac{x+5}{x-1} \quad \text{Ans.}$$

(b) $\quad gof(x) = g(f(x))$

$$= g(2x+1)$$

$$= \frac{3}{2x+1-1} \quad = \frac{3}{2x} \quad \text{Ans.}$$

(c) $\quad fof(x) = f(f(x))$

$$= f(2x+1)$$

$$= 2(2x+1) + 1$$

$$= 4x + 2 + 1$$

$$= 4x + 3 \quad \text{Ans.}$$

(d) $\quad gog(x) = g(g(x))$

$$= g\left(\frac{3}{x-1}\right)$$

$$= \frac{3}{\dfrac{3}{x-1} - 1}$$

$$= \frac{3}{\frac{3-(x-1)}{x-1}}$$

$$= \frac{3(x-1)}{3-x+1}$$

$$= \frac{3(x-1)}{4-x} \quad \text{Ans.}$$

(ii) $\mathbf{f(x) = \sqrt{x+1} \quad ; \quad g(x) = \frac{1}{x^2} \quad , \quad x \neq 0}$

(a) $fog(x) = f(g(x))$

$$= f\left(\frac{1}{x^2}\right)$$

$$= \sqrt{\frac{1}{x^2}+1}$$

$$= \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{x} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\sqrt{x+1}\right)$$

$$= \frac{1}{\left(\sqrt{x+1}\right)^2} = \frac{1}{x+1} \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f\left(\sqrt{x+1}\right)$$

$$= \sqrt{\sqrt{x+1}+1} \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4 \quad \text{Ans.}$$

(iii) $\mathbf{f(x) = \frac{1}{\sqrt{x-1}} \quad ; \quad x \neq 1 \quad ; \quad g(x) = (x^2+1)^2}$

(a) $fog(x) = f(g(x))$

$$= f((x^2+1)^2)$$

$$= \frac{1}{\sqrt{(x^2+1)^2-1}}$$

$$= \frac{1}{\sqrt{x^4 + 1 + 2x^2 - 1}}$$

$$= \frac{1}{\sqrt{x^2(x^2+2)}} \quad = \frac{1}{x\sqrt{x^2+2}} \qquad \text{Ans.}$$

(b) $\quad gof(x) = g(f(x))$

$$= g\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2$$

$$= \left(\frac{1}{x-1} + 1\right)^2 \quad = \left(\frac{1 + x - 1}{x-1}\right)^2$$

$$= \left(\frac{x}{x-1}\right)^2 \qquad \text{Ans.}$$

(c) $\quad fof(x) = f(f(x))$

$$= f\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}}$$

$$= \frac{1}{\sqrt{\frac{1-\sqrt{x-1}}{\sqrt{x-1}}}} \quad = \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x-1}}} \qquad \text{Ans.}$$

(d) $\quad gog(x) = g(g(x))$

$$= g((x^2+1)^2)$$

$$= [\{(x^2+1)^2\}^2 + 1]^2$$

$$= [(x^2+1)^4 + 1]^2 \qquad \text{Ans.}$$

(iv) $\quad \mathbf{f(x) = 3x^4 - 2x^2 \quad ; \quad g(x) = \frac{2}{\sqrt{x}} \quad , \quad x \neq 0}$

(a) $\quad fog(x) = f(g(x))$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2}$$

$$= \frac{8(6 - x)}{x^2} \qquad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g(3x^4 - 2x^2)$$

$$= \frac{2}{\sqrt{3x^4 - 2x^2}}$$

$$= \frac{2}{\sqrt{x^2(3x^2 - 2)}} \qquad = \frac{2}{x\sqrt{3x^2 - 2}} \qquad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f(3x^4 - 2x^2)$$

$$= 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)^2 \qquad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{2}{\sqrt{x}}\right)$$

$$= \frac{2}{\sqrt{2/\sqrt{x}}}$$

$$= 2\sqrt{\frac{\sqrt{x}}{2}}$$

$$= \sqrt{2} \times \sqrt{2}\ \frac{\sqrt{\sqrt{x}}}{\sqrt{2}}$$

$$= \sqrt{2\sqrt{x}} \quad \text{Ans.}$$

Q.2 For the real valued function, f defined below, find:

(a) $f^{-1}(x)$

(b) $f^{-1}(-1)$ **and verify** $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

(i) $f(x) = -2x + 8$ ***(Lahore Board 2007,2009)*** **(ii)** $f(x) = 3x^3 + 7$

(iii) $f(x) = (-x + 9)^3$ **(iv)** $f(x) = \dfrac{2x + 1}{x - 1}$ **, $x > 1$**

Solution:

(i) $\quad \mathbf{f(x) = -2x + 8}$

(a) Since $\quad y = f(x)$

$$\Rightarrow x = f^{-1}(y)$$

Now,

$$f(x) = -2x + 8$$

$$y = -2x + 8$$

$$2x = 8 - y$$

$$x = \frac{8-y}{2}$$

$$f^{-1}(y) = \frac{8-y}{2}$$

Replacing y by x

$$f^{-1}(x) = \frac{8-x}{2}$$

Replacing y by x.

$$\boxed{f^{-1}(x) = \frac{8-x}{2}}$$

(b) Put, $\quad x = -1$

$$f^{-1}(-1) = \frac{8-(-1)}{2} = \frac{8+1}{2} = \frac{9}{2}$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{8-x}{2}\right)$$

$$= -2\left(\frac{8-x}{2}\right) + 8$$

$$= -8 + x + 8$$

$$= x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(-2x + 8)$$

$$= \frac{8-(-2x+8)}{2}$$

$$= \frac{8+2x-8}{2}$$

$$= \frac{2x}{2} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ $\qquad$ Hence proved.

(ii) $\mathbf{f(x) = 3x^3 + 7}$

(a) Since $y = f(x)$

$\Rightarrow x = f^{-1}(y)$

Now

$$f(x) = 3x^3 + 7$$

$$y = 3x^3 + 7$$

$$3x^3 = y - 7$$

$$x^3 = \frac{y-7}{3}$$

$$x = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

$$f^{-1}(y) = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{\frac{1}{3}}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{-8}{3}\right)^{\frac{1}{3}}$$

$$f\left(f^{-1}(x)\right) = f\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]$$

$$= 3\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]^3 + 7$$

$$= 3\left(\frac{x-7}{3}\right) + 7$$

$$= x - 7 + 7 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(3x^3 + 7)$$

$$= \left(\frac{3x^3 + 7 - 7}{3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{3x^3}{3}\right)^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iii) $\mathbf{f(x) = (-x+9)^3}$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$f(x) = (-x+9)^3$

$y = (-x+9)^3$

$y^{\frac{1}{3}} = -x+9$

$x = 9 - y^{\frac{1}{3}}$

Replacing y by x

$f^{-1}(x) = 9 - x^{\frac{1}{3}}$

(b) Put $x = -1$

$f^{-1}(-1) = 9-(-1)^{\frac{1}{3}}$

$$f\left(f^{-1}(x)\right) = f\left(9 - x^{\frac{1}{3}}\right)$$

$$= \left[-\left(9 - x^{\frac{1}{3}}\right) + 9\right]^3$$

$$= \left(-9 + x^{\frac{1}{3}} + 9\right)^3$$

$$= \left(x^{\frac{1}{3}}\right)^3 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left((-x+9)^3\right)$$

$$= 9 - \left[(-x+9)^3\right]^{\frac{1}{3}}$$

$$= 9 - (-x+9)$$

$$= 9 + x - 9$$

$$= x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iv) $\mathbf{f(x) = \dfrac{2x+1}{x-1} \quad , \quad x > 1}$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$$f(x) = \frac{2x+1}{x-1}$$

$$y = \frac{2x+1}{x-1}$$

$$y(x-1) = 2x+1$$

$$yx - y = 2x+1$$

$$yx - 2x = 1 + y$$

$$x(y-2) = y+1$$

$$x = \frac{y+1}{y-2}$$

$$f^{-1}(y) = \frac{y+1}{y-2}$$

Replacing y by x

$$f^{-1}(x) = \frac{x+1}{x-2}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \frac{-1+1}{-1-2}$$

$$= \frac{0}{-3} = 0$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{x+1}{x-2}\right)$$

$$= \frac{2\left(\dfrac{x+1}{x-2}\right)+1}{\dfrac{x+1}{x-2}-1}$$

$$= \frac{\dfrac{2(x+1)+(x-2)}{x-2}}{\dfrac{x+1-(x-2)}{x-2}}$$

$$= \frac{2x + 2 + x - 2}{x + 1 - x + 2}$$

$$= \frac{3x}{3} = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{2x+1}{x-1}\right)$$

$$= \frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2}$$

$$= \frac{\frac{2x + 1 + x - 1}{x - 1}}{\frac{2x + 1 - 2(x - 1)}{x - 1}}$$

$$= \frac{3x}{2x + 1 - 2x + 2}$$

$$= \frac{3x}{3} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

Q.3 Without finding the inverse, state the domain and range of f^{-1}.

(i) $f(x) = \sqrt{x + 2}$ **(ii)** $f(x) = \frac{x - 1}{x - 4}, x \neq 4$

(iii) $f(x) = \frac{1}{x + 3}, x \neq -3$ **(iv)** $f(x) = (x - 5)^2, x \geq 5$

Solution:

(i) $f(x) = \sqrt{x + 2}$

Domain of f(x) = $[-2, +\infty)$

Range of f(x) = $[0, +\infty)$

Domain of $f^{-1}(x)$ = Range of f(x) = $[0, +\infty)$

Range of $f^{-1}(x)$ = Domain of f(x) = $[-2, +\infty)$

(ii) $f(x) = \frac{x - 1}{x - 4}, x \neq 4$

Domain of f(x) = $R - \{4\}$

Range of f(x) = $R - \{1\}$

Domain of $f^{-1}(x)$ = Range of f(x) = $R - \{1\}$

Range of $f^{-1}(x)$ = Domain of f(x) = $R - \{4\}$

(iii) $\mathbf{f(x) = \dfrac{1}{x+3}}$ **, $\mathbf{x \neq -3}$**

Domain of f(x) $= R - \{-3\}$

Range of f(x) $= R - \{0\}$

Domain of $f^{-1}(x)$ = Range of f(x) $= R - \{0\}$

Range of $f^{-1}(x)$ = Domain of f(x) $= R - \{-3\}$

(iv) $\mathbf{f(x) = (x-5)^2}$ **, $\mathbf{x \geq 5}$** ***(Gujranwala Board 2007)***

Domain of f(x) $= [5, +\infty)$

Range of f(x) $= [0, +\infty)$

Domain of $f^{-1}(x)$ = Range of f(x) $= [0, +\infty)$

Range of $f^{-1}(x)$ = Domain of f(x) $= [5, +\infty)$

Limit of a Function:

Let a function f(x) be defined in an open interval near the number 'a' (need not at a) if, as x approaches 'a' from both left and right side of 'a', f(x) approaches a specific number 'L' then 'L', is called the limit of f(x) as x approaches a symbolically it is written as.

$\lim\limits_{x \to a} f(x) = L$ read as "Limit of f(x) as $x \to a$, is L"

Theorems on Limits of Functions:

Let f and g be two functions, for which $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then

Theorem 1: The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$= L + M$$

Theorem 2: The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
$$= L - M$$

Theorem 3: If K is any real numbers, then.

$$\lim_{x \to a} [kf(x)] = K \lim_{x \to a} f(x) = kL$$

Theorem 4: The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x \to a} [f(x) \,.\, g(x)] = [\lim_{x \to a} f(x)]\,[\lim_{x \to a} g(x)] = LM$$

Theorem 5: The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad , \quad g(x) \neq 0,\ M \neq 0$$

Theorem 6: Limit of $[f(x)]^n$, where n is an integer.

$$\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$$

The Sandwich Theorem:

Let f, g and h be functions such that $f(x) \leq g(x) \leq h(x)$ for all number x in some open interval containing "C", except possibly at C itself.

If, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$

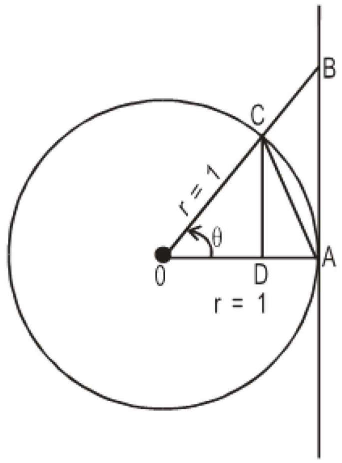
Prove that

If θ is measured in radian, then

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Proof:

Take θ a positive acute central angle of a circle with radius r = 1. OAB represents the sector of the circle.



$$|OA| = |OC| = 1 \qquad \text{(radii of unit circle)}$$

From right angle ΔODC

$$\text{Sin}\theta = \frac{|DC|}{|OC|} = |DC| \qquad (\because \ |OC| = 1)$$

From right angle ΔOAB

$$\text{Tan}\theta = \frac{|AB|}{|OA|} = AB \qquad (\because \ |OA| = 1)$$

In terms of θ, the areas are expressed as

$$\text{Area of } \Delta OAC = \frac{1}{2}\ |OA|\ |CD| = \frac{1}{2}(1)\sin\theta = \frac{1}{2}\sin\theta$$

Area of sector OAC $= \frac{1}{2} r^2\theta = \frac{1}{2} (1)(\theta) = \frac{1}{2}\theta$

Area of ΔOAB $= \frac{1}{2} |OA|\,|AB| = \frac{1}{2} (1) \tan\theta = \frac{1}{2}\tan\theta$

From figure

Area of ΔOAB > Area of sector OAC > Area of ΔOAC

$$\frac{1}{2}\tan\theta > \frac{1}{2}\theta > \frac{1}{2}\sin\theta$$

$$\frac{1}{2}\frac{\sin\theta}{\cos\theta} > \frac{\theta}{2} > \frac{\sin\theta}{2}$$

As $\sin\theta$ is positive, so on division by $\frac{1}{2}\sin\theta$, we get.

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1 \qquad (0 < \theta < \pi/2)$$

i.e.

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

When, $\theta \to 0$, $\cos\theta \to 1$

Since $\frac{\sin\theta}{\theta}$ is sandwitched between 1 and a quantity approaching 1 itself.

So by the sandwitch theorem it must also approach 1.

i.e.

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

Theorem: Prove that

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof:

Taking

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots\ldots\ldots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots\ldots$$

Taking $\lim\limits_{n\to+\infty}$ on both sides.

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\ldots\ldots$$

$$= 1 + 1 + 0.5 + 0.166667 + 0.0416667 + \ldots\ldots\ldots$$

$$= 2.718281\ldots\ldots\ldots$$

As approximate value of e is $= 2.718281$

$$\boxed{\therefore \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^2 = e}$$

Deduction:

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

We know that.

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Put $\quad x = \frac{1}{n}$ then $\frac{1}{x} = n$

As $\quad n \to +\infty \quad , \quad x \to 0$

$$\boxed{\therefore \lim_{n \to +\infty} (1 + x)^{1/x} = e}$$

Theorem:

Prove that:

$$\lim_{x \to a} \frac{a^x - 1}{x} = \log_e a$$

Proof:

Taking,

$$\lim_{x \to a} \frac{a^x - 1}{x}$$

Let $\quad a^x - 1 = y$

$$a^x = 1 + y$$

$$x = \log_a (1 + y)$$

As, $\quad x \to a \quad , \quad y \to 0$

$$\lim_{x \to a} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a(1 + y)}$$

$$= \lim_{y \to 0} \frac{1}{\frac{1}{y} \log_a(1 + y)} \qquad = \lim_{y \to 0} \frac{1}{\log_a(1 + y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a e} \qquad \because \lim_{y \to 0} (1 + y)^{1/y} = e$$

$$= \log_e a$$

Deduction

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

We know that

$$\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$$

Put $\quad a = e$

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

Important results to remember

(i) $\lim_{x \to +\infty} (e^x) = \infty$ $\qquad$ (ii) $\lim_{x \to -\infty} (e^x) = \lim_{x \to -\infty} \left(\frac{1}{e^{-x}}\right) = 0$

(iii) $\lim_{x \to \pm\infty} \left(\frac{a}{x}\right) = 0$, where a is any real number.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits.

(i) $\lim_{x\to 3} (2x + 4)$ $\qquad$ **(ii)** $\lim_{x\to 1} (3x^2 - 2x + 4)$

(iii) $\lim_{x\to 3} \sqrt{x^2 + x + 4}$ $\qquad$ **(iv)** $\lim_{x\to 2} x\sqrt{x^2 - 4}$

(v) $\lim_{x\to 2} \left(\sqrt{x^3 + 1} - \sqrt{x^2 + 5}\right)$ $\qquad$ **(iv)** $\lim_{x\to 2} \frac{2x^3 + 5x}{3x - 2}$

Solution:

(i) $\lim_{x\to 3} (2x + 4) = \lim_{x\to 3} (2x) + \lim_{x\to 3} (4)$

$= 2 \lim_{x\to 3} x + 4$

$= 2(3) + 4 = 6 + 4 = 10$ $\quad$ Ans.

(ii) $\lim_{x\to 1} (3x^2 - 2x + 4) = \lim_{x\to 1} (3x^2) - \lim_{x\to 1} (2x) + \lim_{x\to 1} (4)$

$= 3 \lim_{x\to 1} x^2 - 2 \lim_{x\to 1} x + 4$

$= 3(1)^2 - 2(1) + 4$

$= 3 - 2 + 4$

$= 5$ $\quad$ Ans.

(iii) $\lim_{x\to 3} \sqrt{x^2 + x + 4} = [\lim_{x\to 3} (x^2 + x + 4)]^{1/2}$

Important Limits

I. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, where n is integer and $a>0$.

II. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+a}-\sqrt{a}}{x}=\dfrac{1}{2\sqrt{a}}$.

III. $\lim\limits_{n\to 0}\left(1+\dfrac{1}{n}\right)^n=e$.

IV. $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}}=e$.

V. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$, where $a>0$.

VI. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\ln e=1$.

VII. If θ is measured in radian, then $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$.

Question # 1

(i) $\lim\limits_{x\to 3}(2x+4)\ =\lim\limits_{x\to 3}(2x)+\lim\limits_{x\to 3}(4)\ =2\lim\limits_{x\to 3}(x)+4\ =2(3)+4\ =10$.

(ii) $\lim\limits_{x\to 1}\left(3x^2-2x+4\right)\ =3(1)^2-2(1)+4\ =3-2+4=5$.

(iii) $\lim\limits_{x\to 3}\sqrt{x^2+x+4}\ =\sqrt{(3)^2+(3)+4}\ =\sqrt{9+3+4}\ =\sqrt{16}=4$.

(iv) $\lim\limits_{x\to 2}x\sqrt{x^2-4}\ =2\sqrt{2^2-4}=0$.

(v) $$\begin{aligned}\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)&=\lim_{x\to 2}\left(\sqrt{x^3+1}\right)-\lim_{x\to 2}\left(\sqrt{x^2+5}\right)\\&=\left(\sqrt{(2)^3+1}\right)-\left(\sqrt{(2)^2+5}\right)\\&=\sqrt{8+1}-\sqrt{4+5}\ =\sqrt{9}-\sqrt{9}\ =0.\end{aligned}$$

(vi) $\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2}\ =\dfrac{2(-2)^3+5(-2)}{3(-2)-2}\ =\dfrac{-16-10}{-6-2}\ =\dfrac{-26}{-8}\ =\dfrac{13}{4}$.

Question # 2

(i) $$\begin{aligned}\lim_{x\to -1}\frac{x^3-x}{x+1}&=\lim_{x\to -1}\frac{x(x^2-1)}{x+1}=\lim_{x\to -1}\frac{x(x+1)(x-1)}{x+1}\\&=\lim_{x\to -1}x(x-1)\ =(-1)(-1-1)\ =2\end{aligned}$$

(ii) $\lim\limits_{x\to 0}\left(\dfrac{3x^3+4x}{x^2+x}\right)\ =\lim\limits_{x\to 0}\dfrac{x(3x^2+4)}{x(x+1)}\ =\lim\limits_{x\to 0}\dfrac{3x^2+4}{x+1}\ =\dfrac{3(0)+4}{0+1}\ =4$.

(iii) $\lim_{x\to 2}\dfrac{x^3-8}{x^2+x-6}$

$$=\lim_{x\to 2}\frac{x^3-(2)^3}{x^2+3x-2x-6} \quad =\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x(x+3)-2(x+3)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x+3)(x-2)} \quad =\lim_{x\to 2}\frac{(x^2+2x+4)}{(x+3)}$$

$$=\frac{(2)^2+2(2)+4)}{(2+3)} \quad =\frac{12}{5}$$

(iv) $\lim_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x^3-x}$

$$=\lim_{x\to 1}\frac{(x-1)^3}{x(x^2-1)} \quad =\lim_{x\to 1}\frac{(x-1)^3}{x(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)^2}{x(x+1)} \quad =\lim_{x\to 1}\frac{(1-1)^2}{(1)(1+1)} \quad =0$$

(v) $\lim_{x\to -1}\left(\dfrac{x^3+x^2}{x^2-1}\right) \quad =\lim_{x\to -1}\dfrac{x^2(x+1)}{(x+1)(x-1)} \quad =\lim_{x\to -1}\dfrac{x^2}{(x-1)}$

$$=\frac{(-1)^2}{(-1-1)}=-\frac{1}{2}$$

(vi) $\lim_{x\to 4}\dfrac{2x^2-32}{x^3-4x^2} \quad =\lim_{x\to 4}\dfrac{2(x^2-16)}{x^2(x-4)}$

$$=\lim_{x\to 4}\frac{2(x+4)(x-4)}{x^2(x-4)} \quad =\lim_{x\to 4}\frac{2(x+4)}{x^2}$$

$$=\frac{2(4+4)}{4^2} \quad =\frac{16}{16} \quad =1.$$

(vii) $\lim_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2} \quad =\lim_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2}\left(\dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right)$

$$=\lim_{x\to 2}\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{2}\right)^2}{(x-2)\left(\sqrt{x}+\sqrt{2}\right)} \quad =\lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{x}+\sqrt{2}\right)}$$

$$=\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} \quad =\frac{1}{\sqrt{2}+\sqrt{2}} \quad =\frac{1}{2\sqrt{2}}$$

(viii) $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} \quad =\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \quad =\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \qquad = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} \qquad = \frac{1}{2\sqrt{x}}$$

(ix) $\displaystyle \lim_{x\to a}\frac{x^n - a^n}{x^m - a^m}$

$$= \lim_{x\to a}\frac{(x-a)\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+....+a^{n-1}\right)}{(x-a)\left(x^{m-1}+x^{m-2}a+x^{m-3}a^2+....+a^{m-1}\right)}$$

$$= \lim_{x\to a}\frac{\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+....+a^{n-1}\right)}{\left(x^{m-1}+x^{m-2}a+x^{m-3}a^2+....+a^{m-1}\right)}$$

$$= \frac{a^{n-1}+a^{n-2}a+a^{n-3}a^2+....+a^{n-1}}{a^{m-1}+a^{m-2}a+a^{m-3}a^2+....+a^{m-1}}$$

$$= \frac{a^{n-1}+a^{n-1}+a^{n-1}+....+a^{n-1} \quad (n \text{ terms})}{a^{m-1}+a^{m-1}+a^{m-1}+....+a^{m-1} \quad (m \text{ terms})}$$

$$= \frac{n\,a^{n-1}}{m\,a^{m-1}} \qquad = \frac{n}{m}a^{n-1-m+1} \qquad = \frac{n}{m}a^{n-m}$$

Law of Sine

If θ is measured in radian, then $\displaystyle \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

See proof on book at page 25

Question # 3

(i) $\displaystyle \lim_{x\to 0}\frac{\sin 7x}{x}$

Put $t = 7x \quad \Rightarrow \quad \frac{t}{7} = x$

When $x \to 0$ then $t \to 0$, so

$$\lim_{x\to 0}\frac{\sin 7x}{x} = \lim_{t\to 0}\frac{\sin t}{t/7}$$

$$= 7\lim_{t\to 0}\frac{\sin t}{t} = 7(1) = 7 \qquad \text{By law of sine.}$$

(ii) $\displaystyle \lim_{x\to 0}\frac{\sin x^\circ}{x}$

Since $180^\circ = \pi$ rad $\quad \Rightarrow 1^\circ = \frac{\pi}{180}$ rad $\quad \Rightarrow x^\circ = \frac{x\pi}{180}$ rad

So $\displaystyle \lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin \pi x/180}{x}$

Now put $\dfrac{\pi x}{180} = t$ i.e. $x = \dfrac{180t}{\pi}$

When $x \to 0$ then $t \to 0$, so

$$\lim_{x\to 0}\frac{\sin \pi x/180}{x} = \lim_{x\to 0}\frac{\sin t}{180t/\pi}$$

$$= \frac{\pi}{180}\lim_{x\to 0}\frac{\sin t}{t} = \frac{\pi}{180}(1) = \frac{\pi}{180}$$ by law of sine

(iii) $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\sin\theta}$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{(1+\cos\theta)} = \frac{\sin(0)}{1+\cos(0)} = \frac{0}{1+1} = 0$$

(iv) $\lim\limits_{x\to \pi}\dfrac{\sin x}{\pi - x}$

Put $t = \pi - x \Rightarrow x = \pi - t$

When $x \to \pi$ then $t \to 0$, so

$$\lim_{x\to \pi}\frac{\sin x}{\pi - x} = \lim_{t\to 0}\frac{\sin(\pi - t)}{t}$$

$$= \lim_{t\to 0}\frac{\sin t}{t}$$ $\because \sin(\pi - t) = \sin\left(2\cdot\frac{\pi}{2} - t\right) = \sin t$

$= 1$ By law of sine.

(v) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx} = \lim\limits_{x\to 0}\sin ax\cdot\dfrac{1}{\sin bx}$

$$= \lim_{x\to 0}\sin ax\cdot\left(\frac{ax}{ax}\right)\frac{1}{\sin bx\cdot\left(\frac{bx}{bx}\right)} = \lim_{x\to 0}\frac{\sin ax}{ax}\cdot ax\frac{1}{\frac{\sin bx}{bx}\cdot bx}$$

$$= \frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\frac{1}{\lim\limits_{x\to 0}\frac{\sin bx}{bx}} = \frac{a}{b}\cdot(1)\cdot\frac{1}{(1)} = \frac{a}{b}$$ by law of sine

(vi) $\lim\limits_{x\to 0}\dfrac{x}{\tan x} = \lim\limits_{x\to 0}\dfrac{x}{\frac{\sin x}{\cos x}} = \lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot\cos x$

$$= \lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}\cdot\cos x = \frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\cdot\lim_{x\to 0}\cos x = \frac{1}{1}\cdot 1 = 1$$

(vii) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$\because \sin^2 x = \dfrac{1-\cos 2x}{2}$

$\therefore 2\sin^2 x = 1 - \cos 2x$

$$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

(vii) *Do yourself by rationalizing*

(viii) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\sin\theta$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\sin\theta = (1)\cdot(0) = 0$$

(x) $\lim_{x\to 0}\frac{\sec x - \cos x}{x}$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos x} - \cos x}{x} = \lim_{x\to 0}\frac{\frac{1-\cos^2 x}{\cos x}}{x}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x\cos x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{\cos x} = (1)\,\frac{\sin(0)}{\cos(0)} = (1)\cdot\frac{0}{1} = 0$$

(xi) $\lim_{x\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{p\theta}{2}}{2\sin^2\frac{q\theta}{2}} \qquad \because \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$= \lim_{x\to 0}\ \sin^2\frac{p\theta}{2}\cdot\frac{1}{\sin^2\frac{q\theta}{2}} = \lim_{x\to 0}\ \sin^2\frac{p\theta}{2}\cdot\frac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2}\cdot\frac{1}{\sin^2\frac{q\theta}{2}\cdot\frac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}}$$

$$= \lim_{x\to 0}\ \frac{\sin^2\frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2}\cdot\left(\frac{p\theta}{2}\right)^2\cdot\frac{1}{\frac{\sin^2\frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2}\cdot\left(\frac{q\theta}{2}\right)^2} = \lim_{x\to 0}\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\frac{1}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}\cdot\frac{p^2\theta^2/4}{q^2\theta^2/4}$$

$$= \frac{p^2}{q^2}\left(\lim_{x\to 0}\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\frac{1}{\left(\lim_{x\to 0}\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2} = \frac{p^2}{q^2}(1)^2\cdot\frac{1}{(1)^2} = \frac{p^2}{q^2}$$

(xii) $\lim_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$

$$= \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\sin^3\theta} \quad = \lim_{\theta\to 0}\frac{\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}}{\sin^3\theta}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta-\sin\theta\cos\theta}{\sin^3\theta\cos\theta} \quad = \lim_{\theta\to 0}\frac{\sin\theta\left(1-\cos\theta\right)}{\sin^3\theta\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin^2\theta\cos\theta} \quad = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin^2\theta\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin^2\theta\cos\theta\left(1+\cos\theta\right)} \quad = \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin^2\theta\cos\theta\left(1+\cos\theta\right)}$$

$$= \lim_{\theta\to 0}\frac{1}{\cos\theta\left(1+\cos\theta\right)} \quad = \lim_{x\to 0}\frac{1}{\cos\theta\left(1+\cos\theta\right)}$$

$$= \frac{1}{\cos(1)\left(1+\cos(1)\right)} \quad = \frac{1}{1\cdot\left(1+1\right)} \quad = \frac{1}{2}$$

Note:

a) $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$

b) $\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}} = e$ where $e = 2.718281...$

See proof of (a) and (b) on book at page 23

c) $\lim_{x\to 0}\dfrac{a^x-1}{x} = \log_e a$ or $\ln a$

Proof:

Put $y = a^x - 1$ …….. (i)

When $x\to 0$ then $y\to 0$

Also from (i) $1+y = a^x$

Taking log on both sides

$$\ln\left(1+y\right) = \ln a^x \quad \Rightarrow \quad \ln(1+y) = x\ln a \qquad \because \ \ln x^m = m\ln x$$

$$\Rightarrow \ x = \frac{\ln\left(1+y\right)}{\ln a}$$

Now $\lim_{x\to 0}\dfrac{a^x-1}{x} = \lim_{y\to 0}\dfrac{y}{\dfrac{\ln\left(1+y\right)}{\ln a}}$

$$= \lim_{y\to 0}\frac{y\ln a}{\ln\left(1+y\right)} \quad = \lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln\left(1+y\right)}$$

$$= \lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} = \frac{\ln a}{\lim\limits_{y\to 0}\ln(1+y)^{\frac{1}{y}}} \qquad \because \ln x^m = m\ln x$$

$$= \frac{\ln a}{\ln\left(\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}\right)} = \frac{\ln a}{\ln(e)} \qquad \because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$= \frac{\ln a}{1} = \ln a \qquad \because \ln e = 1$$

Question # 4

(i) $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\right]^2 = e^2$$

(ii) $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

(iii) $$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n} = \left[\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{-n}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(iv) $$\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{n}$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{\frac{3n}{3}} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}} = e^{\frac{1}{3}}$$

(v) $$\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n} = \lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{\frac{4n}{4}} = \left[\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right]^4 = e^4.$$

(vi) $$\lim_{x\to 0}(1+3x)^{\frac{2}{x}}$$

$$= \lim_{x\to 0}(1+3x)^{\frac{6}{3x}} = \left[\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right]^6 = e^6$$

(vii) $$\lim_{x\to 0}\left(1+2x^2\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(1+2x^2\right)^{\frac{2}{2x^2}} = \left[\lim_{x\to 0}\left(1+2x^2\right)^{\frac{1}{2x^2}}\right]^2 = e^2$$

(viii) $$\lim_{h\to 0}(1-2h)^{\frac{1}{h}} = \lim_{h\to 0}(1-2h)^{\frac{-2}{-2h}} = \left[\lim_{h\to 0}(1-2h)^{\frac{1}{-2h}}\right]^{-2} = e^{-2} = \frac{1}{e^2}$$

(ix) $$\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$$

$$= \lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x} = \lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x} = \lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(x) $\displaystyle\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}$; $x<0$

Put $x=-t$ where $t>0$

When $x\to 0$ then $t\to 0$, so

$$\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{t\to 0}\frac{e^{-1/t}-1}{e^{-1/t}+1} = \frac{e^{-1/0}-1}{e^{-1/0}+1}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} \qquad \because e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0$$

$$=-1$$

(xi) $\displaystyle\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}$; $x>0$

$$= \lim_{x\to 0}\frac{e^{1/x}\left(1-\frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1+\frac{1}{e^{1/x}}\right)} = \lim_{x\to 0}\frac{\left(1-\frac{1}{e^{1/x}}\right)}{\left(1+\frac{1}{e^{1/x}}\right)}$$

$$= \frac{1-\frac{1}{e^{1/0}}}{1+\frac{1}{e^{1/0}}} = \frac{1-\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}} = \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1-0}{1+0} = 1$$

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<u>*Book:*</u> *Exercise 1.3 (Page 27)*
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Question # 1:

$(i) \qquad f(x) = 2x^2 + x - 5 \qquad c = 1$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-}(2x^2 + x - 5) = 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}(2x^2 + x - 5) = 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$

$\Rightarrow \qquad \lim\limits_{x\to1^{-1}} f(x) = \lim\limits_{x\to1^+} f(x) = -2 \qquad \therefore \qquad \lim\limits_{x\to1} f(x) = -2$

$(ii) \qquad f(x) = \dfrac{x^2 - 9}{x - 3} \qquad C = -3$

$$\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-} \frac{x^2 - 9}{x - 3} = \frac{\lim\limits_{x\to-3^-}(x^2 - 9)}{\lim\limits_{x\to-3^-}(x - 3)} = \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0$$

$$Now \quad \lim_{x\to-3^+} f(x) = \lim_{x\to-3^+} \frac{x^2 - 9}{x - 3} = \frac{\lim\limits_{x\to-3^+}(x^2 - 9)}{\lim\limits_{x\to-3^+}(x - 3)} = \frac{(-3)^2 - 9}{-3 - 3} = \frac{9 - 9}{-6} = \frac{0}{-6} = 0$$

$\Rightarrow \qquad \lim\limits_{x\to-3^-} f(x) = \lim\limits_{x\to-3^+} f(x) = 0 \qquad \therefore \qquad \lim\limits_{x\to-3} f(x) = 0$

$(iii) \qquad f(x) = |x - 5| \qquad C = 5$

$\lim\limits_{x\to5^-} f(x) = \lim\limits_{x\to5^-} |x - 5| \qquad\qquad |x - 5| = \pm(x - 5)$

$$\begin{array}{ccc} & -(x-5) & & +(x-5) & \\ \hline -\infty & & 5 & & +\infty \end{array}$$

$= \lim\limits_{x\to5^-}[-(x - 5)] = -\lim\limits_{x\to5}(x - 5) = -(5 - 5) = 0$

$\lim\limits_{x\to5^+} f(x) = \lim\limits_{x\to5^+} |x - 5| = \lim\limits_{x\to5^+}(x - 5) = 5 - 5 = 0$

$\Rightarrow \qquad \lim\limits_{x\to5^-} f(x) = \lim\limits_{x\to5^+} f(x) = 0$

$\qquad \lim\limits_{x\to5} f(x) = 0$

Question # 2:

Discuss the continuity of $\quad f(x) \quad at \quad x = c$

$$(i) \qquad f(x) = \begin{cases} 2x + 5 & if \quad x \le 2 \\ 4x + 1 & if \quad x > 2 \\ & c = 2 \end{cases}$$

We have to discuss the continuity of $f(x) \quad at \quad x = 2$

$(a) \qquad f(2) = 2(2) + 5 = 4 + 5 = 9 \qquad ....(1)$

$(b) \qquad \lim\limits_{x\to2} f(x) = ?$

$$\begin{array}{ccc} & f(x) = 2x + 5 & & f(x) = 4x + 1 & \\ \hline -\infty & & 2 & & +\infty \end{array}$$

$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2}(2x + 5) = 2(2) + 5 = 4 + 5 = 9$

and $\quad \lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2}(4x + 1) = 4(2) + 1 = 8 + 1 = 9$

$\therefore \qquad \lim\limits_{x\to2} f(x) = 9 \qquad ....(2)$

(c) *from* (1) *and* (2) *we get*

$\lim_{x\to 2} f(x) = f(2)$

$\therefore$ $f(x)$ *is continous at* $x = 2$

(ii) $f(x) = \begin{cases} 3x-1 & if \quad x<1 \\ 4 & x=1 \quad c=2 \\ 2x & x>1 \end{cases}$

if $c = 2$ $f(c) = f(2)$

is not defined so given function is discontinous

(ii) *Correction*

$$f(x) = \begin{cases} 3x-1 & if \quad x<1 \\ 4 & if \quad x=1 \\ 2x & if \quad x>1 \end{cases}$$

$c = 1$ $(correction)$

$$f(1) = 4$$

$-\infty$ 1 $+\infty$

$f(x) = 3x-1$ $f(x) = 2x$

(a) $f(1) = 4$ $(given)$

(b) $\lim_{x\to 1} f(x) = ?$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(3x-1) = 3(1)-1 = 2$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x) = 2(1) = 2$

$\Rightarrow$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$

$\therefore$ $\lim_{x\to 1} f(x) = 2$ ....(2)

(c) *From* (1) *and* (2) *we get*

$\lim_{x\to 1} f(x) \neq f(1)$

$\therefore$ $f(x)$ *is discontinous at* $x = 1$

(iii) $f(x) = \begin{cases} 3x-1 & if \quad x<1 \\ \\ 2x & if \quad x>1 \end{cases}$ $c = 1$

(a) $f(1)$ *is not defined*

$\therefore$ $f(x)$ *is discontinous at* $x = 1$

Question # 3:

Given that

$$f(x) = \begin{cases} 3x & if & x \le -2 \\ x^2 - 1 & if & -2 < x < 2 \\ 3 & if & x \ge 2 \end{cases}$$

$-\infty$ $+\infty$

$f(x) = 3x$ -2 $f(x) = x^2 - 1$ 2 $f(x) = 3$

(i) *We check continuity at* $x = 2$

(a) $f(2) = 3$ ....(1) $(given)$

(b) $\quad \lim_{x\to 2} f(x) = ?$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^2-1) = (2)^2 - 1 = 4-1 = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(3) = 3$

$\Rightarrow \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$

$\therefore \quad \lim_{x\to 2} f(x) = 3 \quad ....(2)$

(c) $\quad$ *From* (1) *and* (2), $\quad$ *we get*

$\lim_{x\to 2} f(x) = f(2)$

$\therefore \quad f(x)$ *is continuous at* $\quad x = 2$

(ii) $\quad$ *At* $\quad x = -2$

(a) $\quad f(-2) = 3(-2) = -6 \quad ....(1)$

(b) $\quad \lim_{x\to -2} f(x) = ?$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2}(3x) = 3(-2) = -6$

and $\quad \lim_{x\to -2^+} f(x) = \lim_{x\to -2}(x^2-1) = (-2)^2 - 1 = 4-1 = 3$

$\Rightarrow \quad \lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x) \qquad \Rightarrow \lim_{x\to -2} f(x)$ *does not exist*

$\therefore \quad f(x)$ *is discontinuous at* $\quad x = -2$

Question # 4:

Given that

$$f(x) = \begin{cases} x+2 & x \le -1 \\ c+2 & x > -1 \end{cases}$$

$c = ?$

$$\begin{array}{ccc} -\infty & & +\infty \\ \hline f(x) = x+2 & -1 & f(x) = c+2 \end{array}$$

$\because \quad \lim_{x\to -1} f(x)$ $\quad$ *exists*

$\therefore \quad \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\Rightarrow \quad \lim_{x\to -1}(x+2) = \lim_{x\to -1}(c+2)$

$\Rightarrow \quad -1+2 = c+2$

$\Rightarrow \quad 1 = c+2$

$\Rightarrow \quad c = 1-2 \qquad \Rightarrow \qquad c = -1$

Question # 5:

(i)

$$f(x) = \begin{cases} mx & if \quad x < 3 \\ n & if \quad x = 3 \\ -2x+9 & if \quad x > 3 \end{cases}$$

here $\qquad f(3) = n \qquad$ *(given)*

$\because \quad f(x)$ *is continuous at* $\quad x = 3$

$\therefore \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \quad \lim_{x\to 3}(mx) = \lim_{x\to 3}(-2x+9) = n$

$\Rightarrow \quad (m)(3) = -2(3) + 9 = n$

$\Rightarrow \quad 3m = -6 + 9 = n$

$\Rightarrow \quad 3m = 3 = n$

$\Rightarrow \quad 3m = 3 \quad , \quad n = 3$

$\Rightarrow \quad m = 1 \quad , \quad n = 3$

$$(ii) \quad f(x) = \begin{cases} mx & if \quad x < 4 \\ x^2 & if \quad x \geq 4 \end{cases}$$

$here \quad f(4) = (4)^2 = 16$

$\because \quad f(x) \quad is\ continuous\ at \quad x = 4$

$\because \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$

$\Rightarrow \quad \lim_{x \to 4} (mx) = \lim_{x \to 4} (x^2) = 16$

$\Rightarrow \quad 4m = (4)^2 = 16$

$\Rightarrow \quad 4m = 16 = 16 \quad \Rightarrow \quad 4m = 16$

$\Rightarrow \quad m = 4$

Question # 6:

$Given \ that$

$$f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & x \neq 2 \\ K & x = 2 \end{cases}$$

$K = ?$

$here \quad f(2) = K \quad given$

$\because \quad f(x) \quad is\ continuous\ at \quad x = 2$

$\because \quad \lim_{x \to 2} f(x) = f(2)$

$$\lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = K \quad \Rightarrow \quad \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} = K$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{\left(\sqrt{2x+5}\right)^2 - \left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K \quad \Rightarrow \quad \frac{(2x+5) - (x+7)}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K$$

$$\Rightarrow \quad \frac{2x+5-x-7}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K \quad \Rightarrow \quad \frac{(x-2)}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = K \quad \Rightarrow \quad \frac{1}{\lim_{x \to 2} \left[\sqrt{2x+5} + \sqrt{x+7}\right]} = K$$

$$\Rightarrow \quad \frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}} = K \quad \Rightarrow \quad \frac{1}{\sqrt{9} + \sqrt{9}} = K$$

$$\Rightarrow \quad \frac{1}{3+3} = K$$

$$\Rightarrow \quad K = \frac{1}{6}$$

EXERCISE 1.5

Q.1 Draw the graphs of the following equations.

(i) $x^2 + y^2 = 9$ **(ii)** $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(iii) $y = e^{2x}$ **(iv)** $y = 3^x$

Solution:

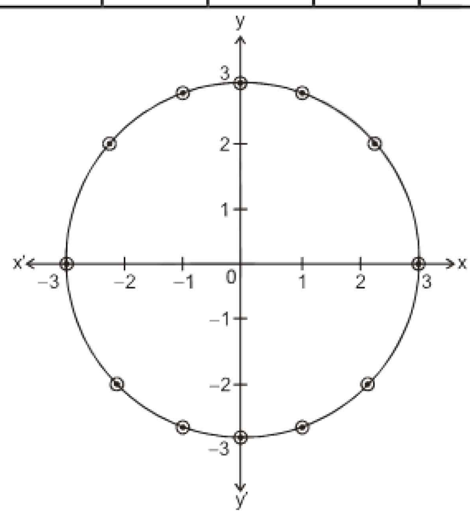
(i) $x^2 + y^2 = 9$

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

Its domain is $-3 \le x \le 3+$

x	−3	−2	−1	0	1	2	3
$y = \pm\sqrt{9 - x^2}$	0	± 2.2	± 2.8	± 3	± 2.8	± 2.2	0



(ii) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}$$

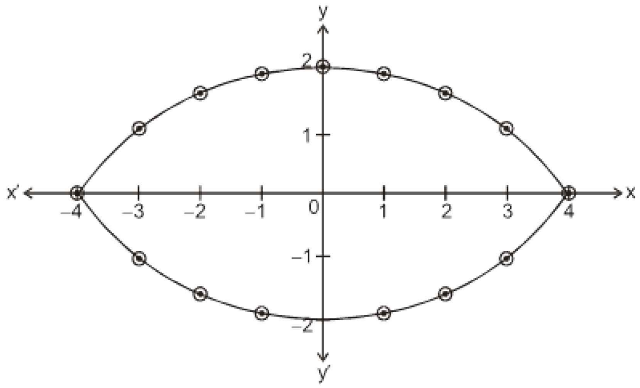
$$y^2 = 4\left(\frac{16 - x^2}{16}\right)$$

$$y^2 = \frac{16 - x^2}{4}$$

$$y = \pm\frac{\sqrt{16 - x^2}}{2}$$

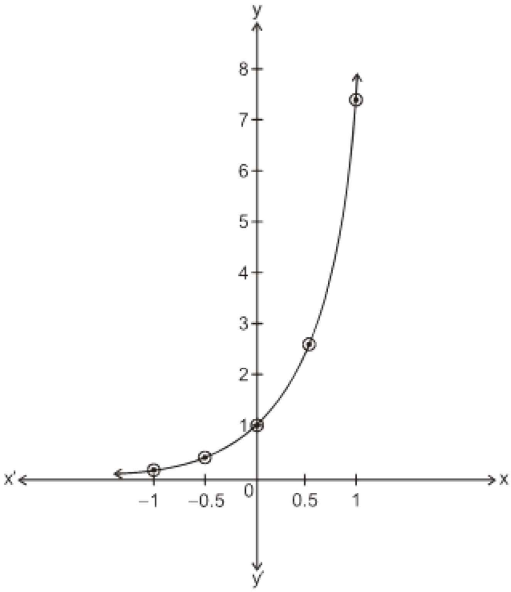
Its domain is $-4 \le x \le 4$.

x	−4	−3	−2	−1	0	1	2	3	4
$y = \pm \frac{\sqrt{9 - x^2}}{2}$	0	± 1.3	± 1.7	± 1.9	± 2	± 1.9	± 1.7	± 1.3	0



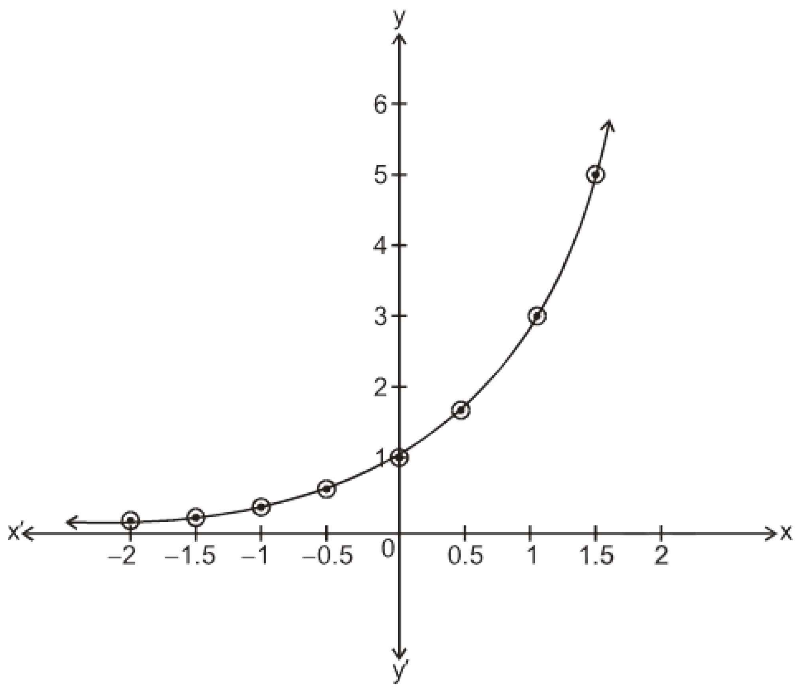
(iii) $\mathbf{y = e^{2x}}$

x	−1	−0.5	0	0.5	1
$y = e^{2x}$	0.1	0.4	1	2.7	7.4



(iv) $\mathbf{y = 3^x}$

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5
$y = 3^x$	0.1	0.2	0.3	0.6	1	1.7	3	5.2



Q.2 **Graph the curves that has the parametric equations given below.**

(i) $x = t$, $y = t^2$, $-3 \le t \le 3$ **where 't' is a parameter**

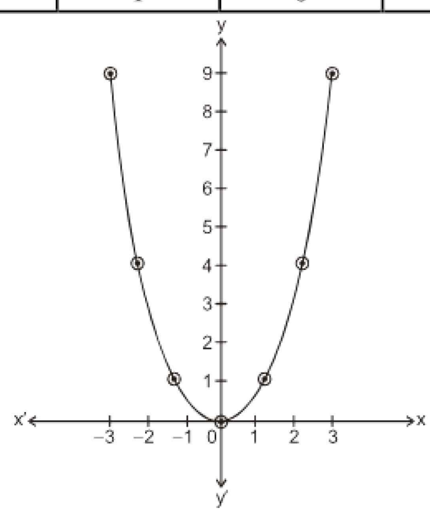
(ii) $x = t - 1$, $y = 2t - 1$, $-1 < t < 5$ **where 't' is a parameter**

(iii) $x = \sec\theta$, $y = \tan\theta$ **where 'θ' is a parameter**

Solution:

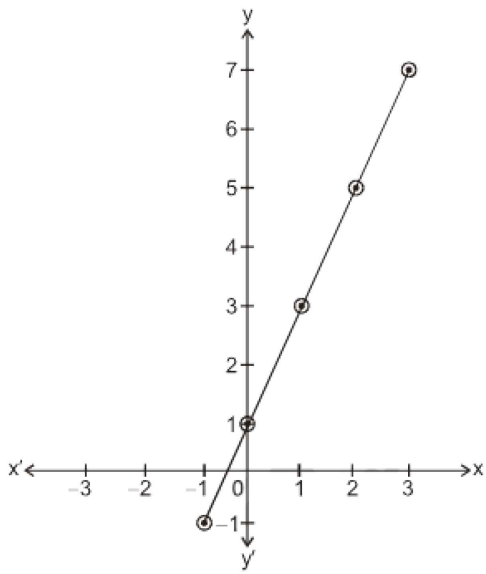
(i) $x = t$, $y = t^2$, $-3 \le t \le 3$ where 't' is a parameter

t	−3	−2	−1	0	1	2	3
x = t	−3	−2	−1	0	1	2	3
$y = t^2$	9	4	1	0	1	4	9



(ii) $x = t - 1$, $y = 2t - 1$, $-1 < t < 5$ where 't' is a parameter

t	0	1	2	3	4
$x = t - 1$	-1	0	1	2	3
$y = 2t - 1$	-1	1	3	5	7



(iii) $x = \sec\theta$, $y = \tan\theta$ where 'θ' is a parameter

$$x^2 = \sec^2\theta , \quad y^2 = \tan^2\theta$$

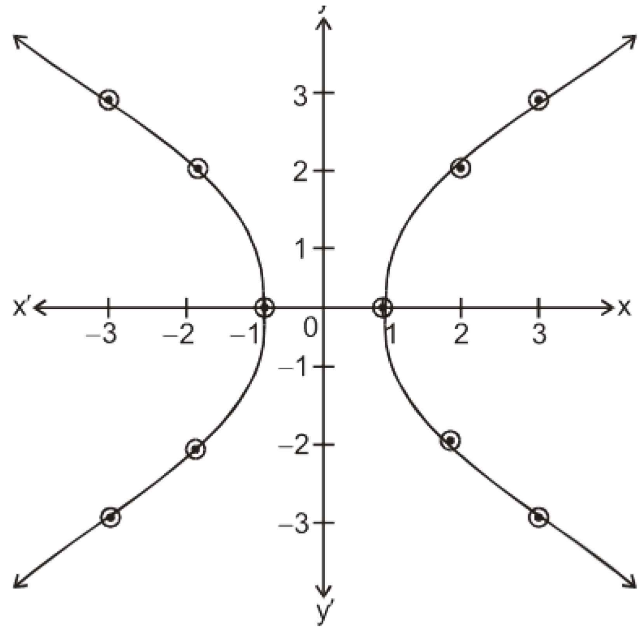
$$x^2 - y^2 = \sec^2\theta - \tan^2\theta$$

$$x^2 - y^2 = 1 \qquad (\because \ 1 + \tan^2\theta = \sec^2\theta \Rightarrow 1 = \sec^2\theta - \tan^2\theta)$$

$$y^2 = x^2 - 1$$

$$y = \pm\sqrt{x^2 - 1}$$

x	-3	-2	-1	1	2	3
$y = \sqrt{x^2 - 1}$	± 2.8	± 1.7	0	0	± 1.7	± 2.8



Q.3 **Draw the graphs of the functions defined below and find whether they are continuous.**

(i) $y = \begin{cases} x - 1 & \text{if } x < 3 \\ 2x + 1 & \text{if } x \geq 3 \end{cases}$ **(ii)** $y = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$

(iii) $y = \begin{cases} x + 3 & , x \neq 3 \\ 2 & , x = 3 \end{cases}$ **(iv)** $y = \dfrac{x^2 - 16}{x - 4}$, $x \neq 4$

Solution:

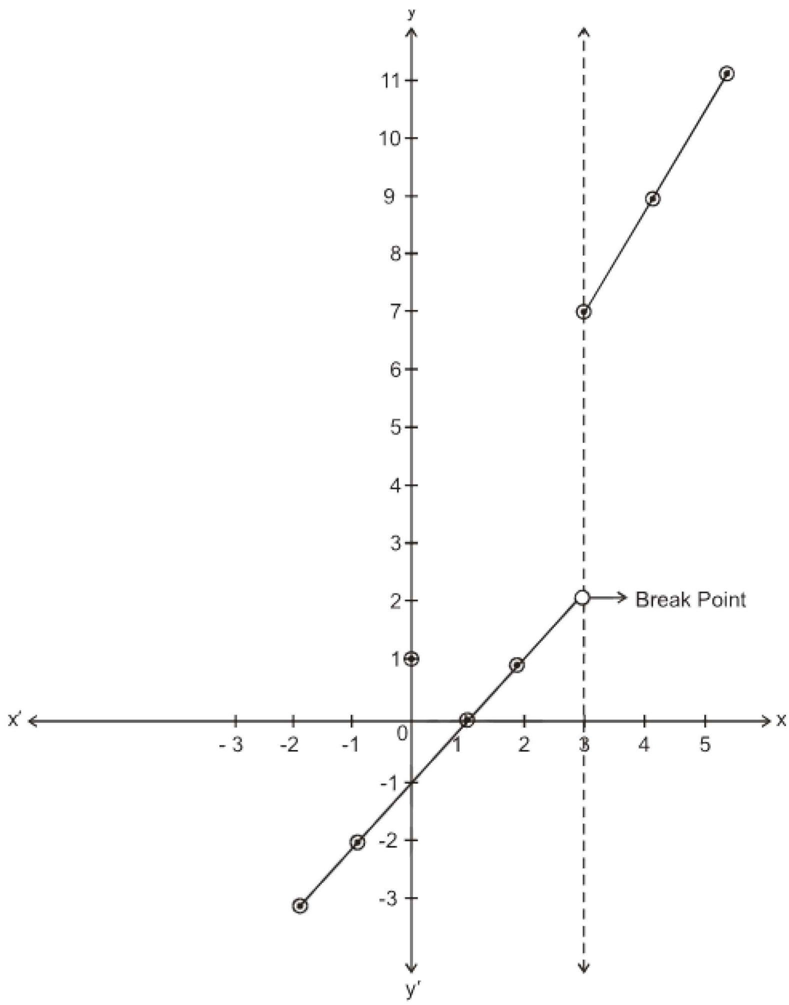
(i) $y = \begin{cases} x - 1 & \text{if } x < 3 \\ 2x + 1 & \text{if } x \geq 3 \end{cases}$

$y = x - 1 \quad , \quad x < 3$

x	–2	–1	0	1	2
y = x – 1	–3	–2	–1	0	1

$y = 2x + 1 \quad , \quad x \geq 3$

x	3	4	5
y = 2x + 1	7	9	11



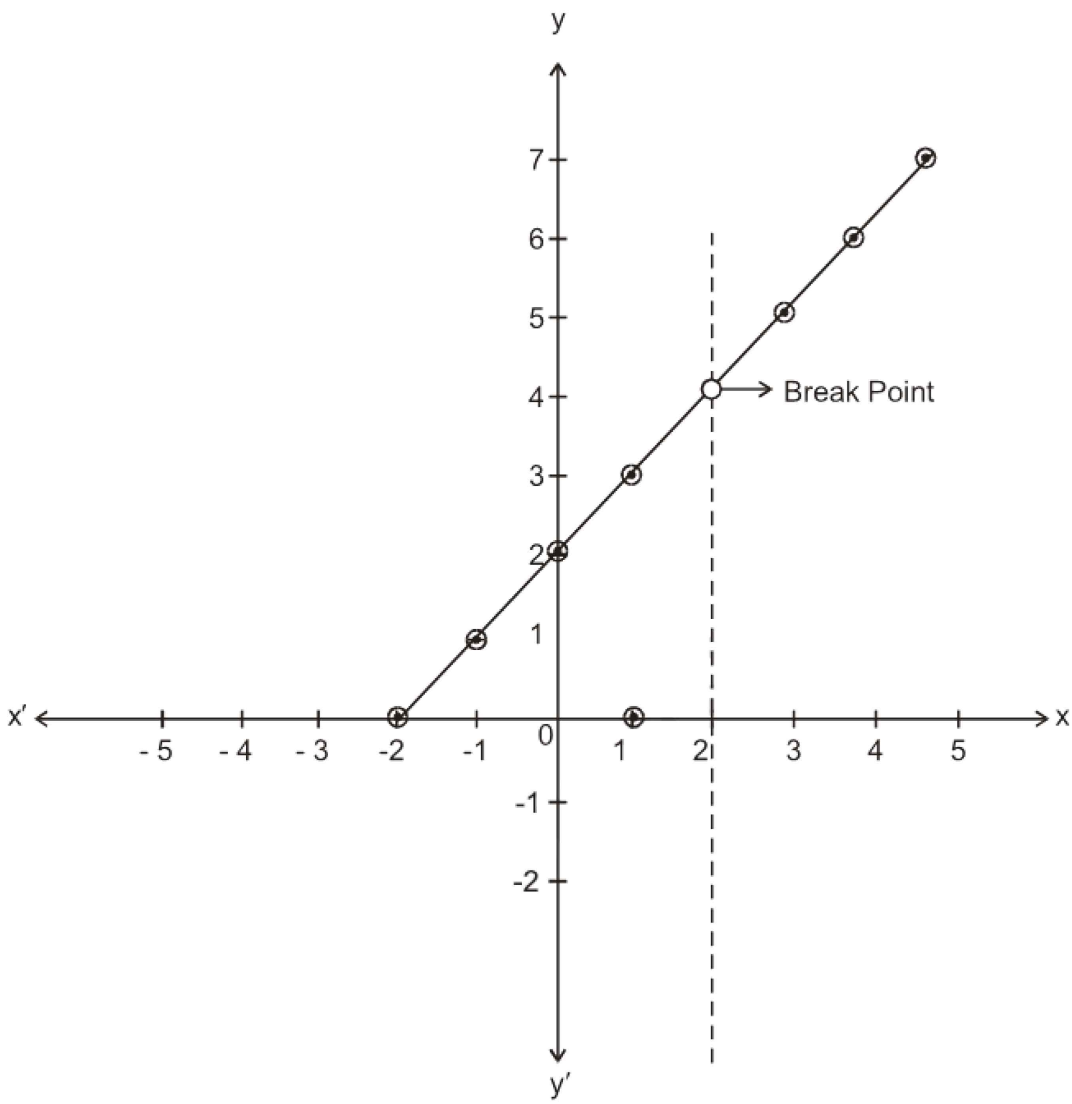
Since there is a break in a graph. So this function is not continuous.

(ii) $\mathbf{y = \dfrac{x^2 - 4}{x - 2}, \quad x \neq 2}$

$$= \frac{(x+2)(x-2)}{x-2}, \quad x \neq 2$$

$$y = x + 2, \quad x \neq 2$$

x	– 3	– 2	– 1	0	1	3	4	5
y	– 1	0	1	2	3	5	6	7



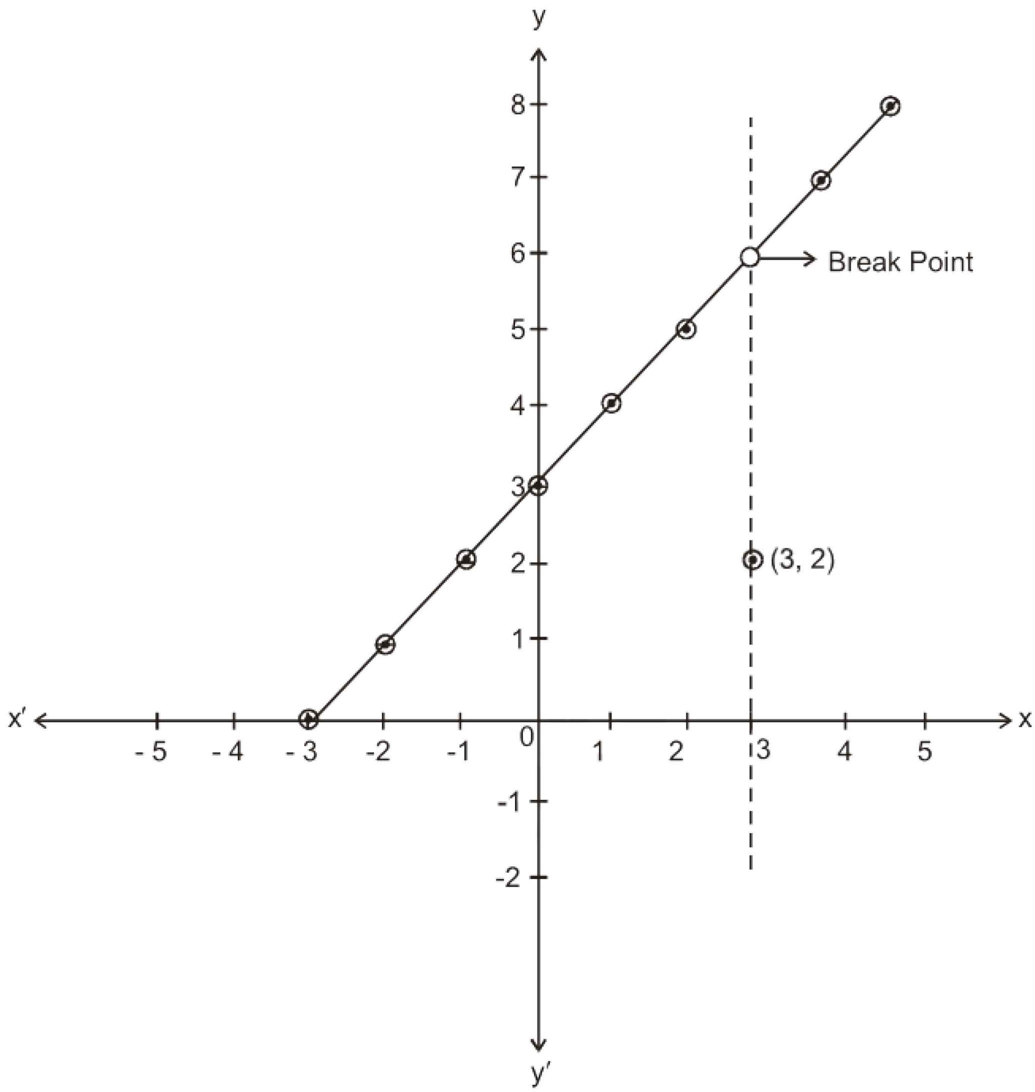
Since there is a break in a graph so this function is not continuous.

(iii) $\mathbf{y = \begin{cases} x + 3 & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}}$

$y = x + 3 \quad \text{if} \quad x \neq 3$

x	– 3	– 2	– 1	0	1	3	4	5
y	0	1	2	3	4	5	7	8

$y = 2 \quad \text{if} \quad x = 3$

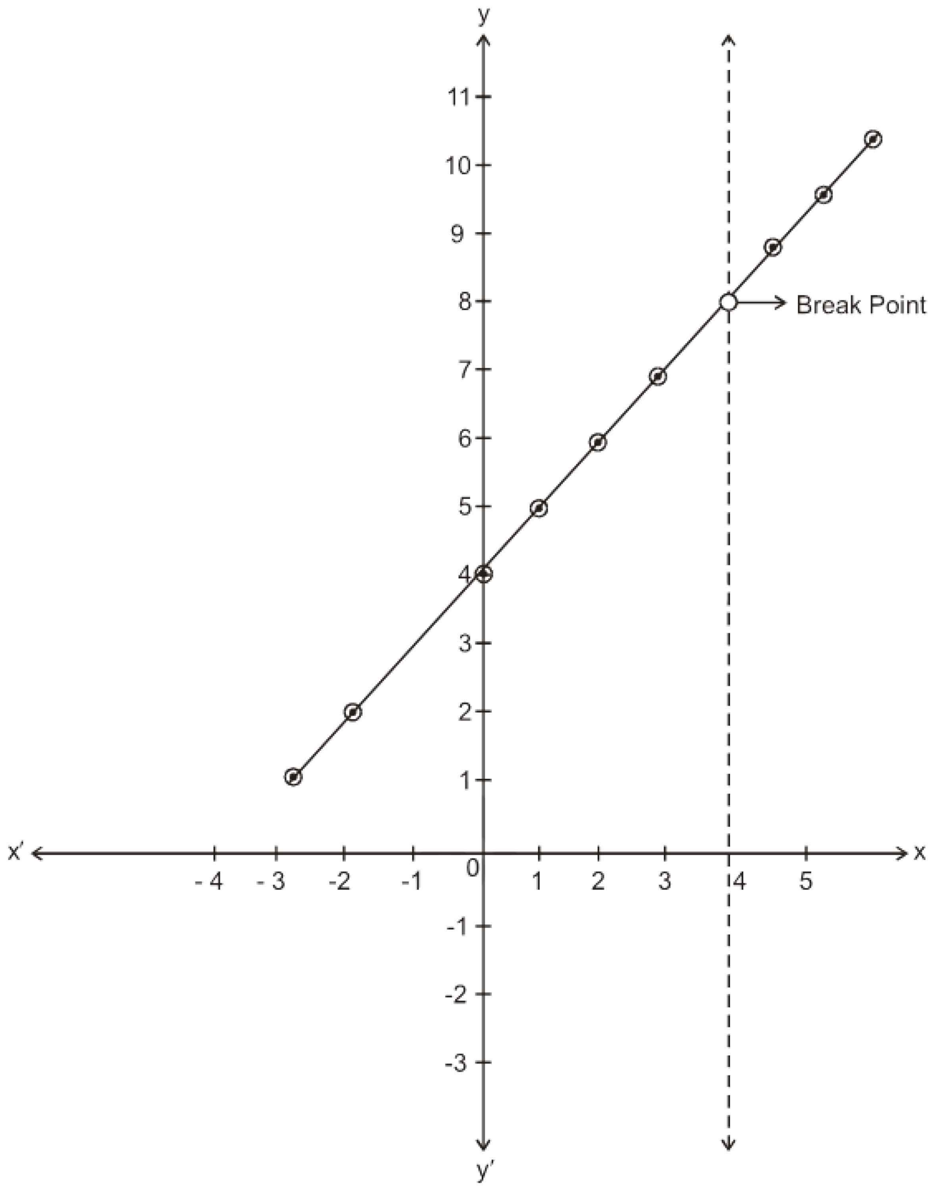


Since there is a break in a graph. So this function is not continuous at x = 3.

(iv) $\mathbf{y = \dfrac{x^2 - 16}{x - 4}, \quad x \neq 4}$

$$= \frac{(x + 4)(x - 4)}{x - 4}, \quad x \neq 4$$

x	– 3	– 2	– 1	0	1	2	3	5	6
y	1	2	3	4	5	6	7	9	10



Since there is a break in a graph. So this function is not continuous at x = 4.

Q.4 Find the graphical solution of the following equations.

(i) $x = \sin 2x$ **(ii)** $\frac{x}{2} = \cos x$ **(iii)** $2x = \tan x$

Solution:

(i) Let $y = x = \sin 2x$

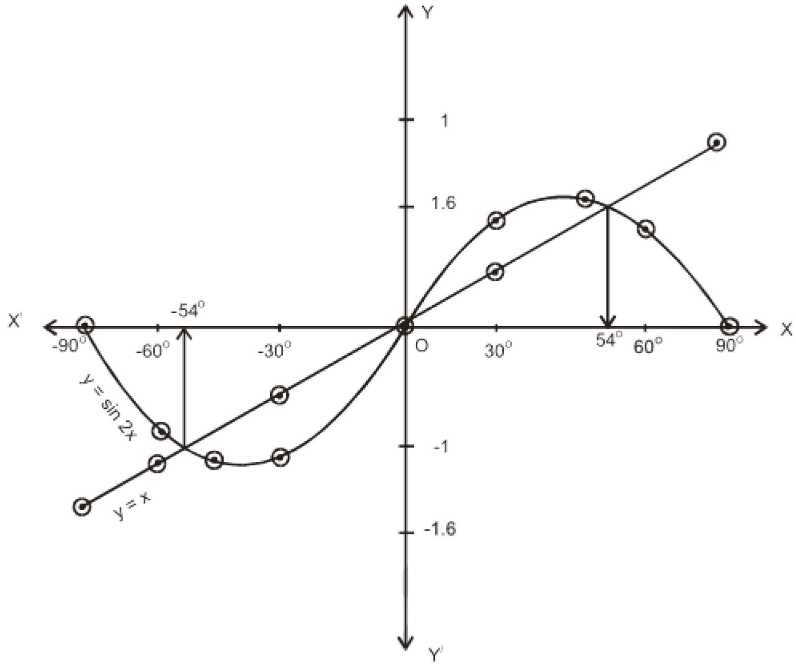
Therefore $y = x$ and $y = \sin 2x$

x	– 90º	– 60º	– 30º	0º	30º	60º	90º
y = x	– π/2 = – 1.6	– π/3 = – 1.05	– π/6 = – 0.52	0	π/6 = 0.52	π/3 = 1.05	π/2 = 1.6

$y = \sin 2x$

x	-90^o	-60^o	-30^o	0^o	30^o	60^o	90^o
$y = \sin 2x$	0	-0.87	-0.87	0	0.87	0.87	0

The graphical solution is the points of intersection of two graphs, i.e. $x = 0^o$, 54^o



(ii) **Let** $y = \dfrac{x}{2} = \cos x$

Therefore $y = \dfrac{x}{2}$ and $y = \cos x$

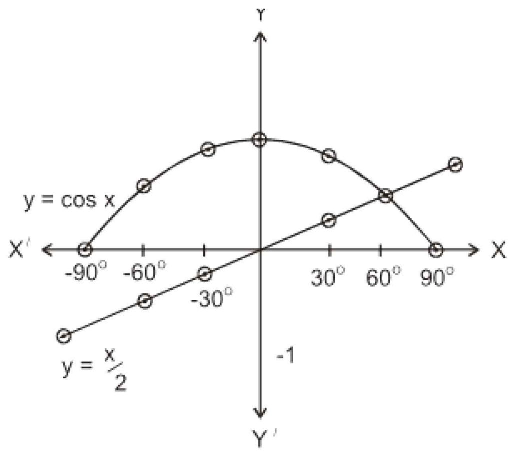
$y = \dfrac{x}{2}$

x	-90^o	-60^o	-30^o	0^o	30^o	60^o	90^o
$y = \dfrac{x}{2}$	$-\pi/4$ $= -.79$	$-\pi/6$ $= -0.52$	$-\pi/12$ $= -0.26$	0	$\pi/6$ $= 0.26$	$\pi/6$ $= 0.52$	$\pi/4$ $= 0.79$

$y = \cos x$

x	-90^o	-60^o	-30^o	0^o	30^o	60^o	90^o
$y = \cos x$	0	0.5	0.87	1	0.87	0.5	0

The graphical solution is the point on x-axis, which is just below the point of intersection of two graphs. Hence $x = 60^o$.



(iii) **Let** $\mathbf{y = 2x = \tan x}$

Therefore $y = 2x$ and $y = \tan x$

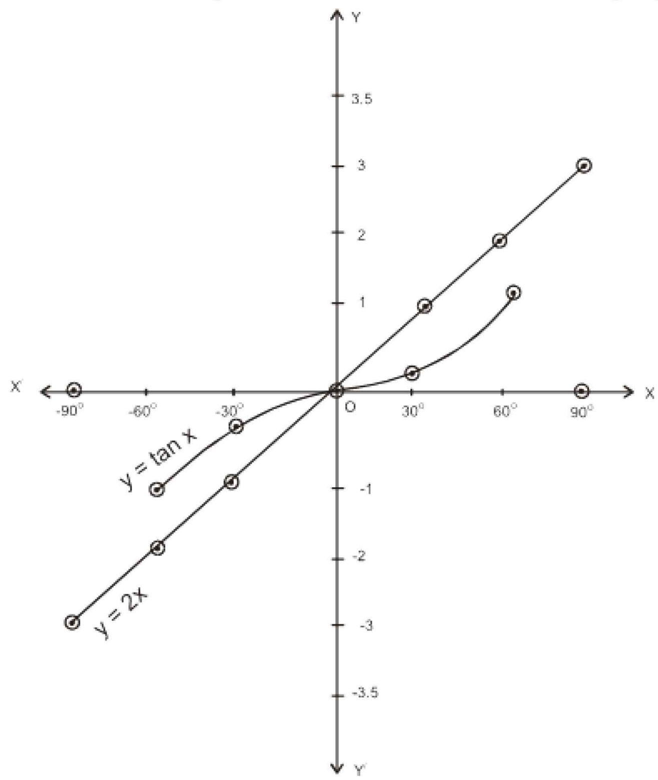
$y = 2x$

x	– 90º	– 60º	– 30º	0º	30º	60º	90º
y = 2x	$-\pi = -3.14$	$-2\pi/3 = -2.09$	$-\pi/3 = -1.05$	0	$\pi/3 = 1.05$	$2\pi/3 = 2.09$	$\pi = 3.14$

$y = \tan x$

x	– 90º	– 60º	– 30º	0º	30º	60º	90º
y = tan x	∞	– 1.73	– 0.58	0	0.58	1.73	∞

The graphical solution is the point of intersection of two graphs, i.e. x = 0º.



Question # 1
Find the definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2+1$ (ii) $2-\sqrt{x}$ (iii) $\dfrac{1}{\sqrt{x}}$ (iv) $\dfrac{1}{x^3}$

(v) $\dfrac{1}{x-a}$ (vi) $x(x-3)$ (vii) $\dfrac{2}{x^4}$ (viii) $(x+4)^{\frac{1}{3}}$

(ix) $x^{\frac{3}{2}}$ (x) $x^{5/2}$ (xi) $x^m, m\in N$ (xii) $\dfrac{1}{x^m}, m\in N$

(xiii) x^{40} (xiv) x^{-100}

Solution

(i) Let $y=2x^2+1$

$\Rightarrow\ y+\delta y=2(x+\delta x)^2+1 \qquad \Rightarrow\ \delta y=2(x+\delta x)^2+1-y$

$\Rightarrow\ \delta y=2\left(x^2+2x\delta x+\delta x^2\right)+1-2x^2-1 \qquad \because\ y=2x^2+1$

$\Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2 \quad \Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2$

$\Rightarrow\ \delta y=4x\delta x+2\delta x^2$

$=\delta x(4x+2\delta x)$

Dividing by δx

$$\frac{\delta y}{\delta x}=4x+2\delta x$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(4x+2\delta x)$$

$$\Rightarrow\ \frac{dy}{dx}=4x+2(0)$$

$$\Rightarrow\ \frac{dy}{dx}=4x \quad \text{i.e.} \quad \boxed{\frac{d}{dx}\left(2x^2+1\right)=4x}$$

(ii) Let $y=2-\sqrt{x}$

$\Rightarrow\ y+\delta y=2-\sqrt{x+\delta x} \qquad \Rightarrow\ \delta y=2-\sqrt{x+\delta x}-y$

$\Rightarrow\ \delta y=2-\sqrt{x+\delta x}-2+\sqrt{x} \qquad \Rightarrow\ \delta y=x^{\frac{1}{2}}-(x+\delta x)^{\frac{1}{2}}$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{1}{2}\cdot\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\right)$$

$$= x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{1}{2}}\left(\frac{\delta x}{2x} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{\delta x^2}{x^2} + ...\right)$$

$$= -x^{\frac{1}{2}}\delta x\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

Taking limit as

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\lim_{\delta x \to 0}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

$$\Rightarrow \frac{dy}{dx} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - 0 + 0 - ....\right)$$

$$= -x^{\frac{1}{2}} \cdot \frac{1}{2x} = -\frac{1}{2}x^{\frac{1}{2}-1} \Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}}$$

(iii) Let $y = \dfrac{1}{\sqrt{x}} \Rightarrow y = x^{-\frac{1}{2}}$

Now do yourself as above

(iv) Let $y = \dfrac{1}{x^3} \Rightarrow y = x^{-3}$

$$\Rightarrow y + \delta y = (x + \delta x)^{-3}$$

$$\Rightarrow \delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\Rightarrow \delta y = x^{-3}\left[\left(1 + \frac{\delta x}{x}\right)^{-3} - 1\right]$$

$$= x^{-3}\left[\left(1 - \frac{3\delta x}{x} + \frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + ...\right) - 1\right]$$

$$= x^{-3}\left[1 - \frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + ... - 1\right]$$

$$= x^{-3}\left[-\frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + ...\right]$$

$$= x^{-3} \cdot \frac{\delta x}{x}\left[-3 + 6\left(\frac{\delta x}{x}\right) - .....\right]$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x}=x^{-3-1}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

Taking limit on both sided, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{-4}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

$$\Rightarrow \frac{dy}{dx}=x^{-4}\left[-3+0-0+.....\right]$$

$$\Rightarrow \frac{dy}{dx}=-3x^{-4} \quad \text{or} \quad \boxed{\frac{dy}{dx}=-\frac{3}{x^4}}$$

(v) Let $y=\dfrac{1}{x-a}$

$$\Rightarrow y=(x-a)^{-1}$$

$$\Rightarrow y+\delta y=(x+\delta x-a)^{-1}$$

$$\Rightarrow \delta y=(x-a+\delta x)^{-1}-y$$

$$\Rightarrow \delta y=(x-a+\delta x)^{-1}-(x-a)^{-1}$$

$$=(x-a)^{-1}\left[\left(1+\frac{\delta x}{x-a}\right)^{-1}-1\right]$$

$$=(x-a)^{-1}\left[\left(1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....\right)-1\right]$$

$$\Rightarrow \delta y=(x-a)^{-1}\left[1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....-1\right]$$

$$=(x-a)^{-1}\left[-\frac{\delta x}{x-a}+\frac{-1(-2)}{2}\left(\frac{\delta x}{x-a}\right)^2+....\right]$$

$$=(x-a)^{-1}\cdot\frac{\delta x}{x-a}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Taking limit when $\delta x\to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

$$\Rightarrow \frac{dy}{dx}=(x-a)^{-2}\left[-1+0-0+....\right] \quad \Rightarrow \boxed{\frac{dy}{dx}=\frac{-1}{(x-a)^2}}$$

(vi) Let $y = x(x-3)$

$= x^2 - 3x$

Do yourself

(vii) Let $y = \dfrac{2}{x^4} = 2x^{-4}$

$\Rightarrow y+\delta y = 2(x+\delta x)^{-4}$

Do yourself

(viii) Let $y = (x+4)^{\frac{1}{3}}$

$$\Rightarrow y+\delta y = (x+\delta x+4)^{\frac{1}{3}}$$

$$\Rightarrow \delta y = (x+\delta x+4)^{\frac{1}{3}} - y$$

$$= (x+4+\delta x)^{\frac{1}{3}} - (x+4)^{\frac{1}{3}}$$

$$= (x+4)^{\frac{1}{3}}\left[\left(1+\frac{\delta x}{x+4}\right)^{\frac{1}{3}} - 1\right]$$

$$= (x+4)^{\frac{1}{3}}\left[\left(1+\frac{1}{3}\frac{\delta x}{x+4}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2+\dots\right)-1\right]$$

$$= (x+4)^{\frac{1}{3}}\left[\frac{\delta x}{3(x+4)}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)^2+\dots\right]$$

$$= (x+4)^{\frac{1}{3}}\cdot\frac{\delta x}{x+4}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = (x+4)^{\frac{1}{3}-1}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-0+0-\dots\right] \qquad \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{3}(x+4)^{-\frac{2}{3}}}$$

(ix) Let $y = x^{\frac{3}{2}}$

$$\Rightarrow y+\delta y = (x+\delta x)^{\frac{3}{2}}$$

$$\Rightarrow \delta y=(x+\delta x)^{\frac{3}{2}}-x^{\frac{3}{2}}$$
$$=x^{\frac{3}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{3}{2}}-1\right]$$
$$=x^{\frac{3}{2}}\left[\left(1+\frac{3}{2}\frac{\delta x}{x}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+....\right)-1\right]$$
$$=x^{\frac{3}{2}}\left[\frac{3\delta x}{2x}+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2}\left(\frac{\delta x}{x}\right)^2+....\right]$$
$$=x^{\frac{3}{2}}\cdot\frac{\delta x}{x}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=x^{\frac{3}{2}-1}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{1}{2}}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$
$$\Rightarrow \frac{dy}{dx}=x^{\frac{1}{2}}\left[\frac{3}{2}-0+0-....\right] \qquad \Rightarrow \boxed{\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}}$$

(x) Let $y = x^{5/2}$

Do yourself as above.

(xi) Let $y = x^m$

$$\Rightarrow y+\delta y = (x+\delta x)^m$$
$$\Rightarrow \delta y = (x+\delta x)^m-x^m$$
$$= x^m\left[\left(1+\frac{\delta x}{x}\right)^m-1\right]$$
$$= x^m\left[\left(1+m\cdot\frac{\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+....\right)-1\right]$$
$$= x^m\left[\frac{m\delta x}{x}+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)^2+....\right]$$
$$= x^m\cdot\frac{\delta x}{x}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = x^{m-1}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{m-1}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{m-1}\left[m+0+0....\right] \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = m x^{m-1}}$$

(xii) Let $y = \dfrac{1}{x^m} = x^{-m}$

Do yourself as above, just change the m by $-m$ in above question.

(xiii) Let $y = x^{40}$

$$\Rightarrow y+\delta y = (x+\delta x)^{40}$$

$$\Rightarrow \delta y = (x+\delta x)^{40} - x^{40}$$

$$= \left[\binom{40}{0}x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ...+ \binom{40}{40}\delta x^{40}\right] - x^{40}$$

$$= (1)x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ...+ \binom{40}{40}\delta x^{40} - x^{40}$$

$$= \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ...+ \binom{40}{40}\delta x^{40}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + ...+ \binom{40}{40}\delta x^{39}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + ...+ \binom{40}{40}\delta x^{39}\right]$$

$$\frac{dy}{dx} = \left[\binom{40}{1}x^{39} + 0 + 0 + ...+ 0\right]$$

$$\Rightarrow \frac{dy}{dx} = \binom{40}{1}x^{39} \quad \text{or} \quad \boxed{\frac{dy}{dx} = 40x^{39}}$$

(xiv) Let $y = x^{-100}$

Do yourself Question # 1(xii), Replace m by -100.

Question # 2

Find $\frac{dy}{dx}$ from the first principles if

(i) $\sqrt{x+2}$ (ii) $\frac{1}{\sqrt{x+a}}$

Solution

(i) Let $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$

Now do yourself as Question # 1(ix)

(ii) Let $y = \frac{1}{\sqrt{x+a}} = (x+a)^{-\frac{1}{2}}$

Now do yourself as Question # 1 (ix)

Question # 1

Find from first principles, the derivatives of the following expensions w.r.t. their respective independent variables:

(i) $(ax+b)^3$ (ii) $(2x+3)^5$ (iii) $(3t+2)^{-2}$

(iv) $(ax+b)^{-5}$ (v) $\dfrac{1}{(az-b)^7}$

Solution

(i) Let $y = (ax+b)^3$

$$\Rightarrow y+\delta y = \left(a(x+\delta x)+b\right)^3$$

$$\Rightarrow \delta y = \left(ax+b+a\delta x\right)^3 - y$$

$$= \left((ax+b)+a\delta x\right)^3 - (ax+b)^3$$

$$= \left[(ax+b)^3 + 3(ax+b)^2(a\delta x) + 3(ax+b)(a\delta x)^2 + (a\delta x)^3\right] - (ax+b)^3$$

$$= 3a(ax+b)^2\delta x + 3a^2(ax+b)\delta x^2 + a^3\delta x^3$$

$$= \delta x\left(3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2\right]$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2 + 3a^2(ax+b)(0) + a^3(0)^2$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2 + 0 + 0 \qquad \Rightarrow \boxed{\frac{dy}{dx} = 3a(ax+b)^2}$$

(ii) Let $y = (2x+3)^5$

$$\Rightarrow y+\delta y = \left(2(x+\delta x)+3\right)^5$$

$$\Rightarrow \delta y = \left(2x+2\delta x+3\right)^5 - y$$

$$= \left((2x+3)+2\delta x\right)^5 - (2x+3)^5$$

$$= \left[\binom{5}{0}(2x+3)^5 + \binom{5}{1}(2x+3)^4(2\delta x) + \binom{5}{2}(2x+3)^3(2\delta x)^2 + \dots\right.$$

$$\left.\dots + \binom{5}{5}(2\delta x)^5\right] - (2x+3)^5$$

$$= \left[(1)(2x+3)^5 + 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots\right.$$

$$\left. \ldots + 32\binom{5}{5}\delta x^5\right] - (2x+3)^5$$

$$= 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots + 32\binom{5}{5}\delta x^5$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4\right]$$

$$\Rightarrow \frac{dy}{dx} = \left[2\binom{5}{1}(2x+3)^4 + 0 + 0 + \ldots + 0\right]$$

$$\Rightarrow \frac{dy}{dx} = 2(5)(2x+3)^4 \quad \text{or} \quad \boxed{\frac{dy}{dx} = 10(2x+3)^4}$$

(iii) Let $y = (3t+2)^{-2}$

$$\Rightarrow y + \delta y = (3(t+\delta t)+2)^{-2}$$

$$\Rightarrow \delta y = (3t+3\delta t+2)^{-2} - y$$

$$\Rightarrow \delta y = ((3t+2)+3\delta t)^{-2} - (3t+2)^{-2}$$

$$= (3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2} = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$$

$$= (3t+2)^{-2}\left[\left(1+(-2)\frac{3\delta t}{3t+2} + \frac{-2(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right) - 1\right]$$

$$\Rightarrow \delta y = (3t+2)^{-2}\left[1 - \frac{6\delta t}{3t+2} + \frac{-2(-3)}{2}\left(\frac{\delta t}{3t+2}\right)^2 + \ldots - 1\right]$$

$$= (3t+2)^{-2}\left[-\frac{6\delta t}{3t+2} + 3\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right]$$

$$= (3t+2)^{-1}\cdot\frac{3\delta t}{3t+2}\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Dividing by δt

$$\frac{\delta y}{\delta t} = 3(3t+2)^{-2-1}\left[-2 + \left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Taking limit when $\delta t \to 0$, we have

$$\lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \lim_{\delta t \to 0} 3(3t+2)^{-3}\left[-2+\left(\frac{3\delta t}{3t+2}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx} = 3(3t+2)^{-3}\left[-2+0-0+....\right] \Rightarrow \boxed{\frac{dy}{dx} = -6(3t+2)^{-3}}$$

(iv) Let $y = (ax+b)^{-5}$

Do yourself

(v) Let $y = \dfrac{1}{(az-b)^7} = (az-b)^{-7}$

$$\Rightarrow y+\delta y = \left(a(z+\delta z)-b\right)^{-7}$$

$$\Rightarrow \delta y = \left((az-b)+a\delta z\right)^{-7} - \left(az-b\right)^{-7}$$

$$\Rightarrow \delta y = \left(az-b\right)^{-7}\left[\left(1+\frac{a\delta z}{(az-b)}\right)^{-7} - 1\right]$$

Differentiate w.r.t. 'x'

Question # 1

$x^4+2x^3+x^2$

Solution Let $y = x^4+2x^3+x^2$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4+2x^3+x^2\right)$$
$$= \frac{d}{dx}x^4+2\frac{d}{dx}x^3+\frac{d}{dx}x^2$$
$$= 4x^{4-1}+2\left(3x^{3-1}\right)+2x^{2-1}$$
$$= 4x^3+6x^2+2x$$

Question # 2

$x^{-3}+2x^{-\frac{3}{2}}+3$

Solution Let $y=x^{-3}+2x^{-\frac{3}{2}}+3$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{-3}+2x^{-\frac{3}{2}}+3\right)$$
$$=\frac{d}{dx}x^{-3}+2\frac{d}{dx}x^{-\frac{3}{2}}+\frac{d}{dx}(3)$$
$$=-3x^{-3-1}+2\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)+0$$
$$\Rightarrow \frac{dy}{dx}=-3x^{-4}-3x^{-\frac{5}{2}}$$
$$\text{or} \quad \frac{dy}{dx}=-3\left(\frac{1}{x^4}+\frac{1}{x^{5/2}}\right)$$

Question # 3

$\frac{a+x}{a-x}$

Solution Let $y=\frac{a+x}{a-x}$

Now $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{a+x}{a-x}\right) = \frac{(a-x)\frac{d}{dx}(a+x)-(a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$

$$=\frac{(a-x)(0+1)-(a+x)(0-1)}{(a-x)^2}$$

$$=\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$=\frac{a-x+a+x}{(a-x)^2} \quad =\frac{2a}{(a-x)^2} \qquad \textit{Answer}$$

Question # 4

$$\frac{2x-3}{2x+1}$$

Solution Let $y=\dfrac{2x-3}{2x+1}$

Now $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{2x-3}{2x+1}\right)$

$$=\frac{(2x+1)\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2-0)-(2x-3)(2+0)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2)-(2x-3)(2)}{(2x+1)^2}$$

$$=\frac{2(2x+1-2x+3)}{(2x+1)^2}$$

$$=\frac{2(4)}{(2x+1)^2} \quad =\frac{8}{(2x+1)^2} \qquad \textit{Answer}$$

Question # 5

$$(x-5)(3-x)$$

Solution Let $y=(x-5)(3-x)$

$$=3x-x^2-15+5x$$

$$=-x^2+8x-15$$

Now

$$\frac{dy}{dx}=\frac{dy}{dx}\left(-x^2+8x-15\right)$$

$$=\frac{dy}{dx}\left(-x^2\right)+8\frac{d}{dx}(x)-\frac{d}{dx}(15)$$

$$=-2x^{2-1}+8(1)-0 \quad =-2x+8 \qquad \textit{Answer}$$

Question # 6

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$$

Solution Let $y=\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

$$=\left(\sqrt{x}\right)^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\left(\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right)$$

$$=x+\frac{1}{x}-2 \quad =x+x^{-1}-2$$

Now diff. w.r.t x

$$\frac{dy}{dx} =\frac{d}{dx}\left(x+x^{-1}-2\right) \quad =\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})-\frac{d}{dx}(2)$$

$$= 1+(-1\cdot x^{-1-1})-0= 1-x^{-2}$$

$$= 1-\frac{1}{x^2} \quad = \frac{x^2-1}{x^2} \quad \textit{Answer}$$

Question # 7

$$\frac{\left(1+\sqrt{x}\right)\left(x-x^{3/2}\right)}{\sqrt{x}}$$

Solution Consider $y = \dfrac{\left(1+\sqrt{x}\right)\left(x-x^{3/2}\right)}{\sqrt{x}}$

$$= \frac{\left(1+\sqrt{x}\right)\ x\left(1-x^{\frac{1}{2}}\right)}{\sqrt{x}}$$

$$= \frac{x\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)}{\sqrt{x}} \qquad \text{Since } x^{\frac{3}{2}}=x^{1+\frac{1}{2}}$$

$$= \frac{\left(\sqrt{x}\right)^2\left(1-\left(\sqrt{x}\right)^2\right)}{\sqrt{x}}$$

$$= \sqrt{x}\left(1-x\right) \quad = x^{\frac{1}{2}}\left(1-x\right) \quad = x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

Now

$$\frac{dy}{dx}= \frac{d}{dx}\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1}-\frac{3}{2}x^{\frac{3}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{x}}-3\sqrt{x}\right) \quad \textit{Answer}$$

Question # 8

$$\frac{\left(x^2+1\right)^2}{x^2-1}$$

Solution Let $y = \dfrac{\left(x^2+1\right)^2}{x^2-1}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\left(x^2+1\right)^2}{x^2-1}\right)$$

$$= \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)^2 - \left(x^2+1\right)^2\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(x^2-1\right)\,2\left(x^2+1\right)^{2-1}\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)^2(2x)}{\left(x^2-1\right)^2}$$

$$= \frac{\left(x^2-1\right)\,2\left(x^2+1\right)(2x) - \left(x^2+1\right)^2(2x)}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left[2\left(x^2-1\right) - \left(x^2+1\right)\right]}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left[2x^2-2-x^2-1\right]}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left(x^2-3\right)}{\left(x^2-1\right)^2}$$ *Answer*

Question # 9

$$\frac{x^2+1}{x^2-3}$$

Solution Let $y = \dfrac{x^2+1}{x^2-3}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$= \frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2}$$

$$= \frac{(x^2-3)\,(2x)-(x^2+1)(2x)}{(x^2-3)^2} = \frac{2x(x^2-3-x^2-1)}{(x^2-3)^2}$$

$$= \frac{2x(-4)}{(x^2-3)^2} = \frac{-8x}{(x^2-3)^2} \qquad \textit{Answer}$$

Question # 10

$$\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Solution Let $y = \dfrac{\sqrt{1+x}}{\sqrt{1-x}} = \left(\dfrac{1+x}{1-x}\right)^{1/2}$

Now $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1+x}{1-x}\right)^{1/2}$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left(\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right)$$

$$= \frac{1}{2}\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}\left(\frac{1-x+1+x}{(1-x)^2}\right) = \frac{(1-x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{2}{(1-x)^2}\right)$$

$$= \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{2-\frac{1}{2}}} = \frac{1}{\sqrt{1+x}\;(1-x)^{\frac{3}{2}}} \qquad \textit{Answer}$$

Question # 11

$$\frac{2x-1}{\sqrt{x^2+1}}$$

Solution Let $y = \dfrac{2x-1}{\sqrt{x^2+1}}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-1}{(x^2+1)^{1/2}}\right)$$

$$= \frac{(x^2+1)^{1/2}\,\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)^{1/2}}{\left((x^2+1)^{1/2}\right)^2}$$

$$= \frac{\left(x^2+1\right)^{1/2}(2)-(2x-1)\,\frac{1}{2}\left(x^2+1\right)^{-1/2}\frac{d}{dx}(x^2+1)}{\left(x^2+1\right)}$$

$$= \frac{2\left(x^2+1\right)^{1/2}-(2x-1)\frac{1}{2\left(x^2+1\right)^{1/2}}(2x)}{\left(x^2+1\right)}$$

$$= \frac{1}{\left(x^2+1\right)}\left(2\left(x^2+1\right)^{1/2}-\frac{2x^2-x}{\left(x^2+1\right)^{1/2}}\right)$$

$$= \frac{1}{\left(x^2+1\right)}\left(\frac{2x^2+2-2x^2+x}{\left(x^2+1\right)^{1/2}}\right)$$

$$= \frac{x+2}{\left(x^2+1\right)\sqrt{x^2+1}} \quad \text{or} \quad \frac{x+2}{\left(x^2+1\right)^{3/2}} \quad \textit{Answer}$$

Question # 12

$$\frac{\sqrt{a-x}}{\sqrt{a+x}}$$

Solution

Do yourself as Question # 10

Question # 13

$$\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$$

Solution Let $y = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

$$= \left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

Differentiating w.r.t x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{(x^2-1)(2x)-(x^2+1)(2x)}{\left(x^2-1\right)^2}\right)$$

$$=\frac{1}{2}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}\left(\frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}\right)$$

$$=\frac{1}{2}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}\left(\frac{-4x}{\left(x^2-1\right)^2}\right)$$

$$=\frac{-2\sqrt{x^2-1}}{\left(x^2-1\right)^2\sqrt{x^2+1}}=\frac{-2}{\left(x^2-1\right)^{2-\frac{1}{2}}\sqrt{x^2+1}}$$

$$=\frac{-2}{\left(x^2-1\right)^{\frac{3}{2}}\sqrt{x^2+1}} \quad \textit{Answer}$$

Question # 14

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

Solution Assume $y=\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\cdot\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} \qquad \textit{Rationalizing}$$

$$=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}$$

$$=\frac{\left(\sqrt{1+x}\right)^2+\left(\sqrt{1-x}\right)^2-2\left(\sqrt{1+x}\right)\left(\sqrt{1-x}\right)}{1+x-1+x}$$

$$=\frac{1+x+1-x-2\sqrt{(1+x)(1-x)}}{2x}$$

$$=\frac{2-2\sqrt{1-x^2}}{2x}=\frac{2\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)}{2x}$$

$$=\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}$$

Now differentiation w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}\right)$$

$$= \frac{x\,\frac{d}{dx}\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)-\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)\frac{d}{dx}x}{x^2}$$

$$= \frac{1}{x^2}\cdot\left[x\left(0-\frac{1}{2}\left(1-x^2\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(1-x^2\right)\right)-\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)(1)\right]$$

$$= \frac{1}{x^2}\cdot\left[x\left(-\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\right)-1+\left(1-x^2\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{x^2}\cdot\left[\frac{x^2}{\left(1-x^2\right)^{\frac{1}{2}}}-1+\left(1-x^2\right)^{\frac{1}{2}}\right] = \frac{1}{x^2}\cdot\left[\frac{x^2-\left(1-x^2\right)^{\frac{1}{2}}+1-x^2}{\left(1-x^2\right)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{x^2}\cdot\left[\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{1}{2}}}\right] = \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}} \quad \textit{Answer}$$

Question # 15

$$\frac{x\sqrt{a+x}}{\sqrt{a-x}}$$

Solution Let $y = \dfrac{x\sqrt{a+x}}{\sqrt{a-x}} = x\left(\dfrac{a+x}{a-x}\right)^{1/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}x\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$= x\frac{d}{dx}\left(\frac{a+x}{a-x}\right)^{1/2} + \left(\frac{a+x}{a-x}\right)^{1/2}\frac{d}{dx}x \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now $\dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)^{1/2} = \dfrac{1}{2}\left(\dfrac{a+x}{a-x}\right)^{1/2-1}\dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)$

$$= \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2}\left(\frac{(a-x)\frac{d}{dx}(a+x)-(a+x)\frac{d}{dx}(a-x)}{(a-x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{1/2}\left(\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}\right)$$

$$= \frac{1}{2}\frac{(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}}\left(\frac{a-x+a+x}{(a-x)^2}\right) = \frac{1}{2}\frac{1}{(a+x)^{\frac{1}{2}}(a-x)^{-\frac{1}{2}}}\cdot\left(\frac{2a}{(a-x)^2}\right)$$

$$= \frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{2-\frac{1}{2}}} = \frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

Using in eq. (i)

$$\frac{dy}{dx} = x\cdot\frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \left(\frac{a+x}{a-x}\right)^{1/2}(1)$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax+(a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} = \frac{ax+a^2-x^2}{\sqrt{a+x}\,(a-x)^{\frac{3}{2}}} \quad \textit{Answer}$$

Question # 16

If $y=\sqrt{x}-\dfrac{1}{\sqrt{x}}$, show that $2x\dfrac{dy}{dx}+y= 2\sqrt{x}$

Solution Since $y = \sqrt{x}-\dfrac{1}{\sqrt{x}}$

$$= x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

Multiplying by $2x$

$$2x\frac{dy}{dx} = 2x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)+2x\left(\frac{1}{2}x^{-\frac{3}{2}}\right) \Rightarrow 2x\frac{dy}{dx} = x^{-\frac{1}{2}+1}+x^{-\frac{3}{2}+1}$$

$$2x\frac{dy}{dx} = x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$

Adding y on both sides

$$2x\frac{dy}{dx}+y= x^{\frac{1}{2}}+x^{-\frac{1}{2}}+y$$

$$\Rightarrow 2x\frac{dy}{dx}+y= x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{\frac{1}{2}}-x^{-\frac{1}{2}} \qquad \because y=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$$

$$\Rightarrow 2x\frac{dy}{dx}+y= 2x^{\frac{1}{2}} \qquad \Rightarrow 2x\frac{dy}{dx}+y= 2\sqrt{x} \qquad \textit{Proved}$$

Question # 17

If $y = x^4+2x^2+2$, prove that $\dfrac{dy}{dx} = 4x\sqrt{y-1}$

Solution Since $y = x^4 + 2x^2 + 2$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(x^4 + 2x^2 + 2\right)$

$$\Rightarrow \frac{dy}{dx} = 4x^{4-1} + 2\left(2x^{2-1}\right) + 0$$

$$= 4x^3 + 4x$$

$$\Rightarrow \frac{dy}{dx} = 4x\left(x^2 + 1\right) \text{ .... (i)}$$

Now $y = x^4 + 2x^2 + 2$

$$\Rightarrow y - 1 = x^4 + 2x^2 + 2 - 1$$

$$= x^4 + 2x^2 + 1 = \left(x^2 + 1\right)^2$$

$$\Rightarrow \sqrt{y-1} = \left(x^2 + 1\right) \quad \text{i.e.} \quad \left(x^2 + 1\right) = \sqrt{y-1}$$

Using it in eq. (i), we have

$$\Rightarrow \frac{dy}{dx} = 4x\sqrt{y-1} \qquad \text{as required.}$$

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Question # 1

Find by making suitable substitution in the following functions defined as:

(i) $y = \sqrt{\dfrac{1-x}{1+x}}$

(ii) $y = \sqrt{x+\sqrt{x}}$

(iii) $y = x\sqrt{\dfrac{a+x}{a-x}}$

(iv) $y=\left(3x^2-2x+7\right)^6$

(v) $\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}$

Solution

(i)

$$y = \sqrt{\frac{1-x}{1+x}}$$

Put $u = \dfrac{1-x}{1+x}$

So $y = \sqrt{u} \quad \Rightarrow \quad y = u^{\frac{1}{2}}$

Now diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2}$$

$$\Rightarrow \frac{du}{dx} = \frac{-2}{(1+x)^2}$$

Now diff. y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\cdot\frac{-2}{(1+x)^2}$$

$$= \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\cdot\frac{-1}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{2-\frac{1}{2}}}$$

$$= \frac{-1}{\sqrt{1-x}\,(1+x)^{\frac{3}{2}}} \qquad \textit{Answer}$$

(ii)

$$y = \sqrt{x+\sqrt{x}}$$

Let $u = x+\sqrt{x} = x+x^{\frac{1}{2}}$

$$\Rightarrow y = \sqrt{u} = u^{\frac{1}{2}}$$

Diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(x+x^{\frac{1}{2}}\right)$$

$$= 1+\frac{1}{2}x^{-\frac{1}{2}} = 1+\frac{1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

Now diff. y w.r.t. x

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2u^{\frac{1}{2}}} = \frac{1}{2\left(x+\sqrt{x}\right)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{x+\sqrt{x}}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{4\sqrt{x}\cdot\sqrt{x+\sqrt{x}}} \quad \textit{Answer}$$

(iii)

$$y = x\sqrt{\frac{a+x}{a-x}}$$

Put $\quad u = \dfrac{a+x}{a-x}$

So $\quad y = x\sqrt{u} = x(u)^{\frac{1}{2}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}x(u)^{\frac{1}{2}}$$

$$= x\frac{d}{dx}(u)^{\frac{1}{2}} + (u)^{\frac{1}{2}}\frac{d}{dx}x$$

$$= x\frac{1}{2}(u)^{-\frac{1}{2}}\frac{du}{dx} + (u)^{\frac{1}{2}}(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}(u)^{-\frac{1}{2}}\frac{du}{dx} + (u)^{\frac{1}{2}} \quad \ldots\ldots. (i)$$

Now diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$= \frac{(a-x)\frac{d}{dx}(a+x)-(a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(0+1)-(a+x)(0-1)}{(a-x)^2}$$

$$= \frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$= \frac{a-x+a+x}{(a-x)^2} = \frac{2a}{(a-x)^2}$$

Using value of u and $\dfrac{du}{dx}$ in eq. (i)

$$\frac{dy}{dx} = \frac{x}{2}\left(\frac{a+x}{a-x}\right)^{-\frac{1}{2}}\frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$$

$$= \frac{(a+x)^{-\frac{1}{2}}}{(a-x)^{-\frac{1}{2}}}\cdot\frac{ax}{(a-x)^2} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{2-\frac{1}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax+(a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{ax+a^2-x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}}$$

(iv)

Do yourself as above

(v)

Do yourself as above

Question # 2

Find $\dfrac{dy}{dx}$ if:

(i) $3x+4y+7 = 0$

(ii) $xy+y^2 = 2$

(iii) $x^2-4xy-5y=0$

(iv) $4x^2+2hxy+by^2+2gx+2fy+c = 0$

(v) $x\sqrt{1+y}+y\sqrt{1+x} = 0$

(vi) $y\left(x^2-1\right)=x\sqrt{x^2+4}$

Solution

(i)
$$3x+4y+7 = 0$$
Diff. w.r.t. x.
$$\frac{d}{dx}(3x+4y+7) = \frac{d}{dx}(0)$$
$$\Rightarrow 3(1)+4\frac{dy}{dx}+0 = 0 \quad \Rightarrow 4\frac{dy}{dx} = -3$$
$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{3}{4}}$$

(ii) $xy+y^2 = 2$

Differentiating w.r.t. x
$$\frac{d}{dx}\left(xy+y^2\right) = \frac{d}{dx}(2)$$
$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}y^2 = 0$$
$$\Rightarrow x\frac{dy}{dx}+y\frac{dx}{dx}+2y\frac{dy}{dx} = 0$$
$$\Rightarrow (x+2y)\frac{dy}{dx}+y(1) = 0$$
$$\Rightarrow (x+2y)\frac{dy}{dx} = -y$$
$$\Rightarrow (x+2y)\frac{dy}{dx} = -y$$
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-y}{x+2y}}$$

(iii)

Do yourself

(iv)
$$4x^2+2hxy+by^2+2gx+2fy+c = 0$$
Differentiating w.r.t. x
$$\frac{d}{dx}\left(4x^2+2hxy+by^2+2gx+2fy+c\right) = \frac{d}{dx}(0)$$
$$\Rightarrow 4\frac{d}{dx}(x^2)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y^2)$$
$$+2g\frac{d}{dx}(x)+2f\frac{d}{dx}(y)+\frac{d}{dx}(c) = 0$$
$$\Rightarrow 4(2x)+2h\left(x\frac{dy}{dx}+y(1)\right)+b\cdot 2y\frac{dy}{dx}$$
$$+2g(1)+2f\frac{dy}{dx}+0 = 0$$
$$\Rightarrow 8x+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}$$
$$+2g+2f\frac{dy}{dx} = 0$$
$$\Rightarrow 2(hx+by+f)\frac{dy}{dx}+2(4x+hy++g)=0$$
$$\Rightarrow 2(hx+by+f)\frac{dy}{dx}=-2(4x+hy++g)$$
$$\Rightarrow (hx+by+f)\frac{dy}{dx}=-(4x+hy++g)$$
$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{4x+hy++g}{hx+by+f}}$$

(v)
$$x\sqrt{1+y}+y\sqrt{1+x} = 0 \quad \Rightarrow x(1+y)^{\frac{1}{2}}+y(1+x)^{\frac{1}{2}} = 0$$
Differentiating w.r.t. x
$$\Rightarrow \frac{d}{dx}\left[x(1+y)^{\frac{1}{2}}\right]+\frac{d}{dx}\left[y(1+x)^{\frac{1}{2}}\right] = \frac{d}{dx}(0)$$
$$\Rightarrow x\frac{d}{dx}(1+y)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}\frac{dx}{dx}+y\frac{d}{dx}(1+x)^{\frac{1}{2}}+(1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\Rightarrow\ x\cdot\frac{1}{2}(1+y)^{-\frac{1}{2}}\frac{dy}{dx}+(1+y)^{\frac{1}{2}}(1)\ +y\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}(1)+(1+x)^{\frac{1}{2}}\frac{dy}{dx}\ =\ 0$$

$$\Rightarrow\ \frac{x}{2(1+y)^{\frac{1}{2}}}\frac{dy}{dx}+(1+y)^{\frac{1}{2}}\ +\frac{y}{2(1+x)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\frac{dy}{dx}\ =\ 0$$

$$\Rightarrow\ \left[\frac{x}{2(1+y)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\right]\frac{dy}{dx}\ =-\left[(1+y)^{\frac{1}{2}}+\frac{y}{2(1+x)^{\frac{1}{2}}}\right]$$

$$\Rightarrow\ \left[\frac{x+2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}{2(1+y)^{\frac{1}{2}}}\right]\frac{dy}{dx}\ =-\left[\frac{2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}+y}{2(1+x)^{\frac{1}{2}}}\right]$$

$$\Rightarrow\ \left[\frac{x+2\sqrt{(1+x)(1+y)}}{2\sqrt{1+y}}\right]\frac{dy}{dx}\ =-\left[\frac{2\sqrt{(1+x)(1+y)}+y}{2\sqrt{1+x}}\right]$$

$$\Rightarrow\ \frac{dy}{dx}\ =-\frac{2\sqrt{(1+x)(1+y)}+y}{2\sqrt{1+x}}\cdot\frac{2\sqrt{1+y}}{x+2\sqrt{(1+x)(1+y)}}$$

$$\Rightarrow\ \frac{dy}{dx}\ =-\frac{\sqrt{1+y}\left(2\sqrt{(1+x)(1+y)}+y\right)}{\sqrt{1+x}\left(x+2\sqrt{(1+x)(1+y)}\right)}\qquad \textit{Answer}$$

(vi)

$$y\left(x^2-1\right)=x\sqrt{x^2+4}$$

Differentiating w.r.t x

$$\frac{d}{dx}y\left(x^2-1\right)=\frac{d}{dx}x\left(x^2+4\right)^{\frac{1}{2}}$$

$$\Rightarrow\ y\frac{d}{dx}\left(x^2-1\right)+\left(x^2-1\right)\frac{dy}{dx}=x\frac{d}{dx}\left(x^2+4\right)^{\frac{1}{2}}+\left(x^2+4\right)^{\frac{1}{2}}\frac{dx}{dx}$$

$$\Rightarrow\ y(2x)+\left(x^2-1\right)\frac{dy}{dx}=x\frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}(2x)+\left(x^2+4\right)^{\frac{1}{2}}(1)$$

$$\Rightarrow\ 2xy+\left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}-2xy$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}-2xy$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2+x^2+4-2xy\left(x^2+4\right)^{\frac{1}{2}}}{\left(x^2+4\right)^{\frac{1}{2}}}\qquad\Rightarrow\ \boxed{\frac{dy}{dx}=\frac{2x^2+4-2xy\sqrt{x^2+4}}{\left(x^2-1\right)\sqrt{x^2+4}}}$$

Question # 3

Find $\dfrac{dy}{dx}$ of the following parametric functions:

(i) $x = \theta + \dfrac{1}{\theta}$ and $y = \theta + 1$

(ii) $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$

Solution

(i) Since $x = \theta + \dfrac{1}{\theta}$

$$\Rightarrow x = \theta + \theta^{-1}$$

Differentiating x w.r.t. θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(\theta + \theta^{-1}\right)$$

$$= 1 - \theta^{-2} = 1 - \frac{1}{\theta^2} = \frac{\theta^2 - 1}{\theta^2}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{\theta^2}{\theta^2 - 1}$$

Now $y = \theta + 1$

Diff. w.r.t. θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1) \Rightarrow \frac{dy}{d\theta} = 1$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}}$$

(ii) Since $x = \dfrac{a(1-t^2)}{1+t^2}$

Diff. w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= a\,\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= a\,\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= a\,\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-4at}{(1+t^2)^2} \Rightarrow \frac{dt}{dx} = \frac{(1+t^2)^2}{-4at}$$

Now $y = \dfrac{2bt}{1+t^2}$y

Diff. w.r.t. t

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2bt}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}2bt - 2bt\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)2b(1) - 2bt(2t)}{(1+t^2)^2}$$

$$= \frac{2b + 2bt^2 - 4bt^2}{(1+t^2)^2} = \frac{2b - 2bt^2}{(1+t^2)^2}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{-4at}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{b(1-t^2)}{2at}}$$

Question # 4

Prove that $y\dfrac{dy}{dx}+x=0$ if $x = \dfrac{1-t^2}{1+t^2}$,

$y = \dfrac{2t}{1+t^2}$

Solution Since $x = \dfrac{1-t^2}{1+t^2}$

Differentiating w.r.t. t, we get (*solve yourself as above*)

$$\frac{dx}{dt} = \frac{-4t}{\left(1+t^2\right)^2} \Rightarrow \frac{dt}{dx} = \frac{\left(1+t^2\right)^2}{-4t}$$

Now $y = \dfrac{2t}{1+t^2}$

Differentiating w.r.t. t, we get (*solve yourself as above*)

$$\frac{dy}{dt} = \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}\cdot\frac{\left(1+t^2\right)^2}{-4t}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1-t^2}{2t}$$

Multiplying both sides by y

$$\Rightarrow y\frac{dy}{dx} = -y\cdot\frac{1-t^2}{2t}$$

$$= -\frac{2t}{1+t^2}\cdot\frac{1-t^2}{2t}$$

$$\Rightarrow y\frac{dy}{dx} = -\frac{1-t^2}{1+t^2}$$

$$\Rightarrow y\frac{dy}{dx} = -x \qquad \because\ x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow y\frac{dy}{dx}+x = 0 \qquad \textit{Proved.}$$

Question # 5

Differentiate

(i) $x^2-\dfrac{1}{x^2}$ w.r.t. x^4

(ii) $\left(1+x^2\right)^n$ w.r.t. x^2

(iii) $\dfrac{x^2+1}{x^2-1}$ w.r.t. $\dfrac{x-1}{x+1}$

(iv) $\dfrac{ax+b}{cx+d}$ w.r.t. $\dfrac{ax^2+b}{ax^2+d}$

(v) $\dfrac{x^2+1}{x^2-1}$ w.r.t. x^3

Solution

(i) Suppose $y = x^2-\dfrac{1}{x^2}$ and $u=x^4$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2-\frac{1}{x^2}\right)$$

$$= \frac{d}{dx}\left(x^2-x^{-2}\right) = 2x+2x^{-3}$$

$$= 2\left(x+\frac{1}{x^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{x^4+1}{x^3}\right)$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(x^4\right)$$

$$\Rightarrow \frac{du}{dx} = 4x^3$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{dy}{dx}\cdot\frac{1}{\frac{du}{dx}}$$

$$\Rightarrow \frac{dy}{du} = 2\left(\frac{x^4+1}{x^3}\right)\cdot\frac{1}{4x^3}$$

$$\Rightarrow \boxed{\frac{dy}{du} = \frac{x^4+1}{2x^6}}$$

(ii) Let $y=\left(1+x^2\right)^n$ and $u = x^2$

Differentiation y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(1+x^2\right)^n$$
$$= n\left(1+x^2\right)^{n-1}\frac{d}{dx}\left(1+x^2\right)$$
$$= n\left(1+x^2\right)^{n-1}(2x)$$
$$= 2nx\left(1+x^2\right)^{n-1}$$

Now differentiating u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}x^2$$
$$= 2x \quad \Rightarrow \frac{dx}{du}=\frac{1}{2x}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$\Rightarrow \frac{dy}{du} = 2nx\left(1+x^2\right)^{n-1}\cdot\frac{1}{2x}$$
$$\Rightarrow \boxed{\frac{dy}{du} = n\left(1+x^2\right)^{n-1}}$$

(iii) Let $y=\dfrac{x^2+1}{x^2-1}$ and $u = \dfrac{x-1}{x+1}$

Diff. y w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$
$$= \textit{Solve yourself} = \frac{-4x}{\left(x^2-1\right)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$
$$= \textit{Solve yourself} = \frac{2}{(x+1)^2}.$$
$$\Rightarrow \frac{dx}{du} = \frac{(x+1)^2}{2}.$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= \frac{-4x}{\left(x^2-1\right)^2}\cdot\frac{(x+1)^2}{2}$$
$$= \frac{-2x}{(x-1)^2(x+1)^2}\cdot(x+1)^2$$
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2x}{(x-1)^2}}$$

(iv) Let $y=\dfrac{ax+b}{cx+d}$ and $u=\dfrac{ax^2+b}{ax^2+d}$

Diff. y w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)$$
$$=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$
$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{ad-bc}{(cx+d)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{ax^2+b}{ax^2+d}\right)$$
$$=\frac{(ax^2+d)\frac{d}{dx}(ax^2+b)-(ax^2+b)\frac{d}{dx}(ax^2+d)}{\left(ax^2+d\right)^2}$$
$$=\frac{\left(ax^2+d\right)(2ax)-\left(ax^2+b\right)(2ax)}{\left(ax^2+d\right)^2}$$
$$=\frac{2ax\left(ax^2+d-ax^2-b\right)}{\left(ax^2+d\right)^2}$$
$$=\frac{2ax(d-b)}{\left(ax^2+d\right)^2}$$

$$\Rightarrow \frac{dx}{du} = \frac{\left(ax^2+d\right)^2}{2ax(d-b)}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{ad-bc}{(cx+d)^2}\cdot\frac{\left(ax^2+d\right)^2}{2ax(d-b)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{(ad-bc)\left(ax^2+d\right)^2}{2ax(cx+d)^2(d-b)}}$$

(v) Let $y=\dfrac{x^2+1}{x^2-1}$ and $u=x^3$

Diff. y w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$= Solve\ yourself$

$$=\frac{-4x}{\left(x^2-1\right)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx}=\frac{d}{dx}x^3$$

$$=3x^2$$

$$\Rightarrow \frac{dx}{du}=\frac{1}{3x^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$=\frac{-4x}{\left(x^2-1\right)^2}\cdot\frac{1}{3x^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-4}{3x\left(x^2-1\right)^2}}$$

Some Important Derivative Formulas

- $\frac{d}{dx}c=0$ where c is constant
- $\frac{d}{dx}x^n=nx^{n-1}$
- $\frac{d}{dx}\sin x=\cos x$
- $\frac{d}{dx}\cos x=-\sin x$
- $\frac{d}{dx}\tan x=\sec^2 x$
- $\frac{d}{dx}\cot x=-\csc^2 x$
- $\frac{d}{dx}\csc x=-\csc x\cot x$
- $\frac{d}{dx}\sec x=\sec x\tan x$
- $\frac{d}{dx}Sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Tan^{-1}x=\frac{1}{1+x^2}$
- $\frac{d}{dx}Cot^{-1}x=\frac{-1}{1+x^2}$
- $\frac{d}{dx}Sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}Csc^{-1}x=\frac{-1}{x\sqrt{x^2-1}}$

Question # 1

Difference the following trigonometric functions from the first principles.

(i) $\sin 2x$ (ii) $\tan 3x$ (iii) $\sin 2x+\cos 2x$ (iv) $\cos x^2$

(v) $\tan^2 x$ (vi) $\sqrt{\tan x}$ (vii) $\cos\sqrt{x}$

Solution

(i) Suppose $y=\sin 2x$

$$\Rightarrow y+\delta y=\sin 2(x+\delta x)$$

$$\Rightarrow \delta y=\sin 2(x+\delta x)-y$$

$$=\sin 2(x+\delta x)-\sin 2x$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{\sin(2x+2\delta x)-\sin 2x}{\delta x}$$

$$=\frac{2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)}{\delta x}$$

$$=\frac{2\cos(2x+\delta x)\sin(\delta x)}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{2\cos(2x+\delta x)\sin(\delta x)}{\delta x}$$

$$\frac{dy}{dx}=2\lim_{\delta x\to 0}\cos(2x+\delta x)\cdot\frac{\sin(\delta x)}{\delta x}$$

$$=2\lim_{\delta x\to 0}\cos(2x+\delta x)\cdot\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$=2\cos(2x+0)\cdot(1) \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii) Let $\quad y = \tan 3x$

$$\Rightarrow y+\delta y = \tan 3(x+\delta x)$$

$$\Rightarrow \delta y = \tan(3x+3\delta x)-\tan 3x$$

$$= \frac{\sin(3x+3\delta x)}{\cos(3x+3\delta x)}-\frac{\sin 3x}{\cos 3x} = \frac{\sin(3x+3\delta x)\cos 3x-\cos(3x+3\delta x)\sin 3x}{\cos(3x+3\delta x)\cos 3x}$$

$$= \frac{\sin(3x+3\delta x-3x)}{\cos(3x+3\delta x)\cos 3x} = \frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\cdot\frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x\ \cos(3x+3\delta x)\cos 3x}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x}\cdot\frac{1}{\cos(3x+3\delta x)\cos 3x}\cdot\frac{3}{3} \qquad \times\text{ing and } \div\text{ing 3 on R.H.S}$$

$$= 3\lim_{\delta x\to 0}\frac{\sin(3\delta x)}{3\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos(3x+3\delta x)\cos 3x}$$

$$= 3(1)\cdot\frac{1}{\cos(3x+3(0))\cos 3x}$$

$$= \frac{3}{\cos 3x\cos 3x} = \frac{3}{\cos^2 3x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3\sec^2 3x}$$

(iii) Let $\quad y = \sin 2x+\cos 2x$

$$\Rightarrow y+\delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)$$

$$\Rightarrow \delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)-y$$

$$= \sin 2(x+\delta x)+\cos 2(x+\delta x)-\sin 2x-\cos 2x$$

$$= \left[\sin(2x+2\delta x)-\sin 2x\right]+\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$$= \left[2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$+\left[-2\sin\left(\frac{2x+2\delta x-2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$= 2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

$$\frac{dy}{dx} = 2\lim_{\delta x\to 0}\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x} - 2\lim_{\delta x\to 0}\sin(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$= 2\cos(2x+0)\cdot(1) - 2\sin(2x+0)\cdot(1) \qquad \text{Since } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x - 2\sin 2x}$$

(iv) Let $y = \cos x^2$

$$\Rightarrow y+\delta y = \cos(x+\delta x)^2$$

$$\Rightarrow \delta y = \cos(x+\delta x)^2 - \cos x^2$$

$$= -2\sin\left(\frac{(x+\delta x)^2 + x^2}{2}\right)\sin\left(\frac{(x+\delta x)^2 - x^2}{2}\right)$$

$$= -2\sin\left(\frac{x^2+2x\delta x+\delta x^2+x^2}{2}\right)\sin\left(\frac{x^2+2x\delta x+\delta x^2-x^2}{2}\right)$$

$$= -2\sin\left(\frac{2x^2+2x\delta x+\delta x^2}{2}\right)\cdot\sin\left(\frac{2x\delta x+\delta x^2}{2}\right)$$

$$= -2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{1}{\delta x}\cdot 2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x$$

$\times$ing and $\div$ing $\left(x+\frac{\delta x}{2}\right)$ on R.H.S

$$\Rightarrow \frac{\delta y}{\delta x} = -\left[\frac{2}{\delta x}\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x\right]\cdot\frac{\left(x+\frac{\delta x}{2}\right)}{\left(x+\frac{\delta x}{2}\right)}$$

$$= -\left[2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x+\frac{\delta x}{2}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\left[2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x+\frac{\delta x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\cdot\lim_{\delta x\to 0}\left(x+\frac{\delta x}{2}\right)$$

$$= -2\sin\left(x^2+(0)+(0)\right)\cdot(1)\cdot\left(x+(0)\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(v) Let $y = \tan^2 x$

$$\Rightarrow y+\delta y = \tan^2(x+\delta x)$$

$$\Rightarrow \delta y = \tan^2(x+\delta x)-\tan^2 x$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\tan(x+\delta x)-\tan x\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)\cos x-\sin x\cos(x+\delta x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\tan(x+\delta x)+\tan x}{\cos(x+\delta x)\cos x}\right)\cdot\lim_{\delta x\to 0}\left(\frac{\sin\delta x}{\delta x}\right)$$

$$= \left(\frac{\tan(x+0)+\tan x}{\cos(x+0)\cos x}\right)\cdot(1) = \frac{\tan x+\tan x}{\cos x\cdot\cos x} = \frac{2\tan x}{\cos^2 x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\tan x\ \sec^2 x}$$

(vi) Let $y = \sqrt{\tan x}$

$$\Rightarrow y+\delta y = \sqrt{\tan(x+\delta x)}$$

$$\Rightarrow \delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

$$= \left(\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right)\cdot\left(\frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\right)$$

$$= \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$= \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right)$$

Now do yourself as above.

(vii) Let $y = \cos\sqrt{x}$

$$\Rightarrow y+\delta y = \cos\sqrt{x+\delta x}$$

$$\Rightarrow \delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x}$$

$$= -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

As $\delta x = \left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)$, putting in above

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)}$$

$$= -\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}\cdot\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

$$= -\frac{\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+0}+\sqrt{x}\right)}\cdot(1) \qquad \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}}$$

Question # 2

Differentiate the following w.r.t. the variable involved.

(i) $x^2\sec 4x$ (ii) $\tan^3\theta\ \sec^2\theta$ (iii) $\left(\sin 2\theta-\cos 3\theta\right)^2$ (iv) $\cos\sqrt{x}+\sqrt{\sin x}$

Solution

(i) Assume $y = x^2\sec 4x$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2\sec 4x$$

$$= x^2\frac{d}{dx}\sec 4x+\sec 4x\frac{d}{dx}x^2$$

$$= x^2\sec 4x\tan 4x\frac{d}{dx}(4x)+\sec 4x\ (2x)$$

$$= x^2\sec 4x\tan 4x(4)+2x\sec 4x$$

$$= 2x\sec 4x\left(2x\tan 4x+1\right)$$

(ii) Let $y = \tan^3\theta\ \sec^2\theta$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\tan^3\theta\ \sec^2\theta$$

$$= \tan^3\theta\frac{d}{d\theta}\sec^2\theta+\sec^2\theta\frac{d}{d\theta}\tan^3\theta$$

$$= \tan^3\theta\left(2\sec\theta\frac{d}{d\theta}\sec\theta\right)+\sec^2\theta\left(3\tan^2\theta\frac{d}{d\theta}\tan\theta\right)$$

$$= \tan^3\theta\left(2\sec\theta\cdot\sec\theta\tan\theta\right)+\sec^2\theta\left(3\tan^2\theta\cdot\sec^2\theta\right)$$

$$= \sec^2\theta\tan^2\theta\left(2\tan^2\theta+3\sec^2\theta\right)$$

(iii) Let $y = \left(\sin 2\theta-\cos 3\theta\right)^2$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)^2$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot\frac{d}{d\theta}(2\theta)+\sin 3\theta\cdot\frac{d}{d\theta}(3\theta)\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot(2)+\sin 3\theta\cdot(3)\right)$$

$$= 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$

(iv) Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

$$= \cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}\right)$$

$$= -\sin(x)^{\frac{1}{2}}\frac{d}{dx}x^{\frac{1}{2}} + \frac{1}{2}(\sin x)^{-\frac{1}{2}}\frac{d}{dx}(\sin x)$$

$$= -\sin(x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$$

$$= \frac{1}{2}\left(\frac{\cos x}{\sqrt{\sin x}} - \frac{\sin\sqrt{x}}{\sqrt{x}}\right)$$

Question # 3

Find $\frac{dy}{dx}$ if

(i) $y = x\cos y$ (ii) $x = y\cos y$

Solution

(i) Since $y = x\cos y$

$$\frac{dy}{dx} = \frac{d}{dx}x\cos y$$

$$= x\frac{d}{dx}\cos y + \cos y\frac{dx}{dx}$$

$$= x(-\sin y)\frac{dy}{dx} + \cos y(1)$$

$$\Rightarrow \frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y \quad \Rightarrow (1 + x\sin y)\frac{dy}{dx} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

(ii) *Do yourself as above*

Question # 4

Find the derivative w.r.t. "x"

(i) $\cos\sqrt{\dfrac{1+x}{1+2x}}$ (ii) $\sin\sqrt{\dfrac{1+2x}{1+x}}$

Solution

(i) Since $y = \cos\sqrt{\dfrac{1+x}{1+2x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\sqrt{\frac{1+x}{1+2x}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\sqrt{\frac{1+x}{1+2x}}\right) = -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+2x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1+2x)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{1+2x-2-2x}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{-1}{(1+2x)^2}\right)$$

$$= \frac{1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}(1+2x)^{2-\frac{1}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}(1+2x)^{\frac{3}{2}}}\sin\sqrt{\frac{1+x}{1+2x}}}$$

(ii) *Do yourself as above.*

Question # 5

Differentiate

(i) $\sin x$ w.r.t. $\cot x$ (ii) $\sin^2 x$ w.r.t. $\cos^4 x$

Solution

(i) Let $y = \sin x$ and $u = \cot x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin x$$
$$= \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cot x$$
$$= -\csc^2 x$$

$$\Rightarrow \frac{dx}{du} = -\frac{1}{\csc^2 x}$$
$$= -\sin^2 x$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (\cos x)(-\sin^2 x) = -\sin^2 x \cos x$$

(ii) Let $y = \sin^2 x$ and $u = \cos^4 x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2 x$$
$$= 2\sin x \frac{d}{dx}(\sin x) = 2\sin x \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cos^4 x$$
$$= 4\cos^3 x \frac{d}{dx}(\cos x) = 4\cos^3 x(-\sin x)$$
$$= -4\sin x \cos^3 x$$
$$\Rightarrow \frac{dx}{du} = -\frac{1}{4\sin x \cos^3 x}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (2\sin x \cos x)\left(-\frac{1}{4\sin x \cos^3 x}\right)$$
$$= -\frac{1}{2}\sec^2 x$$

Question # 6

If $\tan y(1+\tan x) = 1-\tan x$, show that $\frac{dy}{dx} = -1$

Solution

Since $\tan y(1+\tan x) = 1-\tan x$

$$\Rightarrow \tan y = \frac{1-\tan x}{1+\tan x}$$
$$= \frac{1-\tan x}{1+1\cdot\tan x} = \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x} = \tan\left(\frac{\pi}{4}-x\right)$$
$$\Rightarrow y = \frac{\pi}{4}-x$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$$

$$= 0 - 1 \Rightarrow \frac{dy}{dx} = -1$$

Question # 7

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$, Prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

Solution

Since $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$

Taking square on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}$$

$$= \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$$

$$\Rightarrow y^2 = \tan x + y$$

Diff. w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \sec^2 x$$

Question # 8

If $x = a\cos^3\theta$, $y = b\sin^3\theta$, Show that $a\frac{dy}{dx} + b\tan\theta = 0$

Solution

$$x = a\cos^3\theta, \quad y = b\sin^3\theta$$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right)$$

$$= a\cdot 3\cos^2\theta\frac{d}{d\theta}(\cos\theta) = 3a\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta \quad \Rightarrow \frac{d\theta}{dx} = \frac{-1}{3a\sin\theta\cos^2\theta}$$

Now diff. y w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\sin^3\theta\right)$$

$$= b\cdot 3\sin^2\theta\frac{d}{d\theta}(\sin\theta) = 3b\sin^2\theta\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$
$$= 3b\sin^2\theta\cos\theta\cdot-\frac{1}{3a\sin\theta\cos^2\theta}$$
$$= -\frac{b}{a}\tan\theta$$
$$\Rightarrow a\frac{dy}{dx} = -b\tan\theta \qquad \Rightarrow a\frac{dy}{dx} + b\tan\theta = 0$$

Question # 9

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Solution

$x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Do yourself

Derivative of inverse trigonometric formulas

(i) $\frac{d}{dx}Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

See proof on book page 76

(ii) $\frac{d}{dx}Cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Proof

Let $y = \cos^{-1}x$ where $x \in [0,\pi]$

$\Rightarrow \cos y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cos y = \frac{dx}{dx} \qquad \Rightarrow -\sin y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$
$$= \frac{-1}{\sqrt{1-\cos^2 y}} \qquad \text{Since } \sin y \text{ is positive for } x \in [0,\pi]$$
$$= \frac{-1}{\sqrt{1-x^2}}$$

(iii) $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$

See proof on book at page 77

(iv) $\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$

Proof

Let $y = \cot^{-1}x$

$\Rightarrow \cot y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cot y = \frac{d}{dx}x \quad \Rightarrow -\csc^2 y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

$$= \frac{-1}{1+\cot^2 y} \qquad \because\ 1+\cot^2 y = \csc^2 y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

(v) $\frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

Proof

Let $y = \sec^{-1}x \quad \Rightarrow \sec y = x$

Diff. w.r.t x

$$\frac{d}{dx}\sec y = \frac{d}{dx}x \quad \Rightarrow \sec y\tan y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}} \qquad \because\ 1+\tan^2 y = \sec^2 y$$

$$\Rightarrow \frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \because\ \sec y = x$$

(vi) $\frac{d}{dx}Csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

See on book at page 77

Question # 10

Differentiate w.r.t. "x"

(i) $Cos^{-1}\frac{x}{a}$ (ii) $\cot^{-1}\frac{x}{a}$ (iii) $\frac{1}{a}Sin^{-1}\frac{a}{x}$

(iv) $Sin^{-1}\sqrt{1-x^2}$ (v) $Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ (vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution

(i) Let $y = Cos^{-1}\frac{x}{a}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Cos^{-1}\frac{x}{a}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\frac{d}{dx}x$$

$$= \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a}(1) = \frac{-a}{\sqrt{a^2-x^2}}\cdot\frac{1}{a} = \frac{-1}{\sqrt{a^2-x^2}} \quad Ans$$

(ii) Let $y = \cot^{-1}\frac{x}{a}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}\frac{x}{a}$$

$$= \frac{-1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}\frac{d}{dx}(x)$$

$$= \frac{-a^2}{a^2+x^2}\cdot\frac{1}{a}(1) = \frac{-a}{a^2+x^2}.$$

(iii) Let $y = \frac{1}{a}Sin^{-1}\frac{a}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{a}\frac{d}{dx}Sin^{-1}\frac{a}{x}$$

$$= \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\frac{d}{dx}\left(\frac{a}{x}\right) = \frac{1}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}\cdot a\frac{d}{dx}\left(x^{-1}\right)$$

$$= \frac{x}{\sqrt{x^2-a^2}}\left(-x^{-2}\right) = \frac{x}{\sqrt{x^2-a^2}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x\sqrt{x^2-a^2}} \quad Ans$$

(iv) Let $y = Sin^{-1}\sqrt{1-x^2}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sin^{-1}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{1}{2}\frac{1}{\left(1-x^2\right)^{\frac{1}{2}}}(-2x) = -\frac{1}{x}\cdot\frac{x}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

(v) Let $y = Sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\cdot\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{\left(x^2+1\right)^2-\left(x^2-1\right)^2}{(x^2-1)^2}}}\cdot\left(\frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\cdot\frac{\sqrt{\left(x^4+2x^2+1\right)-\left(x^4+2x^2+1\right)}}{(x^2-1)}}\cdot\left(\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{\left(x^2-1\right)^2}{\left(x^2+1\right)\cdot\sqrt{x^4+2x^2+1-x^4+2x^2-1}}\cdot\left(\frac{2x\left(x^2-1-x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(x^2+1\right)\cdot\sqrt{4x^2}}\cdot\left(2x(-2)\right) = \frac{-4x}{\left(x^2+1\right)\cdot 2x} = \frac{-2}{\left(x^2+1\right)} \quad Ans$$

(vi) *Do yourself as above.*

(vii) *Do yourself as above.*

Question # 11

Show that $\dfrac{dy}{dx}=\dfrac{y}{x}$ if $\dfrac{y}{x}=\tan^{-1}\dfrac{x}{y}$

Solution

Since $\dfrac{y}{x} = Tan^{-1}\dfrac{x}{y} \quad \Rightarrow \quad y = x\,Tan^{-1}\dfrac{x}{y}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\,Tan^{-1}\frac{x}{y}\right)$$

$$= x\frac{d}{dx}\left(Tan^{-1}\frac{x}{y}\right)+Tan^{-1}\frac{x}{y}\cdot\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{1+\left(\dfrac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)\right)+Tan^{-1}\frac{x}{y}\cdot(1)$$

$$=x\left(\frac{1}{\dfrac{y^2+x^2}{y^2}}\left(\frac{y(1)-x\dfrac{dy}{dx}}{y^2}\right)\right)+Tan^{-1}\frac{x}{y} = \frac{x}{y^2+x^2}\left(y-x\frac{dy}{dx}\right)+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{y^2+x^2}-\frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}+\frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx} = \frac{xy}{y^2+x^2}+\frac{y}{x} \qquad \Rightarrow \left(1+\frac{x^2}{y^2+x^2}\right)\cdot\frac{dy}{dx} = \frac{y}{x}\left(\frac{x^2}{y^2+x^2}+1\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \quad \textit{Proved}$$

Question # 12

If $y = \tan\left(pTan^{-1}x\right)$, show that $\left(1+x^2\right)y_1-p\left(1+y^2\right) = 0$

Solution

Since $y = \tan\left(pTan^{-1}x\right) \quad \Rightarrow Tan^{-1}y = pTan^{-1}x$

Differentiating w.r.t x

$$\frac{d}{dx}Tan^{-1}y = p\frac{d}{dx}Tan^{-1}x$$

$$\Rightarrow \frac{1}{1+y^2}\frac{dy}{dx} = p\cdot\frac{1}{1+x^2} \qquad \Rightarrow \left(1+x^2\right)\frac{dy}{dx} = p\left(1+y^2\right)$$

$$\Rightarrow \left(1+x^2\right)y_1-p\left(1+y^2\right) = 0 \qquad \text{Since } \frac{dy}{dx}=y_1$$

2.10 Derivative of General Exponential Function (Page 80)

A function define by

$$f(x)=a^x \text{ where } a>0,\ a\neq 1$$

is called general exponential function.

Suppose $\quad y=a^x$

$$\Rightarrow\quad y+\delta y=a^{x+\delta x} \quad \Rightarrow \delta y=a^{x+\delta x}-y$$

$$\Rightarrow\quad \delta y=a^{x+\delta x}-a^x \qquad \text{Since } y=a^x$$

$$\Rightarrow\quad \delta y=a^x(a^{\delta x}-1)$$

Dividing by δx

$$\frac{\delta y}{\delta x}=\frac{a^x(a^{\delta x}-1)}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{a^x(a^{\delta x}-1)}{\delta x}$$

$$\Rightarrow\quad \frac{dy}{dx}=\lim_{\delta x\to 0}a^x\left(\frac{a^{\delta x}-1}{\delta x}\right) \quad \Rightarrow \quad \frac{dy}{dx}=a^x\lim_{\delta x\to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right)$$

$$\Rightarrow\quad \boxed{\frac{d}{dx}(a^x)=a^x.\ln a} \qquad \text{Since } \lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

Derivative of Natural Exponential Function

The exponential function $f(x)=e^x$, where $e=2.71828...$, is called Natural Exponential Function.

Suppose $\quad y = e^x$

Do yourself … Just Change *a* by *e* in above article. You'll get

$$\boxed{\frac{d}{dx}e^x = e^x}$$

2.11 Derivative of General Logarithmic Function (page 81)

If $a>0,\ a\neq 1$ and $x=a^y$, then the function defined by $y=\log_a x \quad (x>0)$ is called General Logarithmic Function.

Suppose $\quad y = \log_a x$

$$\Rightarrow\quad y+\delta y = \log_a(x+\delta x) \quad \Rightarrow \quad \delta y = \log_a(x+\delta x)-y$$

$$\Rightarrow\quad \delta y = \log_a(x+\delta x)-\log_a x$$

$$= \log_a\left(\frac{x+\delta x}{x}\right) \qquad \text{Since } \log_a m-\log_a n=\log_a\frac{m}{n}$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$=\lim_{\delta x\to 0}\frac{x}{x}\cdot\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right) \qquad \div\text{ing and }\times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\log_a x=\log_a x^m$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\log_a\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\log_a e \qquad \text{Since } \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x)=\frac{1}{x}\,\frac{1}{\log_e a} \qquad \text{Since } \log_a b=\frac{1}{\log_b a}$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x)=\frac{1}{x\ln a} \qquad \text{Since } \log_e a=\ln a$$

Derivative of Natural Logarithmic Function

The logarithmic function $f(x)=\log_e x$ where $e=2.71828...$ is called Natural Logarithmic Function. And we write $\ln x$ instead of $\log_e x$ for our ease.

Suppose $\quad y=\ln x$

$$\Rightarrow \quad y+\delta y=\ln(x+\delta x) \quad \Rightarrow \quad \delta y=\ln(x+\delta x)-y$$

$$\Rightarrow \quad \delta y=\ln(x+\delta x)-\ln x$$

$$\Rightarrow \quad \delta y=\ln\left(\frac{x+\delta x}{x}\right) \qquad \text{Since} \Rightarrow \quad \ln m-\ln n=\ln\frac{m}{n}$$

$$=\ln\left(1+\frac{\delta x}{x}\right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}=\ln\left(1+\frac{\delta x}{x}\right)$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{x}{x} \cdot \frac{1}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) \qquad \div\text{ing and } \times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \frac{x}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\ln x = \ln x^m$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \ln\left[\lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \ln e \qquad \text{Since } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x) = \frac{1}{x} \cdot 1 \qquad \text{Since } \ln e = \log_e e = 1$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Exercise 2.6 (Questions)

Question # 1

Find $f'(x)$ if

(i) $f(x) = e^{\sqrt{x}-1}$ (ii) $f(x) = x^3 e^{\frac{1}{x}}, (x \neq 0)$ (iii) $f(x) = e^x(1 + \ln x)$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$ (v) $f(x) = \ln\left(e^x + e^{-x}\right)$ (vi) $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

(vii) $f(x) = \sqrt{\ln\left(e^{2x} + e^{-2x}\right)}$ (viii) $f(x) = \ln\sqrt{\left(e^{2x} + e^{-2x}\right)}$

Solution

(i) $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t x

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{\sqrt{x}-1}$$

$$\Rightarrow f'(x) = e^{\sqrt{x}-1} \frac{d}{dx}\left(\sqrt{x} - 1\right)$$

$$= e^{\sqrt{x}-1}\left(\frac{1}{2}x^{-\frac{1}{2}} - 0\right) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \quad \textit{Ans.}$$

(ii) $f(x) = x^3 e^{\frac{1}{x}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^3e^{\frac{1}{x}}$$

$$\Rightarrow f'(x) = x^3\frac{d}{dx}e^{\frac{1}{x}} + e^{\frac{1}{x}}\frac{d}{dx}x^3$$

$$= x^3e^{\frac{1}{x}}\frac{d}{dx}\left(\frac{1}{x}\right) + e^{\frac{1}{x}}\left(3x^2\right)$$

$$= x^3e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + e^{\frac{1}{x}}\left(3x^2\right) \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$= -xe^{\frac{1}{x}} + 3x^2e^{\frac{1}{x}} = xe^{\frac{1}{x}}(3x-1) \quad \textit{Ans.}$$

(iii) $f(x) = e^x(1+\ln x)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x(1+\ln x)$$

$$\Rightarrow f'(x) = e^x\frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}e^x$$

$$= e^x\left(0+\frac{1}{x}\right) + (1+\ln x)e^x$$

$$\Rightarrow f'(x) = e^x\left(\frac{1}{x}+1+\ln x\right) \quad \text{or} \quad f'(x) = e^x\left(\frac{1+x(1+\ln x)}{x}\right)$$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$

$$\Rightarrow f'(x) = \frac{\left(e^{-x}+1\right)\frac{d}{dx}e^x - e^x\frac{d}{dx}\left(e^{-x}+1\right)}{\left(e^{-x}+1\right)^2}$$

$$= \frac{\left(e^{-x}+1\right)e^x - e^x\left(e^{-x}(-1)+0\right)}{\left(e^{-x}+1\right)^2} = \frac{e^x\left(e^{-x}+1+e^{-x}\right)}{\left(e^{-x}+1\right)^2}$$

$$\Rightarrow f'(x) = \frac{e^x\left(2e^{-x}+1\right)}{\left(e^{-x}+1\right)^2} \quad \textit{Ans.}$$

(v) $f(x) = \ln\left(e^{x}+e^{-x}\right)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\ln\left(e^{x}+e^{-x}\right)$$

$$\Rightarrow f'(x) = \frac{1}{\left(e^{x}+e^{-x}\right)}\frac{d}{dx}\left(e^{x}+e^{-x}\right)$$

$$= \frac{1}{\left(e^{x}+e^{-x}\right)}\left(e^{x}+e^{-x}(-1)\right)$$

$$\Rightarrow f'(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \quad \text{or} \quad f'(x) = \tanh x \qquad \because \tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

(vi) $f(x) = \dfrac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}\right)$$

$$= \frac{\left(e^{ax}+e^{-ax}\right)\frac{d}{dx}\left(e^{ax}-e^{-ax}\right)-\left(e^{ax}-e^{-ax}\right)\frac{d}{dx}\left(e^{ax}+e^{-ax}\right)}{\left(e^{ax}+e^{-ax}\right)^{2}}$$

$$= \frac{\left(e^{ax}+e^{-ax}\right)\left(e^{ax}(a)-e^{-ax}(-a)\right)-\left(e^{ax}-e^{-ax}\right)\left(e^{ax}(a)+e^{-ax}(-a)\right)}{\left(e^{ax}+e^{-ax}\right)^{2}}$$

$$= \frac{a\left(e^{ax}+e^{-ax}\right)\left(e^{ax}+e^{-ax}\right)-a\left(e^{ax}-e^{-ax}\right)\left(e^{ax}-e^{-ax}\right)}{\left(e^{ax}+e^{-ax}\right)^{2}}$$

$$= \frac{a\left[\left(e^{ax}+e^{-ax}\right)^{2}-\left(e^{ax}-e^{-ax}\right)^{2}\right]}{\left(e^{ax}+e^{-ax}\right)^{2}}$$

$$= \frac{a\left[(e^{2ax}+e^{-2ax}+2e^{ax}e^{-ax})-(e^{2ax}+e^{-2ax}-2e^{ax}e^{-ax})\right]}{\left(e^{ax}+e^{-ax}\right)^{2}}$$

$$= \frac{a\left[e^{2ax}+e^{-2ax}+2-e^{2ax}-e^{-2ax}+2\right]}{\left(e^{ax}+e^{-ax}\right)^{2}} \qquad \because e^{ax}e^{-ax} = e^{0} = 1$$

$$\Rightarrow f'(x) = \frac{4a}{\left(e^{ax}+e^{-ax}\right)^{2}} \quad \textit{Ans.}$$

(vii) $f(x) = \sqrt{\ln\left(e^{2x}+e^{-2x}\right)}$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{2}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{-\frac{1}{2}}\frac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$$

$$= \frac{1}{2\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}}\cdot\frac{1}{\left(e^{2x}+e^{-2x}\right)}\frac{d}{dx}\left(e^{2x}+e^{-2x}\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\frac{1}{\left(e^{2x}+e^{-2x}\right)}\left(e^{2x}(2)+e^{-2x}(-2)\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\frac{2\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)} = \frac{\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}} \qquad \textit{Ans.}$$

(viii) $f(x) = \ln\sqrt{\left(e^{2x}+e^{-2x}\right)}$

$$= \ln\left(e^{2x}+e^{-2x}\right)^{\frac{1}{2}} \quad \Rightarrow f(x) = \frac{1}{2}\ln\left(e^{2x}+e^{-2x}\right) \qquad \because \ln x^m = m\ln x$$

Now diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{1}{2}\frac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$$

Now do yourself

Question # 2

Find $\dfrac{dy}{dx}$ if

(i) $y = x^2\ln\sqrt{x}$ (ii) $y = x\sqrt{\ln x}$ (iii) $y = \dfrac{x}{\ln x}$

(iv) $y = x^2\ln\dfrac{1}{x}$ (v) $y = \ln\sqrt{\dfrac{x^2-1}{x^2+1}}$ (vi) $y = \ln\left(x+\sqrt{x^2+1}\right)$

(vii) $y = \ln\left(9-x^2\right)$ (viii) $y = e^{-2x}\sin 2x$ (ix) $y = e^{-x}\left(x^3+2x^2+1\right)$

(x) $y = xe^{\sin x}$ (xi) $y = 5e^{3x-4}$ (xii) $y = (x+1)^x$

(xiii) $y = (\ln x)^{\ln x}$ (xiv) $y = \dfrac{\sqrt{x^2-1}\,(x+1)}{\left(x^3+1\right)^{3/2}}$

Solution

(i) $y = x^2\ln\sqrt{x}$

$\Rightarrow\ y = x^2\ln(x)^{\frac{1}{2}} \qquad \Rightarrow\ y = \dfrac{1}{2}x^2\ln x \qquad \because\ \ln x^m = m\ln x$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}x^2\ln x$$
$$= \frac{1}{2}\left(x^2\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x^2\right)$$
$$= \frac{1}{2}\left(x^2\cdot\frac{1}{x} + \ln x(2x)\right) = \frac{1}{2}x + x\ln x \text{ or } \frac{1}{2}x + 2x\ln\sqrt{x} \quad \textit{Ans.}$$

(ii) $\quad y = x\sqrt{\ln x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x(\ln x)^{\frac{1}{2}}$$
$$= x\frac{d}{dx}(\ln x)^{\frac{1}{2}} + (\ln x)^{\frac{1}{2}}\frac{d}{dx}(x)$$
$$= x\cdot\frac{1}{2}(\ln x)^{-\frac{1}{2}}\frac{d}{dx}(\ln x) + (\ln x)^{\frac{1}{2}}(1) = \frac{x}{2(\ln x)^{\frac{1}{2}}}\left(\frac{1}{x}\right) + (\ln x)^{\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1+2\ln x}{2\sqrt{\ln x}} \quad \textit{Answer}$$

(iii) $\quad y = \dfrac{x}{\ln x}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\ln x}\right)$$
$$= \frac{\ln x\frac{dx}{dx} - x\frac{d}{dx}\ln x}{(\ln x)^2} = \frac{\ln x(1) - x\cdot\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \qquad \textit{Answer}$$

(iv) $\quad y = x^2\ln\dfrac{1}{x}$

$\Rightarrow\ y = x^2\ln x^{-1} \qquad \Rightarrow\ y = -x^2\ln x$

Now do yourself.

(v) $\quad y = \ln\sqrt{\dfrac{x^2-1}{x^2+1}}$

$$\Rightarrow\ y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}} \quad \Rightarrow\ y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\left(\frac{x^2-1}{x^2+1}\right)}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{x^2+1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)-\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)(2x)-\left(x^2-1\right)(2x)}{\left(x^2+1\right)}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{2x\left(x^2+1-x^2+1\right)}{\left(x^2+1\right)}\right) = \frac{1}{\left(x^2-1\right)}\cdot\left(\frac{x(2)}{\left(x^2+1\right)}\right) = \frac{2x}{\left(x^4-1\right)} \quad \textit{Ans.}$$

(vi) $\quad y = \ln\left(x+\sqrt{x^2+1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\frac{d}{dx}\left(x^2+1\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\left(x^2+1\right)^{\frac{1}{2}}}\cdot(2x)\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}} \quad \textit{Answer}$$

(vii) $\quad y = \ln\left(9-x^2\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(9-x^2\right)$$

$$= \frac{1}{9-x^2} \cdot \frac{d}{dx}\left(9-x^2\right) \quad = \frac{1}{9-x^2} \cdot (-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{9-x^2}$$

(viii) $\quad y = e^{-2x} \sin 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} e^{-2x} \sin 2x$$

$$= e^{-2x} \frac{d}{dx} \sin 2x + \sin 2x \frac{d}{dx} e^{-2x}$$

$$= e^{-2x} \cos 2x\,(2) + \sin 2x\; e^{-2x}(-2) \quad = 2e^{-2x}\left(\cos 2x - \sin 2x\right) \quad \textit{Answer}$$

(ix) $\quad y = e^{-x}\left(x^3 + 2x^2 + 1\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} e^{-x}\left(x^3 + 2x^2 + 1\right)$$

$$= e^{-x} \frac{d}{dx}\left(x^3 + 2x^2 + 1\right) + \left(x^3 + 2x^2 + 1\right) \frac{d}{dx} e^{-x}$$

$$= e^{-x}\left(3x^2 + 4x + 0\right) + \left(x^3 + 2x^2 + 1\right) \cdot e^{-x}\,(-1)$$

$$= e^{-x}\left(3x^2 + 4x\right) - \left(x^3 + 2x^2 + 1\right) \cdot e^{-x} \quad = e^{-x}\left(3x^2 + 4x - x^3 - 2x^2 - 1\right)$$

$$= e^{-x}\left(-x^3 + x^2 + 4x - 1\right) \qquad \textit{Answer}$$

(x) $\quad y = xe^{\sin x}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} xe^{\sin x}$$

$$= x \frac{d}{dx} e^{\sin x} + e^{\sin x} \frac{d}{dx} x$$

$$= x \cdot e^{\sin x} \frac{d}{dx} \sin x + e^{\sin x}(1) \quad = x \cdot e^{\sin x} \cos x + e^{\sin x}$$

$$= e^{\sin x}\left(x \cos x + 1\right) \qquad \textit{Answer}$$

(xi) *Do yourself*

(xii) $\quad y = \left(x+1\right)^x$

Taking log on both sides

$$\ln y = \ln(x+1)^x \quad \Rightarrow \ln y = x\ln(x+1)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln(x+1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(x+1) + \ln(x+1)\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x+1}\frac{d}{dx}(x+1) + \ln(x+1)(1)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{x}{x+1}(1) + \ln(x+1)\right)$$

$$= (x+1)^x\left(\frac{x}{x+1} + \ln(x+1)\right) \qquad \textit{Answer}$$

(xiii) $\quad y = (\ln x)^{\ln x}$

Taking log on both sides

$$\ln y = \ln(\ln x)^{\ln x} \quad \Rightarrow \ln y = (\ln x)\cdot\ln(\ln x)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)\cdot\ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\ln x)\frac{d}{dx}\ln(\ln x) + \ln(\ln x)\frac{d}{dx}(\ln x)$$

$$= (\ln x)\cdot\frac{1}{\ln x}\frac{d}{dx}(\ln x) + \ln(\ln x)\cdot\frac{1}{x}$$

$$= \frac{1}{x} + \frac{\ln(\ln x)}{x} = \frac{1+\ln(\ln x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1+\ln(\ln x)}{x}\right) \qquad \Rightarrow \frac{dy}{dx} = (\ln x)^{\ln x}\left(\frac{1+\ln(\ln x)}{x}\right)$$

(xiv) $\quad y = \dfrac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}} \quad \Rightarrow y = \dfrac{((x+1)(x-1))^{\frac{1}{2}}(x+1)}{\left[(x+1)(x^2-x+1)\right]^{\frac{3}{2}}}$

$$\Rightarrow y = \frac{(x+1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x+1)}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}} \quad \Rightarrow y = \frac{(x+1)^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}}$$

$$\Rightarrow \quad y = \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

Taking log on both sides

$$\ln y = \ln\frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

$$= \ln(x-1)^{\frac{1}{2}} - \ln\left(x^2-x+1\right)^{\frac{3}{2}}$$

$$\Rightarrow \quad \ln y = \frac{1}{2}\ln(x-1) - \frac{3}{2}\ln\left(x^2-x+1\right)$$

Now diff. w.r.t x

$$\frac{d}{dx}\ln y = \frac{1}{2}\frac{d}{dx}\ln(x-1) - \frac{3}{2}\frac{d}{dx}\ln\left(x^2-x+1\right)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x-1}\frac{d}{dx}(x-1) - \frac{3}{2}\frac{1}{\left(x^2-x+1\right)}\frac{d}{dx}\left(x^2-x+1\right)$$

$$= \frac{1}{2(x-1)}(1) - \frac{3}{2\left(x^2-x+1\right)}(2x-1) \quad = \frac{1}{2(x-1)} - \frac{3(2x-1)}{2\left(x^2-x+1\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{x^2-x+1-3(2x-1)(x-1)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}} \cdot \left[\frac{x^2-x+1-3\left(2x^2-x-2x+1\right)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \left[\frac{x^2-x+1-6x^2+3x+6x-3}{2(x-1)^{1-\frac{1}{2}}\left(x^2-x+1\right)^{\frac{3}{2}+1}}\right] \quad = \frac{-5x^2+8x-2}{2(x-1)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{5}{2}}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{5x^2-8x+2}{2\sqrt{x-1}\left(x^2-x+1\right)^{\frac{5}{2}}} \quad Ans.$$

(xv) $$y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+x-2}}$$

$$\Rightarrow \quad y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+2x-x-2}} \quad \Rightarrow \quad y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x(x+2)-1(x+2)}}$$

$$\Rightarrow\ y = \frac{(x+2)^2\cdot\sqrt{x-1}}{\sqrt{(x+2)(x-1)}} \quad \Rightarrow\ y = (x+2)^{2-\frac{1}{2}} \Rightarrow\ y = (x+2)^{\frac{3}{2}}$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)^{\frac{3}{2}}$$

Do yourself

2.1.3 Derivative of Hyperbolic Function (page 85)

The hyperbolic functions are define by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \ x\in R\ ; \qquad \cosh x = \frac{e^x + e^{-x}}{2} \quad , \ x\in R$$

$$\text{and} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad , \ x\in R$$

The reciprocal of these functions are defined as;

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad , \ x\in R-\{0\}; \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad , \ x\in R$$

$$\text{and} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad , \ x\in R-\{0\}$$

and there derivatives are

(i) $\frac{d}{dx}(\sinh x) = \cosh x$ (ii) $\frac{d}{dx}(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ (iv) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ (vi) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Proof:

(i) $$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = \frac{1}{2}\frac{d}{dx}\left(e^x - e^{-x}\right)$$

$$= \frac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) = \frac{1}{2}\left(e^x - e^{-x} \qquad e^{-x}\right)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x$$

(ii) *Similar as above.*

(iii) *See the below (iv) proof.*

(iv) $$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$$

$$=\frac{\left(e^x-e^{-x}\right)\frac{d}{dx}\left(e^x+e^{-x}\right)-\left(e^x+e^{-x}\right)\frac{d}{dx}\left(e^x-e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\left(e^x-e^{-x}\right)\left(e^x+e^{-x}(-1)\right)-\left(e^x+e^{-x}\right)\left(e^x-e^{-x}(-1)\right)}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)-\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\left(e^x-e^{-x}\right)^2-\left(e^x+e^{-x}\right)^2}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{(e^{2x}+e^{-2x}-2e^xe^{-x})-(e^{2x}+e^{-2x}+2e^xe^{-x})}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{e^{2x}+e^{-2x}-2-e^{2x}-e^{-2x}-2}{\left(e^x-e^{-x}\right)^2} \qquad \because\ e^xe^{-x}=e^0=1$$

$$=\frac{-4}{\left(e^x-e^{-x}\right)^2}=-\left(\frac{2}{e^x-e^{-x}}\right)^2=-\operatorname{csch}^2 x$$

(v) $$\frac{d}{dx}(\operatorname{sech} x)=\frac{d}{dx}\left(\frac{2}{e^x+e^{-x}}\right)=\frac{d}{dx}2\left(e^x+e^{-x}\right)^{-1}=2\frac{d}{dx}\left(e^x+e^{-x}\right)^{-1}$$

$$=2\left[(-1)\left(e^x+e^{-x}\right)^{-1-1}\frac{d}{dx}\left(e^x+e^{-x}\right)\right]$$

$$=-2\left(e^x+e^{-x}\right)^{-2}\left(e^x+e^{-x}(-1)\right)=\frac{-2}{\left(e^x+e^{-x}\right)^2}\left(e^x-e^{-x}\right)$$

$$=\frac{-2\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}=-\frac{2}{\left(e^x+e^{-x}\right)}\frac{\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)}$$

$$=-\operatorname{sech} x\tanh x$$

(vi) *Do yourself as above (v).*

2.14 Derivative of Inverse Hyperbolic Function (page 86)

(i) $\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}$ (ii) $\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}}$

(iii) $\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}$ (iv) $\frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^2}$

(v) $\dfrac{d}{dx}\operatorname{sech}^{-1}x=\dfrac{-1}{x\sqrt{1-x^2}}$ (vi) $\dfrac{d}{dx}\operatorname{csch}^{-1}x=\dfrac{-1}{x\sqrt{1+x^2}}$

Proof:

(i) Let $y=\sinh^{-1}x \Rightarrow \sinh y=x$

differentiate w.r.t. x.

$$\frac{d}{dx}\sinh y=\frac{d}{dx}x \quad\Rightarrow\quad \cosh y\frac{dy}{dx}=1 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{\cosh y}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{\sqrt{1+\sinh^2 y}} \qquad \because\ \cosh^2 x-\sinh^2 x=1$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}} \qquad \because\ \sinh y=x$$

(ii) Do yourself as above.

(iii) Do yourself as (iv) below *or* see book at page 88.

(iv) Let $y=\coth^{-1}x \Rightarrow \coth y=x$

differentiate w.r.t. x

$$\frac{d}{dx}\coth y=\frac{d}{dx}x \quad\Rightarrow\quad -\operatorname{csch}^2 y\frac{dy}{dx}=1 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{-\operatorname{csch}^2 y}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{-(\coth^2 y-1)} \qquad \because\ \coth^2 y-1=\operatorname{csch}^2 y$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{-\coth^2 y+1}=\frac{1}{1-\coth^2 y}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{1-x^2} \qquad \because \coth y=x$$

(v) Suppose $y=\operatorname{sech}^{-1}x \Rightarrow \operatorname{sech} y=x$

differentiate w.r.t. x

$$\frac{d}{dx}\operatorname{sech} y=\frac{d}{dx}x \quad\Rightarrow\quad -\operatorname{sech} y\tanh y\frac{dy}{dx}=1 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{-\operatorname{sech} y\tanh y}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-1}{\operatorname{sech} y\sqrt{1-\tanh^2 y}} \qquad \because\ 1-\tanh^2 y=\operatorname{sech}^2 y$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-1}{x\sqrt{1-x^2}} \qquad \because \operatorname{sech} y=x$$

(vi) *Do yourself as above*

Question # 3

Find $\dfrac{dy}{dx}$ if

(i) $y = \cosh 2x$ (ii) $y = \sinh 3x$ (iii) $y = \tanh^{-1}(\sin x), -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(iv) $y = \sinh^{-1}\left(x^3\right)$ (v) $y = \left(\ln \tanh x\right)$ (vi) $y = \sinh^{-1}\left(\dfrac{x}{2}\right)$

Solution

(i) $y = \cosh 2x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cosh 2x \Rightarrow \frac{dy}{dx} = \sinh 2x \frac{d}{dx}(2x) \Rightarrow \frac{dy}{dx} = 2\sinh 2x$$

(ii) *Do yourself*

(iii) $y = \tanh^{-1}(\sin x) \Rightarrow \tanh y = \sin x$

Diff. w.r.t x

$$\frac{d}{dx}\tanh y = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{\operatorname{sech}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{1 - \tanh^2 y}$$

$\because \cosh^2\theta - \sinh^2\theta = 1$

$\therefore 1 - \tanh^2\theta = \operatorname{sech}^2\theta$

$\because \sin x = \tanh y$

$$= \frac{\cos x}{1 - \sin^2 x}$$

$$= \frac{\cos x}{\cos^2 x} \Rightarrow \frac{dy}{dx} = \sec x$$

(iv) $y = \sinh^{-1}\left(x^3\right) \Rightarrow \sinh y = x^3$

$$\Rightarrow \frac{d}{dx}\sinh y = \frac{d}{dx}x^3 \Rightarrow \cosh y \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{\cosh y}$$

$$= \frac{3x^2}{\sqrt{1 + \sinh^2 y}}$$

$\because \cosh^2 y - \sinh^2 y = 1$

$$= \frac{3x^2}{\sqrt{1+\left(x^3\right)^2}} = \frac{3x^2}{\sqrt{1+x^6}}. \quad \textit{Answer}$$

(v) *Do yourself*

(vi) $y = \sinh^{-1}\left(\dfrac{x}{2}\right) \Rightarrow \sinh y = \dfrac{x}{2}$

Now diff w.r.t x

$$\frac{d}{dx}\sinh y = \frac{d}{dx}\left(\frac{x}{2}\right) \Rightarrow \cosh y\frac{dy}{dx} = \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\cosh y}$$

$\because \cosh^2 y - \sinh^2 y = 1$

$\therefore \cosh^2 y = 1 + \sinh^2 y$

$$= \frac{1}{2\sqrt{1+\sinh^2 y}}$$

$$= \frac{1}{2\sqrt{1+(x/2)^2}} = \frac{1}{2\sqrt{(4+x^2)/2}} = \frac{1}{\sqrt{4+x^2}} \quad \textit{Answer.}$$

Question # 1

Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x+5)^{\frac{3}{2}}$ (iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 3x^4 + 4x^3 + x - 2\right)$$

$$\Rightarrow y_1 = 2\left(5x^4\right) - 3\left(4x^3\right) + 4\left(3x^2\right) + 1 - 0$$

$$= 10x^4 - 12x^3 + 12x^2 + 1$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}\left(10x^4 - 12x^3 + 12x^2 + 1\right)$$

$$\Rightarrow y_2 = 10\left(4x^3\right) - 12\left(3x^2\right) + 12(2x) + 0$$

$$= 40x^3 - 36x^2 + 24x \quad Ans.$$

(ii) $y = (2x+5)^{\frac{3}{2}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{\frac{3}{2}}$$

$$\Rightarrow y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1}\frac{d}{dx}(2x+5)$$

$$= \frac{3}{2}(2x+5)^{\frac{1}{2}}(2) = 3(2x+5)^{\frac{1}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}(2x+5)^{\frac{1}{2}}$$

$$\Rightarrow y_2 = 3\cdot\frac{1}{2}(2x+5)^{-\frac{1}{2}}(2) \quad \Rightarrow y_2 = \frac{3}{\sqrt{2x+5}}$$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\Rightarrow y = (x)^{\frac{1}{2}} + (x)^{-\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x)^{\frac{1}{2}}+(x)^{-\frac{1}{2}}\right] \quad \Rightarrow y_1 = \frac{1}{2}(x)^{-\frac{1}{2}}-\frac{1}{2}(x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{1}{2}\frac{d}{dx}\left[(x)^{-\frac{1}{2}}-(x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = \frac{1}{2}\left[-\frac{1}{2}(x)^{-\frac{3}{2}}+\frac{3}{2}(x)^{-\frac{5}{2}}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{x^{\frac{3}{2}}}+\frac{3}{x^{\frac{5}{2}}}\right] = \frac{1}{4}\left[\frac{-x+3}{x^{\frac{5}{2}}}\right] \quad \text{or} \quad y_2 = \frac{3-x}{4x^{\frac{5}{2}}}$$

Question # 2

Find y_2 if

(i) $y = x^2e^{-x}$ (ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

Solution

(i) $y = x^2e^{-x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2e^{-x}$$

$$\Rightarrow y_1 = x^2\frac{d}{dx}e^{-x}+e^{-x}\frac{d}{dx}x^2$$

$$= x^2e^{-x}(-1)+e^{-x}(2x)$$

$$= e^{-x}\left(-x^2+2x\right)$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}e^{-x}\left(-x^2+2x\right)$$

$$y_2 = e^{-x}\frac{d}{dx}\left(-x^2+2x\right)+\left(-x^2+2x\right)\frac{d}{dx}e^{-x}$$

$$= e^{-x}(-2x+2)+\left(-x^2+2x\right)e^{-x}(-1)$$

$$= e^{-x}\left(-2x+2+x^2-2x\right)$$

$$= e^{-x}\left(x^2-4x+2\right)$$

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$\Rightarrow y = \ln(2x+3)-\ln(3x+2)$$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\ln(2x+3)-\frac{d}{dx}\ln(3x+2)$$

$$\Rightarrow y_1 = \frac{1}{2x+3}(2)-\frac{1}{3x+2}(3)$$

$$= 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 2\frac{d}{dx}(2x+3)^{-1} - 3\frac{d}{dx}(3x+2)^{-1}$$

$$\Rightarrow y_2 = 2\left[-(2x+3)^{-2}(2)\right] - 3\left[-(3x+2)^{-2}(3)\right]$$

$$\Rightarrow y_2 = -\frac{4}{(2x+3)^2} + \frac{9}{(3x+2)^2} \quad \textit{Ans.}$$

$$\text{OR} \quad y_2 = \frac{-4(3x+2)^2 + 9(3x+2)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-4\left(9x^2+12x+4\right) + 9\left(4x^2+12x+9\right)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-36x^2 - 48x - 16 + 36x^2 + 108x + 81}{(2x+3)^2(3x+2)^2} = \frac{60x+65}{(2x+3)^2(3x+2)^2} \quad \textit{Ans.}$$

(iii) $y = \sqrt{\dfrac{1-x}{1+x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

By solving, you will get (differentiate here)

$$\Rightarrow y_1 = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}} = -(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left[(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = -(1-x)^{-\frac{1}{2}}\frac{d}{dx}(1+x)^{-\frac{3}{2}} - (1+x)^{-\frac{3}{2}}\frac{d}{dx}(1-x)^{-\frac{1}{2}}$$

$$= -(1-x)^{-\frac{1}{2}}\left(-\frac{3}{2}(1+x)^{-\frac{5}{2}}(1)\right) - (1+x)^{-\frac{3}{2}}\left(-\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\right)$$

$$= \frac{3}{2(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}} - \frac{1}{2(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}$$

$$= \frac{3(1-x)-(1+x)}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{3-3x-1-x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}}$$

$$= \frac{2-4x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{1-2x}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} \quad \textit{Ans.}$$

Question # 3
Find y_2 if

(i) $x^2+y^2 = a^2$ (ii) $x^3-y^3 = a^3$ (iii) $x = a\cos\theta,\ y = a\sin\theta$
(iv) $x = at^2,\ y = bt^4$ (v) $x^2+y^2+2gx+2fy+c=0$

Solution

(i) $x^2+y^2 = a^2$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}a^2 \Rightarrow 2x+2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\,y_1 = -2x \quad \Rightarrow \quad y_1 = -\frac{x}{y}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right) \quad \Rightarrow \quad y_2 = -\left(\frac{y\frac{dx}{dx}-x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow y_2 = -\left(\frac{y(1)-x\left(-\frac{x}{y}\right)}{y^2}\right) \qquad \because \frac{dy}{dx} = -\frac{x}{y}$$

$$= -\left(\frac{y+\frac{x^2}{y}}{y^2}\right) = -\left(\frac{\frac{y^2+x^2}{y}}{y^2}\right)$$

$$= -\left(\frac{x^2+y^2}{y^3}\right) \quad \textit{Ans.}$$

$$\text{OR} \quad y_2 = -\frac{a^2}{y^3} \qquad \because x^2+y^2 = a^2$$

(ii) $x^3-y^3 = a^3$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^3-y^3\right) = \frac{d}{dx}a^3$$

$$3x^2-3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow -3y^2y_1 = -3x^2 \quad \Rightarrow \quad y_1 = \frac{x^2}{y^2}$$

Again diff. w.r.t x

$$\Rightarrow \quad \frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\Rightarrow \quad y_2 = \frac{y^2\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(y^2)}{\left(y^2\right)^2}$$

$$= \frac{y^2(2x) - x^2\left(2y\frac{dy}{dx}\right)}{y^4}$$

$$= \frac{2xy^2 - 2x^2y\left(\frac{x^2}{y^2}\right)}{y^4} \qquad \because \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

$$= \frac{-2x\left(x^3 - y^3\right)}{y^5} \qquad Ans.$$

OR $$y_2 = \frac{-2x\left(a^3\right)}{y^5} \qquad \because x^3 - y^3 = a^3$$

$$\Rightarrow \quad y_2 = -\frac{2a^3x}{y^5}$$

(iii) $x = a\cos\theta \quad , \quad y = a\sin\theta$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\cos\theta$$

$$= -a\sin\theta$$

$$\Rightarrow \quad \frac{d\theta}{dx} = -\frac{1}{a\sin\theta}$$

Diff y w.r.t θ

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}\sin\theta$$

$$= a\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= a\cos\theta\cdot\frac{-1}{a\sin\theta} \qquad \Rightarrow \quad y_1 = -\cot\theta$$

Now diff. y_1 w.r.t θ

$$\frac{dy_1}{dx} = -\frac{d}{dx}\cot\theta$$

$$\Rightarrow \quad y_2 = +\operatorname{cosec}^2\theta\frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta . \left(-\frac{1}{a\sin\theta}\right)$$

$$\Rightarrow y_2 = \frac{-1}{a\sin^3\theta}$$

(iv) $\quad x = at^2 \quad , \quad y = bt^4$

Diff. x w.r.t t

$$\frac{dx}{dt} = a\frac{d}{dt}t^2$$

$$= 2at$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{2at}$$

Diff. y w.r.t t

$$\frac{dy}{dt} = b\frac{d}{dt}(t^4)$$

$$= 4bt^3$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= 4bt^3\cdot\frac{1}{2at} \qquad \Rightarrow y_1 = \frac{2b}{a}t^2$$

Now diff. y_1 w.r.t x

$$\frac{dy_1}{dx} = \frac{2b}{a}\frac{d}{dx}\left(t^2\right) = \frac{2b}{a}\frac{d}{dt}\left(t^2\right)\cdot\frac{dt}{dx}$$

$$\Rightarrow y_2 = \frac{2b}{a}(2t)\cdot\frac{1}{2at} \qquad \Rightarrow y_2 = \frac{2b}{a^2}$$

(v) $\quad x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \frac{d}{dx}\left(x^2 + y^2 + 2gx + 2fy + c\right) = \frac{d}{dx}(0)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow (2y+2f)\frac{dy}{dx} + (2x+2g) = 0$$

$$\Rightarrow (2y+2f)\frac{dy}{dx} = -(2x+2g)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2x+2g)}{(2y+2f)} \Rightarrow y_1 = -\frac{x+g}{y+f}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x+g}{y+f}\right)$$

$$\Rightarrow y_2 = -\left(\frac{(y+f)\frac{d}{dx}(x+g) - (x+g)\frac{d}{dx}(y+f)}{(y+f)^2}\right)$$

$$= -\frac{(y+f)(1)-(x+g)\dfrac{dy}{dx}}{(y+f)^2} = -\frac{(y+f)-(x+g)\left(-\dfrac{x+g}{y+f}\right)}{(y+f)^2}$$

$$= -\frac{\dfrac{(y+f)^2+(x+g)^2}{y+f}}{(y+f)^2} = -\frac{(y+f)^2+(x+g)^2}{(y+f)^3} \qquad \textit{Ans.}$$

OR $$y_2 = -\frac{y^2+2yf+f^2+x^2+2xg+g^2}{(y+f)^3}$$

$$= -\frac{(x^2+y^2+2gx+2fy+c)-c+f^2+g^2}{(y+f)^3}$$

$$= -\frac{0-c+f^2+g^2}{(y+f)^3} \qquad \because\ x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow\ y_2 = \frac{c-f^2-g^2}{(y+f)^3} \qquad \textit{Ans.}$$

Question # 4

Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2-9)$

Solution

(i) $y = \sin 3x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$\Rightarrow\ y_1 = \cos 3x\,(3) \quad \Rightarrow\ y_1 = 3\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}\cos 3x \quad \Rightarrow\ y_2 = 3(-\sin 3x\,(3)) \quad \Rightarrow\ y_2 = -9\sin 3x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -9\frac{d}{dx}\sin 3x$$

$$\Rightarrow\ y_3 = -9\cos 3x\,(3) \quad \Rightarrow\ y_3 = -27\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = -27\frac{d}{dx}\cos 3x \quad \Rightarrow\ y_4 = -27(-\sin 3x\,(3))$$

$$\Rightarrow\ \boxed{y_4 = 81\sin 3x}$$

(ii) $y = \cos^3 x$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^3 x\right)$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\frac{d}{dx}\cos x$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\left(-\sin x\right)$$

$$\Rightarrow y_1 = 3\left(1-\sin^2 x\right)\left(-\sin x\right) \qquad \Rightarrow y_1 = -3\sin x + 3\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -3\frac{d}{dx}\sin x + 3\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_2 = -3\cos x + 9\sin^2 x\frac{d}{dx}\sin x$$

$$\Rightarrow y_2 = -3\cos x + 9\left(1-\cos^2 x\right)\cos x$$

$$= -3\cos x + 9\cos x - 9\cos^3 x \quad = 6\cos x - 9\cos^3 x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = 6\frac{d}{dx}\cos x - 9\frac{d}{dx}\cos^3 x$$

$$\Rightarrow y_3 = 6\left(-\sin x\right) - 9\left(-3\sin x + 3\sin^3 x\right) \qquad \because \frac{d}{dx}(\cos^3 x) = -3\sin x + 3\sin^3 x$$

$$= -6\sin x + 27\sin x - 27\sin^3 x \quad = 21\sin x - 27\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 21\frac{d}{dx}\sin x - 27\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_4 = 21\left(\cos x\right) - 27\left(3\sin^2 x\right)\frac{d}{dx}\sin x$$

$$= 21\cos x - 81\sin^2 x\left(\cos x\right) \quad = 21\cos x - 81\left(1-\cos^2 x\right)\left(\cos x\right)$$

$$= 21\cos x - 81\cos x + 81\cos^3 x \quad = -60\cos x + 54\cos^3 x$$

Alternative:

$$y = \cos^3 x$$

Since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\Rightarrow \cos 3x - 3\cos x = 4\cos^3 x \quad \Rightarrow \cos^3 x = \frac{1}{4}\left(\cos 3x - 3\cos x\right)$$

Therefore

$$y = \frac{1}{4}\left(\cos 3x - 3\cos x\right)$$

Now diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(\frac{d}{dx}\cos 3x - 3\frac{d}{dx}\cos x\right)$$

Do yourself

(iii) $y = \ln\left(x^2-9\right)$

$= \ln\left[(x+3)(x-3)\right] = \ln(x+3)+\ln(x-3)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x+3)+\frac{d}{dx}\ln(x-3)$$

$$\Rightarrow y_1 = \frac{1}{x+3}+\frac{1}{x-3}$$

$$= (x+3)^{-1}+(x-3)^{-1}$$

Again diff w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}(x+3)^{-1}+\frac{d}{dx}(x-3)^{-1}$$

$$\Rightarrow y_2 = -(x+3)^{-2}-(x-3)^{-2}$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -\frac{d}{dx}(x+3)^{-2}-\frac{d}{dx}(x-3)^{-2} \quad \Rightarrow y_3 = 2(x+3)^{-3}+2(x-3)^{-3}$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 2\frac{d}{dx}(x+3)^{-3}+2\frac{d}{dx}(x-3)^{-3}$$

$$\Rightarrow y_4 = 2\left(-3(x+3)^{-4}\right)+2\left(-3(x-3)^{-4}\right)$$

$$= \frac{-6}{(x+3)^4}+\frac{-6}{(x+3)^4} = -6\left[\frac{1}{(x+3)^4}+\frac{1}{(x+3)^4}\right] \quad \textit{Ans.}$$

Question # 5

If $x=\sin\theta$, $y=\sin m\theta$, Show that $\left(1-x^2\right)y_2 - xy_1 + m^2 y = 0$

Solution $x = \sin\theta$ …… (i), $y = \sin m\theta$ ……..(ii)

From (i) $\theta=\sin^{-1}x$, putting in (ii)

$$y = \sin\left(m\sin^{-1}x\right)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin m\left(\sin^{-1}x\right)$$

$$\Rightarrow y_1 = \cos\left(m\sin^{-1}x\right)\frac{d}{dx}m\sin^{-1}x$$

$$= \cos\left(m\sin^{-1}x\right)\cdot m\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1\sqrt{1-x^2} = m\cos\left(m\sin^{-1}x\right)$$

Taking square on both sides.

$$y_1^2\left(1-x^2\right) = m^2\cos^2\left(m\sin^{-1}x\right)$$

$$\Rightarrow y_1^2\left(1-x^2\right) = m^2\left(1-\sin^2(m\sin^{-1}x)\right) \qquad \because \cos^2 x = 1-\sin^2 x$$

$$\Rightarrow\ y_1^2\left(1-x^2\right)=m^2\left(1-y^2\right) \qquad \text{From (ii)}$$

Now again diff. w.r.t x

$$\frac{d}{dx}y_1^2\left(1-x^2\right)=m^2\frac{d}{dx}\left(1-y^2\right)$$

$$\Rightarrow\ y_1^2\frac{d}{dx}\left(1-x^2\right)+\left(1-x^2\right)\frac{d}{dx}y_1^2=m^2\left(0-2y\frac{dy}{dx}\right)$$

$$\Rightarrow\ y_1^2(-2x)+\left(1-x^2\right)2y_1\frac{dy_1}{dx}=-2m^2y\frac{dy}{dx}$$

$$\Rightarrow\ -2xy_1^2+\left(1-x^2\right)2y_1y_2=-2m^2y\,y_1$$

$$\Rightarrow\ 2y_1\left(-xy_1+\left(1-x^2\right)y_2\right)=2y_1\left(-m^2y\right)$$

$$\Rightarrow\ -xy_1+\left(1-x^2\right)y_2=-m^2y$$

$$\Rightarrow\ \left(1-x^2\right)y_2-xy_1+m^2y=0 \qquad \textit{Proved}$$

Question # 6

If $y=e^x\sin x$, show that $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0$

Solution $\quad y=e^x\sin x$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}e^x\sin x$$

$$=e^x\frac{d}{dx}\sin x+\sin x\frac{d}{dx}e^x$$

$$=e^x\cos x+\sin x\,e^x \ =\ e^x\left(\cos x+\sin x\right)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}e^x\left(\cos x+\sin x\right)$$

$$\Rightarrow\ \frac{d^2y}{dx^2}=e^x\frac{d}{dx}\left(\cos x+\sin x\right)+\left(\cos x+\sin x\right)\frac{d}{dx}e^x$$

$$=e^x\left(-\sin x+\cos x\right)+\left(\cos x+\sin x\right)e^x \ =\ e^x\left(-\sin x+\cos x+\cos x+\sin x\right)$$

$$=e^x\left(2\cos x\right)\ =\ 2e^x\cos x$$

Now

$$\text{L.H.S}=\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y$$

$$=2e^x\cos x-2e^x\left(\cos x+\sin x\right)+2e^x\sin x$$

$$=2e^x\left(\cos x-\cos x-\sin x+\sin x\right)$$

$$=0$$

i.e. $\quad \dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0 \qquad \textit{Proved}$

Question # 7

If $y = e^{ax}\sin bx$, show that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + \left(a^2+b^2\right)y = 0$

Solution $y = e^{ax}\sin bx$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}e^{ax}\sin bx$$

$$= e^{ax}\frac{d}{dx}\sin bx + \sin bx\frac{d}{dx}e^{ax} = e^{ax}\cos bx(b) + \sin bx\, e^{ax}(a)$$

$$= e^{ax}\left(b\cos bx + a\sin bx\right)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}e^{ax}\left(b\cos bx + a\sin bx\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{ax}\frac{d}{dx}\left(b\cos bx + a\sin bx\right) + \left(b\cos bx + a\sin bx\right)\frac{d}{dx}e^{ax}$$

$$= e^{ax}\left(-b\sin bx(b) + a\cos bx(b)\right) + \left(b\cos bx + a\sin bx\right)e^{ax}(a)$$

$$= e^{ax}\left(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + 2a^2\sin bx - a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left[2a\left(b\cos bx + a\sin bx\right) - \left(a^2+b^2\right)\sin bx\right]$$

$$= 2ae^{ax}\left(b\cos bx + a\sin bx\right) - \left(a^2+b^2\right)e^{ax}\sin bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - \left(a^2+b^2\right)y \quad \Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = 0$$

Question # 8

If $y = \left(Cos^{-1}x\right)^2$, prove that $\left(1-x^2\right)y_2 - xy_1 - 2 = 0$

Solution $y = \left(Cos^{-1}x\right)^2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(Cos^{-1}x\right)^2 \quad \Rightarrow y_1 = 2\left(Cos^{-1}x\right)\frac{d}{dx}Cos^{-1}x$$

$$\Rightarrow y_1 = 2\left(Cos^{-1}x\right)\cdot\frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow y_1\sqrt{1-x^2} = -2\left(Cos^{-1}x\right)$$

On squaring both sides

$$y_1^2\left(1-x^2\right) = 4\left(Cos^{-1}x\right)^2$$

$$\Rightarrow y_1^2\left(1-x^2\right) = 4y \qquad \because y = \left(Cos^{-1}x\right)^2$$

Again diff. w.r.t x

$$\frac{d}{dx}y_1^2\left(1-x^2\right) = 4\frac{dy}{dx}$$

$$\Rightarrow \left(1-x^2\right)\frac{d}{dx}y_1^2 + y_1^2\frac{d}{dx}\left(1-x^2\right) = 4y_1$$

$$\Rightarrow \left(1-x^2\right)\cdot 2y_1\frac{dy_1}{dx} + y_1^2(-2x) = 4y_1 \quad \Rightarrow 2y_1\left[\left(1-x^2\right)y_2 - xy_1\right] = 4y_1$$

$$\Rightarrow \left(1-x^2\right)y_2 - xy_1 - 2 = 0$$

Question # 9

If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

Solution $\quad y = a\cos(\ln x) + b\sin(\ln x)$

Diff. w.r.t x

$$\frac{dy}{dx} = a\frac{d}{dx}\cos(\ln x) + b\frac{d}{dx}\sin(\ln x)$$

$$= a\left[-\sin(\ln x)\right]\frac{d}{dx}(\ln x) + b\cos(\ln x)\frac{d}{dx}(\ln x)$$

$$= -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = -a\frac{d}{dx}\sin(\ln x) + b\frac{d}{dx}\cos(\ln x)$$

$$\Rightarrow x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\left(\frac{dx}{dx}\right) = -a\cos(\ln x)\frac{d}{dx}(\ln x) + b(-\sin(\ln x))\frac{d}{dx}(\ln x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(1) = -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left(a\cos(\ln x) + b\sin(\ln x)\right) \quad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \textit{Proved}$$

Taylor Series Expansion of Function

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+....$$

Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

Question # 1

Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x)= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$

(ii) $\cos x= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$

(iii) $\sqrt{1+x}= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+.....$

(iv) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....$

(v) $e^{2x} = 1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+.....$

Solution

(i) Let $f(x) = \ln(1+x)$

$\Rightarrow f(0) = \ln(1+0) = 0$

$f'(x) = \frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$\Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$

$\Rightarrow f''(0) = -(1+0)^{-2} = -1$

$f'''(x) = \frac{d}{dx}\left[-(1+x)^{-2}\right] = +2(1+x)^{-3}$

$\Rightarrow f'''(0) = 2(1+0)^{-3} = 2$

$f^{(iv)}(x) = \frac{d}{dx}2(1+x)^{-3} = -6(1+x)^{-4}$

$\Rightarrow f^{(iv)}(0) = -6(1+0)^{-4} = -6$

By Maclaurin series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

$$\Rightarrow \ln(1+x) = 0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+\ldots\ldots$$

$$= x-\frac{x^2}{2\cdot 1}+\frac{x^3}{3\cdot 2\cdot 1}(2)-\frac{x^4}{4\cdot 3\cdot 2\cdot 1}(6)+\ldots\ldots$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

(ii) Let $f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x \qquad \Rightarrow f'(0) = -\sin(0) = 0$$

$$f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \qquad \Rightarrow f''(0) = -\cos(0) = -1$$

$$f'''(x) = \frac{d}{dx}(-\cos x) = +\sin x \qquad \Rightarrow f'''(0) = \sin(0) = 0$$

$$f^{(iv)}(x) = \frac{d}{dx}\sin x = \cos x \qquad \Rightarrow f^{(iv)}(x) = \cos(0) = 1$$

$$f^{(v)}(x) = \frac{d}{dx}\cos x = -\sin x \qquad \Rightarrow f^{(v)}(x) = -\sin(0) = 0$$

$$f^{(vi)}(0) = \frac{d}{dx}(-\sin x) = -\cos x \qquad \Rightarrow f^{(vi)}(0) = -\cos(0) = -1$$

Now by Maclaurin series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

$$\Rightarrow \cos x = 1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+\frac{x^5}{5!}(0)+\frac{x^6}{6!}(-1)+\ldots\ldots$$

$$= 1+0-\frac{x^2}{2!}+0+\frac{x^4}{4!}+0-\frac{x^6}{6!}+\ldots\ldots$$

$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\ldots$$

(iii) Let $f(x) = \sqrt{1+x}$

$$= (1+x)^{\frac{1}{2}} \qquad \Rightarrow f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(x) = \frac{d}{dx}(1+x)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1+x)^{-\frac{1}{2}}(1) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$\Rightarrow f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$

$$f''(x) = \frac{d}{dx}\left[\frac{1}{2}(1+x)^{-\frac{1}{2}}\right] = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$\Rightarrow f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\frac{d}{dx}\left[(1+x)^{-\frac{3}{2}}\right]$$

$$= -\frac{1}{4}\left[-\frac{3}{2}(1+x)^{-\frac{5}{2}}\right] = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$\Rightarrow f'''(0) = \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$$

Now by Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots$$

$$\Rightarrow \sqrt{1+x} = 1 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\cdot\frac{3}{8} + \dots..$$

$$= 1 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{6}\cdot\frac{3}{8} + \dots..$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots..$$

(iv) Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f''(0) = e^0 = 1$$

$$f'''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'''(0) = e^0 = 1$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots$$

$$\Rightarrow e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \dots..$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots..$$

(v) Let $f(x) = e^{2x} \Rightarrow f(0) = e^{2(0)} = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow f'(0) = 2e^{2(0)} = 2(1) = 2$$

$$f''(x) = 2\frac{d}{dx}\left(e^{2x}\right) = 2\left(2e^{2x}\right) = 4e^{2x}$$

$$\Rightarrow f''(0) = 4e^{2(0)} = 4(1) = 4$$

$$f'''(x) = 4\frac{d}{dx}\left(e^{2x}\right) = 4\left(2e^{2x}\right) = 8e^{2x}$$

$$\Rightarrow f'''(0) = 8e^{2(0)} = 8$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \ldots\ldots$$

$$= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots\ldots$$

Question # 2

Show that

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

and evaluate $\cos 61^\circ$.

Solution Let $f(x) = \cos x$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x$$

$$f''(x) = -\frac{d}{dx}\sin x = -\cos x$$

$$f'''(x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x$$

By Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

$$\Rightarrow \cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \ldots\ldots$$

$$\Rightarrow \cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

Put $x = 60^\circ$ and $h = 1^\circ = \frac{\pi}{180} = 0.01745$ rad

$$\cos(60+1) = \cos 60 - (0.01745)\sin 60 - \frac{(0.01745)^2}{\lfloor 2}\cos 60 + \frac{(0.01745)^3}{\lfloor 3}\sin 60 + \ldots\ldots$$

Increasing and Decreasing Function (Page 104)

Let f be defined on an interval (a,b) and let $x_1, x_2 \in (a,b)$. Then

1. f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$
2. f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Theorem (Page 105)

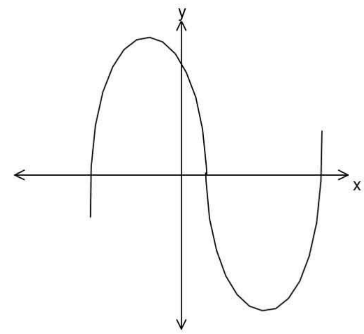
Let f be differentiable on the open interval (a,b).

1- f is increasing on (a,b) if $f'(x) > 0$ for each $x \in (a,b)$.

2- f is decreasing on (a,b) if $f'(x) < 0$ for each $x \in (a,b)$.

First Derivative Test (Page 109)

Let f be differentiable in neighbourhood of c, where $f'(c) = 0$.



1. The function has relative maxima at $x = c$ if $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$.
2. The function has relative minima at $x = c$ if $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$.

Second Derivative Test (Page 111)

Let f be differential function in a neighbourhood of c, where $f'(c) = 0$. Then

1- f has relative maxima at c if $f''(c) < 0$.

2- f has relative minima at c if $f''(c) > 0$.

Question # 1

Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

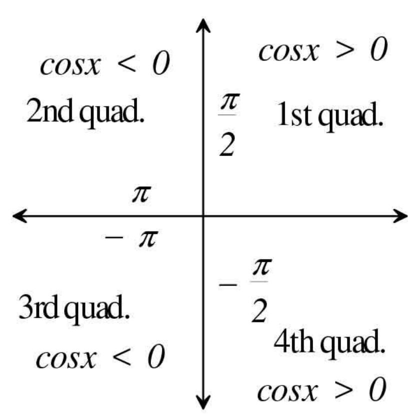
(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

(ii) $f(x) = \cos x$; $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(iii) $f(x) = 4 - x^2$; $x \in [-2, 2]$

(iv) $f(x) = x^2 + 3x + 2$; $x \in [-4, 1]$

Solution



(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

$\Rightarrow f'(x) = \cos x$

Put $f'(x) = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

So we have sub-intervals $\left(-\pi,-\dfrac{\pi}{2}\right)$, $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, $\left(\dfrac{\pi}{2},\pi\right)$

$f'(x)=\cos x<0$ whenever $x\in\left(-\pi,-\dfrac{\pi}{2}\right)$

So f is decreasing on the interval $\left(-\pi,-\dfrac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

So f is increasing on the interval $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(\dfrac{\pi}{2},\pi\right)$

So f is decreasing on the interval $\left(\dfrac{\pi}{2},\pi\right)$.

(ii) $f(x)=\cos x$; $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\Rightarrow f'(x)=-\sin x$

Put $f'(x)=0 \Rightarrow -\sin x=0 \Rightarrow \sin x=0 \Rightarrow x = 0$

So we have sub-intervals $\left(-\dfrac{\pi}{2},0\right)$ and $\left(0,\dfrac{\pi}{2}\right)$.

Now $f'(x)=-\sin x>0$ whenever $x\in\left(-\dfrac{\pi}{2},0\right)$

So f is increasing on $\left(-\dfrac{\pi}{2},0\right)$

$f'(x)=-\sin x<0$ whenever $x\in\left(0,\dfrac{\pi}{2}\right)$

So f is decreasing on $\left(0,\dfrac{\pi}{2}\right)$.

(iii) $f(x)=4-x^2$; $x\in[-2,2]$

$\Rightarrow f'(x) = -2x$

Put $f'(x)=0 \Rightarrow -2x=0 \Rightarrow x=0$

So we have subintervals $(-2,0)$ and $(0,2)$

$\because f'(x)=-2x>0$ whenever $x\in(-2,0)$

$\therefore f$ is increasing on the interval $(-2,0)$

Also $f'(x)=-2x<0$ whenever $x\in(0,2)$

$\therefore f$ is decreasing on $(0,2)$

(iv) $f(x) = x^2+3x+2 \quad ; \quad x\in[-4,1]$

$\Rightarrow f'(x) = 2x+3$

Put $f'(x)=0 \Rightarrow 2x+3=0 \Rightarrow x=-\dfrac{3}{2}$

So we have sub-intervals $\left(-4,-\dfrac{3}{2}\right)$ and $\left(-\dfrac{3}{2},1\right)$

Now $f'(x)=2x+3<0$ whenever $x\in\left(-4,-\dfrac{3}{2}\right)$

So f is decreasing on $\left(-4,-\dfrac{3}{2}\right)$

Also $f'(x)>0$ whenever $x\in\left(-\dfrac{3}{2},1\right)$

Therefore f is increasing on $\left(-\dfrac{3}{2},1\right)$.

Question # 2

Ind the extreme values of the following functions defined as:

(i) $f(x)=1-x^3$

(ii) $f(x) = x^2-x-2$

(iii) $f(x) = 5x^2-6x+2$

(iv) $f(x) = 3x^2$

(v) $f(x) = 3x^2-4x+5$

(vi) $f(x) = 2x^3-2x^2-36x+3$

(vii) $f(x) = x^4-4x^2$

(viii) $f(x) = (x-2)^2(x-1)$

(ix) $f(x) = 5+3x-x^3$

Solution

(i) $f(x)=1-x^3$

Diff. w.r.t x

$f'(x) = -3x^2$ ……. (i)

For stationary points, put $f'(x) = 0$

$\Rightarrow -3x^2 = 0 \quad \Rightarrow x=0$

Diff (i) w.r.t x

$f''(x)=-6x$ ………. (ii)

Now put $x=0$ in (ii)

$f''(0)=-6(0)=0$

So second derivative test fails to determinate the extreme points.

Put $x=0-\varepsilon=-\varepsilon$ in (i)

$f'(x)=-3(-\varepsilon)^2=-3\varepsilon^2<0$

Put $x=0+\varepsilon=\varepsilon$ in (i)

$f'(x)=-3(\varepsilon)^2 =-3\varepsilon^2 <0$

As $f'(x)$ doest not change its sign before and after $x=0$.

Since at $x=0$, $f(x)=1$ therefore $(0,1)$ is the point of inflexion.

(ii) $f(x) = x^2 - x - 2$

Diff. w.r.t. x

$f'(x) = 2x - 1$ ………. (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 2x-1=0 \quad \Rightarrow 2x=1 \quad \Rightarrow x=\frac{1}{2}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(2x-1) = 2$$

As $f''\left(\frac{1}{2}\right) = 2 > 0$

Thus $f(x)$ is minimum at $x=\frac{1}{2}$

Now $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$

(iii) $f(x) = 5x^2 - 6x + 2$

Diff. w.r.t. x

$f'(x) = 10x - 6$ ………. (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 10x-6=0 \quad \Rightarrow 10x=6 \quad \Rightarrow x=\frac{6}{10} \quad \Rightarrow x=\frac{3}{5}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(10x-6) = 10$$

As $f''\left(\frac{3}{5}\right) = 10 > 0$

Thus $f(x)$ is minimum at $x=\frac{3}{5}$

And $f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$

(iv) $f(x) = 3x^2$

Diff. w.r.t x

$f'(x) = 6x$ …….. (i)

For stationary points, put $f'(x)=0$

$$\Rightarrow 6x=0 \quad \Rightarrow x=0$$

Diff. (i) w.r.t x

$f''(x) = 6$

At $x=0$

$$f''(0) = 6 > 0$$

$\Rightarrow$ f has minimum value at $x=0$

And $f(0) = 3(0)^2 = 0$

(v) *Do yourself*

(vi) $f(x) = 2x^3-2x^2-36x+3$

Diff. w.r.t x

$$f'(x) = \frac{d}{dx}\left(2x^3-2x^2-36x+3\right) = 6x^2-4x-36 \text{ ....(i)}$$

For stationary points, put $f'(x) = 0$

$$\Rightarrow 6x^2-4x-36 = 0$$

$$\Rightarrow 3x^2-2x-12 = 0 \qquad \div\text{ing by } 2$$

$$\Rightarrow x = \frac{2\pm\sqrt{4-4(3)(-18)}}{2(3)}$$

$$= \frac{2\pm\sqrt{4+216}}{6} = \frac{2\pm\sqrt{220}}{6} = \frac{2\pm2\sqrt{55}}{6} = \frac{1\pm\sqrt{55}}{3}$$

Diff. (i) w.r.t x

$$f''(x) = \frac{d}{dx}\left(6x^2-4x-36\right) = 12x-4$$

Now $$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right)-4$$

$$= 4\left(1+\sqrt{55}\right)-4 = 4+4\sqrt{55}-4 = 4\sqrt{55} > 0$$

$\Rightarrow$ $f(x)$ has relative minima at $x = \dfrac{1+\sqrt{55}}{3}$.

And $$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3-2\left(\frac{1+\sqrt{55}}{3}\right)^2-36\left(\frac{1+\sqrt{55}}{3}\right)+3$$

$$= \frac{2}{27}\left(1+\sqrt{55}\right)^3-\frac{2}{9}\left(1+\sqrt{55}\right)^2-12\left(1+\sqrt{55}\right)+3$$

$$= \frac{2}{27}\left(1+3\sqrt{55}+3\cdot55+55\sqrt{55}\right)-\frac{2}{9}\left(1+2\sqrt{55}+55\right)-12\left(1+\sqrt{55}\right)+3$$

$$= \frac{2}{27}\left(166+58\sqrt{55}\right)-\frac{2}{9}\left(56+2\sqrt{55}\right)-12\left(1+\sqrt{55}\right)+3$$

$$= \frac{332}{27}+\frac{116}{27}\sqrt{55}-\frac{112}{9}-\frac{4}{9}\sqrt{55}-12-12\sqrt{55}+3$$

$$= -\frac{247}{27}-\frac{220}{27}\sqrt{55} \quad = -\frac{1}{27}\left(247+220\sqrt{55}\right)$$

Also $f''\left(\dfrac{1-\sqrt{55}}{3}\right) = 12\left(\dfrac{1-\sqrt{55}}{3}\right)-4$

$$= 4\left(1-\sqrt{55}\right)-4 = 4-4\sqrt{55}-4 = -4\sqrt{55} < 0$$

$\Rightarrow f(x)$ has relative maxima at $x = \dfrac{1+\sqrt{55}}{3}$.

And Since $f\left(\dfrac{1+\sqrt{55}}{3}\right) = -\dfrac{1}{27}\left(247+220\sqrt{55}\right)$

Therefore by replacing $\sqrt{55}$ by $-\sqrt{55}$, we have

$$f\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{1}{27}\left(247-220\sqrt{55}\right)$$

(vii) $f(x) = x^4-4x^2$

Diff. w.r.t. x

$$f'(x) = 4x^3-8x \text{ .... (i)}$$

For critical points put $f'(x)=0$

$$\Rightarrow 4x^3-8x=0 \quad \Rightarrow 4x\left(x^2-2\right)=0$$

$$\Rightarrow 4x=0 \quad \text{or} \quad x^2-2=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x^2=2 \quad \Rightarrow x=\pm\sqrt{2}$$

Now diff. (i) w.r.t x

$$f''(x) = 12x^2-8$$

For $x=-\sqrt{2}$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=-\sqrt{2}$

And $f\left(-\sqrt{2}\right) = \left(-\sqrt{2}\right)^4-4\left(-\sqrt{2}\right)^2 = 4-8 = -4$

For $x=0$

$$f''(0) = 12(0)-8 = -8 < 0$$

$\Rightarrow f$ has relative maxima at $x=0$

And $f(0) = (0)^4-4(0)^2 = 0$

For $x=\sqrt{2}$

$$f''(\sqrt{2}) = 12(\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=\sqrt{2}$

And $f\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^4-4\left(\sqrt{2}\right)^2 = 4-8 = -4$

(viii) $f(x) = (x-2)^2(x-1)$

$= (x^2-4x+4)(x-1) = x^3-4x^2+4x-x^2+4x-4$

$= x^3-5x^2+8x-4$

Diff. w.r.t. x

$f'(x) = 3x^2-10x+8$

For critical (stationary) points, put $f'(x) = 0$

$\Rightarrow 3x^2-10x+8=0 \quad \Rightarrow 3x^2-6x-4x+8=0$

$\Rightarrow 3x(x-2)-4(x-2)=0 \quad \Rightarrow (x-2)(3x-4)=0$

$\Rightarrow (x-2)=0$ or $(3x-4)=0$

$\Rightarrow x=2$ or $x = \dfrac{4}{3}$

Now diff. (i) w.r.t x

$f''(x) = 6x-10$

For $x = 2$

$f''(2) = 6(2)-10 = 2 > 0$

$\Rightarrow f$ has relative minima at $x=2$

And $f(2) = (2-2)^2(2-1) = 0$

For $x=\dfrac{4}{3}$

$$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)-10 = 8-10 = -2 < 0$$

$\Rightarrow f$ has relative maxima at $x=\dfrac{4}{3}$

And $f\left(\dfrac{4}{3}\right) = \left(\dfrac{4}{3}-2\right)^2\left(\dfrac{4}{3}-1\right) = \left(-\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right) = \left(\dfrac{4}{9}\right)\left(\dfrac{1}{3}\right) = \dfrac{4}{27}$

(ix) $f(x) = 5+3x-x^3$

Diff. w.r.t x

$f'(x) = 3-3x^2$ ……. (i)

For stationary points, put $f'(x)=0$

$\Rightarrow 3-3x^2=0 \quad \Rightarrow 3x^2=3 \quad \Rightarrow x^2=1 \quad \Rightarrow x=\pm 1$

Diff. (i) w.r.t x

$f''(x) = -6x$

For $x=1$

$f''(1) = -6(1) = -6 < 0$

$\Rightarrow f$ has relative maxima at $x=1$

And $f(1) = 5+3(1)-(1)^3 = 5+3-1 = 7$

For $x=-1$

$f''(-1) = -6(-1) = 6 > 0$

$\Rightarrow f$ has relative minima at $x=-1$, and

$f(-1) = 5+3(-1)-(-1)^3 = 5-3+1 = 3$

Question # 3

Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0,2\pi]$

$$f(x) = \sin x+\cos x$$

Solution $f(x) = \sin x+\cos x$ where $x\in[0,2\pi]$

Diff. w.r.t x

$$f'(x) = \cos x-\sin x \text{ .... (i)}$$

For stationary points, put $f'(x)=0$

$$\cos x-\sin x = 0$$

$$\Rightarrow -\sin x = -\cos x \Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$$

$$\Rightarrow x=\tan^{-1}(1) \Rightarrow x=\frac{\pi}{4}, \frac{5\pi}{4} \quad \text{when } x\in[0,2\pi]$$

Now diff. (i) w.r.t x

$$f''(x) = -\sin x-\cos x$$

For $x=\frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) < 0$$

$\Rightarrow f$ has relative maxima at $x=\frac{\pi}{4}$

And $f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) = \left(\sqrt{2}\right)^2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

For $x=\frac{5\pi}{4}$

$$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right)-\cos\left(\frac{5\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right)-\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) > 0$$

$\Rightarrow f$ has relative minima at $x=\frac{5\pi}{4}$

And $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right)+\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$

Question # 4

Show that $y = \frac{\ln x}{x}$ has maximum value at $x=e$

Solution $\quad y = \dfrac{\ln x}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot\frac{1}{x} - \ln x\cdot(1)}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\ln x}{x^2} \quad \ldots\ldots\ldots \text{(i)}$$

For critical points, put $\dfrac{dy}{dx}=0$

$$\Rightarrow \frac{1-\ln x}{x^2} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow \ln x = 1$$

$$\Rightarrow \ln x = \ln e \quad \Rightarrow x = e \qquad \because \ln e = 1$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-\ln x}{x^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2\cdot\left(-\frac{1}{x}\right) - (1-\ln x)\cdot(2x)}{(x^2)^2} = \frac{-x-2x+2x\ln x}{x^4} = \frac{-3x+2x\ln x}{x^4}$$

At $x=e$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3e+2e\cdot\ln e}{e^4}$$

$$= \frac{-3e+2e\cdot(1)}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

$\Rightarrow$ y has a maximum value at $x=e$.

Question # 5

Show that $y = x^x$ has maximum value at $x=\dfrac{1}{e}$.

Solution $\quad y = x^x$

Taking log on both sides

$$\ln y = \ln x^x \Rightarrow \ln y = x\ln x$$

Diff. w.r.t x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}x\ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\cdot\frac{d}{dx}\ln x + \ln x\cdot\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x} + \ln x\cdot(1)$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) \qquad \Rightarrow \frac{dy}{dx} = x^x(1+\ln x) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

For critical point, put $\dfrac{dy}{dx}=0$

$$\Rightarrow\ x^x(1+\ln x)=0 \quad \Rightarrow\ 1+\ln x=0 \ \text{ as } x^x\neq 0$$

$$\Rightarrow\ \ln x=-1 \quad \Rightarrow\ \ln x=-\ln e \qquad \because\ \ln e=1$$

$$\Rightarrow\ \ln x = \ln e^{-1} \quad \Rightarrow\ x = e^{-1} \quad \Rightarrow\ x = \frac{1}{e}$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}x^x(1+\ln x)$$

$$\Rightarrow\ \frac{d^2y}{dx^2} = x^x\frac{d}{dx}(1+\ln x)+(1+\ln x)\frac{d}{dx}x^x$$

$$= x^x\cdot\frac{1}{x}+(1+\ln x)\cdot x^x(1+\ln x) \qquad \text{from (i)}$$

$$= x^x\left(\frac{1}{x}+(1+\ln x)^2\right)$$

At $x=\dfrac{1}{e}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1/e} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(\frac{1}{1/e}+\left(1+\ln\frac{1}{e}\right)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+\left(1+\ln e^{-1}\right)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-\ln e)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-1)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\cdot e > 0$$

$\Rightarrow$ y has a minimum value at $x=\dfrac{1}{e}$

Exercise 2.10 (Solutions)

Calculus and Analytic Geometry, MATHEMATICS 12

Question # 1

Find two positive integers whose sum is 30 and their product will be maximum.

Solution

Let x and $30-x$ be two positive integers and P denotes product integers then

$$P = x(30-x)$$
$$= 30x - x^2$$

Diff. w.r.t. x

$$\frac{dP}{dx} = 30-2x \; \ldots\ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = -2 \; \ldots\ldots\ (ii)$$

For critical points, put $\frac{dP}{dx}=0$

$$\Rightarrow\ 30-2x=0$$
$$\Rightarrow\ -2x=-30 \qquad \Rightarrow\ x=15$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -2 < 0$$

$\Rightarrow$ P is maximum at $x=15$

Other $+$tive integer $= 30-x$
$= 30-15 = 15$

Hence 15 and 15 are the required positive numbers.

Question # 2

Divide 20 into two parts so that the sum of their squares will be minimum.

Solution

Let x be the part of 20 then other is $20-x$.

Let S denotes sum of squares then

$$S = x^2+(20-x)^2$$
$$= x^2+400-40x+x^2$$
$$= 2x^2-40x+400$$

Diff. w.r.t x

$$\frac{dS}{dx} = 4x-40 \; \ldots\ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 4 \; \ldots\ldots\ (ii)$$

For stationary points put $\frac{dS}{dx}=0$

$$\Rightarrow\ 4x-40=0 \qquad \Rightarrow\ 4x=40$$
$$\Rightarrow\ x=10$$

Putting value of x in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=10} = 4 > 0$$

$\Rightarrow$ S is minimum at $x=10$

Other integer $= 20-x = 20-10 = 10$

Hence 10, 10 are the two parts of 20.

Question # 3

Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution

Let x and $12-x$ be two $+$tive integers and P denotes product of one with square of the other then

$$P = x(12-x)^2$$
$$\Rightarrow\ P = x\left(144-24x+x^2\right)$$
$$= x^3-24x^2+144x$$

Diff. w.r.t x

$$\frac{dP}{dx} = 3x^2-48x+144 \; \ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 6x-48 \; \ldots\ (ii)$$

For critical points put $\frac{dP}{dx}=0$

$$3x^2-48x+144 = 0$$
$$\Rightarrow\ x^2-16x+48 = 0$$
$$\Rightarrow\ x^2-4x-12x+48 = 0$$
$$\Rightarrow\ x(x-4)-12(x-4) = 0$$
$$\Rightarrow\ (x-4)(x-12) = 0$$
$$\Rightarrow\ x=4 \quad \text{or} \quad x=12$$

We can not take $x=12$ as sum of integers is 12. So put $x=4$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=4} = 6(4)-48$$

$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=4$.

So the other integer $= 12-4 = 8$

Hence 4, 8 are the required integers.

Alternative Method: (by Irfan Mehmood: Fazaia Degree College Risalpur)

Let x and $12-x$ be two positive integers and P denotes product of one with square of the other then

$$P = x^2(12-x)$$

$$\Rightarrow P = 12x^2-x^3$$

Diff. w.r.t x

$$\frac{dP}{dx} = 24x-3x^2 \;\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 24-6x \;\ldots\ldots\; \text{(ii)}$$

For critical points put $\frac{dP}{dx}=0$

$$24x-3x^2 = 0$$

$$\Rightarrow 3x(x-8) = 0$$

$$\Rightarrow x=0 \text{ or } x=8$$

We cannot take $x=0$ as given integers are positive. So put $x=8$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=8} = 24-6(8)$$

$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=8$.

So the other integer $= 12-8 = 4$

Hence 4, 8 are the required integers.

Question # 4

The perimeter of a triangle is $16cm$. If one side is of length $6cm$, What are length of the other sides for maximum area of the triangle.

Solution

Let the remaining sides of the triangles are x and y

Perimeter $= 16$

$$\Rightarrow 6+x+y = 16$$

$$\Rightarrow x+y = 16-6 \qquad \Rightarrow x+y = 10$$

$$\Rightarrow y = 10-x \;\ldots\ldots\; \text{(i)}$$

Now suppose A denotes the square of the area of triangle then

$$A = s(s-a)(s-b)(s-c)$$

Where $s = \frac{a+b+c}{2} = \frac{6+x+y}{2}$

$$= \frac{6+x+10-x}{2} \qquad \text{from (i)}$$

$$= \frac{16}{2} = 8$$

So $A = 8(8-6)(8-x)(8-y)$

$$= 8(2)(8-x)(8-10+x)$$

$$= 16(8-x)(-2+x)$$

$$= 16\left(-16+2x+8x-x^2\right)$$

$$\Rightarrow A = 16\left(-16+10x-x^2\right)$$

Diff. w.r.t x

$$\frac{dA}{dx} = 16(10-2x) \;\ldots\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = 16(-2) = -32$$

For critical points put $\frac{dA}{dx}=0$

$$16(10-2x) = 0$$

$$\Rightarrow (10-2x) = 0 \quad \Rightarrow -2x = -10$$

$$\Rightarrow x = 5$$

Putting value of x in (ii)

$$\left.\frac{d^2A}{dx^2}\right|_{x=5} = -32 < 0$$

$\Rightarrow$ A is maximum at $x=5$

Putting value of x in (i)

$$y = 10-5 = 5$$

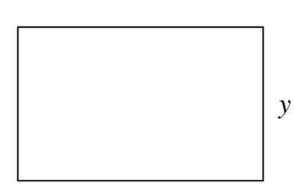
Hence length of remaining sides of triangles are $5cm$ and $5cm$.

Question # 5

Find the dimensions of a rectangle of largest area having perimeter $120cm$.

Solution

Let x and y be the length and breadth of rectangle, then



Area $= A = xy \;\ldots..\;$ (i)

Perimeter $= 120$

$\Rightarrow\ x+x+y+y = 120$

$\Rightarrow\ 2x+2y = 120$

$\Rightarrow\ x+y = 60$

$\Rightarrow\ y = 60-x$ ……. (ii)

Putting in (i)

$$A = x(60-x)$$

$$\Rightarrow\ A = 60x-x^2$$

Diff. w.r.t x

$$\frac{dA}{dx} = 60-2x \ \text{……….. (iii)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = -2 \ \text{………… (iv)}$$

For critical points put $\frac{dA}{dx}=0$

$$60-2x = 0 \quad \Rightarrow -2x = -60$$

$$\Rightarrow\ x = 30$$

Putting value of x in (iv)

$$\left.\frac{d^2A}{dx^2}\right|_{x=30} = -2 < 0$$

$\Rightarrow\ A$ is maximum at $x=30$

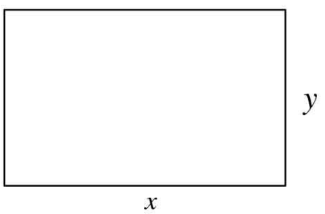
Putting value of x in (ii)

$$y = 60-30 = 30$$

Hence dimension of rectangle is $30cm$, $30cm$.

Question # 6

Find the lengths of the sides of a variable rectangle having area $36cm^2$ when its perimeter is minimum.



Solution

Let x and y be the length and breadth of the rectangle then

Area $= xy$

$\Rightarrow\ 36 = xy$

$\Rightarrow\ y = {}^{36}\!/_{x}$ … (i)

Now perimeter $= 2x+2y$

$$\Rightarrow\ P = 2x+2\left({}^{36}\!/_{x}\right)$$

$$= 2\left(x+36x^{-1}\right)$$

Diff. P w.r.t x

$$\frac{dP}{dx} = 2\left(1-36x^{-2}\right) \ \text{… (ii)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 2\left(0-36\left(-2x^{-3}\right)\right)$$

$$= 2\left(72x^{-3}\right) = \frac{144}{x^3}$$

For critical points put $\frac{dP}{dx}=0$

$$2\left(1-36x^{-2}\right)=0 \quad \Rightarrow 1-\frac{36}{x^2}=0$$

$$\Rightarrow 1=\frac{36}{x^2} \quad \Rightarrow x^2 = 36 \quad \Rightarrow x = \pm 6$$

Since length can not be negative therefore

$$x=6$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{(6)^3} > 0$$

Hence P is minimum at $x=6$.

Putting in eq. (i)

$$y = \frac{36}{6} = 6$$

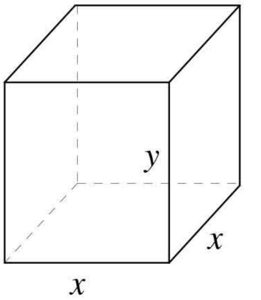
Hence $6cm$ and $6cm$ are the lengths of the sides of the rectangle.

Question # 7

A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.

Solution

Let x be the lengths of the sides of the base and y be the height of the box.



Then Volume

$= x\cdot x\cdot y$

$\Rightarrow\ 4 = x^2y$

$\Rightarrow\ y = \frac{4}{x^2}$ … (i)

Suppose S denotes the surface area of the box, then

$$S = x^2+4xy$$

$$\Rightarrow\ S = x^2+4x\left(\frac{4}{x^2}\right)$$

$$\Rightarrow S = x^2 + 16x^{-1}$$

Diff. S w.r.t x

$$\frac{dS}{dx} = 2x - 16x^{-2} \ldots \text{(ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 16\left(-2x^{-3}\right)$$

$$= 2 + \frac{32}{x^3} \ldots \text{(iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 16x^{-2} = 0 \quad \Rightarrow 2x - \frac{16}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 16}{x^2} = 0$$

$$\Rightarrow 2x^3 - 16 = 0 \quad \Rightarrow 2x^3 = 16$$

$$\Rightarrow x^3 = 8 \quad \Rightarrow x = 2$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=2} = 2 + \frac{32}{(2)^3} > 0$$

$\Rightarrow S$ is min. when $x = 2$

Putting value of x in (i)

$$y = \frac{4}{(2)^2} = 1$$

Hence $2dm$, $2dm$ and $1dm$ are the dimensions of the box.

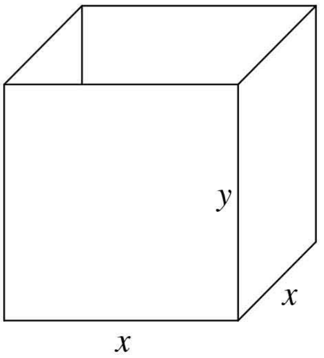
Question # 8

Find the dimensions of a rectangular garden having perimeter 80 meters if its area is to be maximum.

Solution

Do yourself as question # 5.

Question # 9



An open tank of square base of side x and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with lead will be least.

Solution

Let y be the height of the open tank.

Then Volume $= x \cdot x \cdot y$

$$\Rightarrow V = x^2 y$$

$$\Rightarrow y = \frac{V}{x^2} \ldots\ldots\ldots\ldots \text{(i)}$$

If S denotes the surface area the open tank, then

$$S = x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + 4Vx^{-1}$$

Diff. w.r.t x

$$\frac{dS}{dx} = 2x - 4Vx^{-2} \ldots\ldots\ldots \text{(ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 4V\left(-2x^{-3}\right)$$

$$= 2 + \frac{8V}{x^3} \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 4Vx^{-2} = 0 \quad \Rightarrow 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 4V}{x^2} = 0 \quad \Rightarrow 2x^3 - 4V = 0$$

$$\Rightarrow 2x^3 = 4V \quad \Rightarrow x^3 = 2V$$

$$\Rightarrow x = (2V)^{\frac{1}{3}}$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=(2V)^{\frac{1}{3}}} = 2 + \frac{8V}{\left((2V)^{\frac{1}{3}}\right)^3}$$

$$= 2 + \frac{8V}{2V} = 2 + 4 = 6 > 0$$

$\Rightarrow S$ is minimum when $x = (2V)^{\frac{1}{3}}$

i.e. $x^3 = 2V \Rightarrow V = \frac{x^3}{2}$

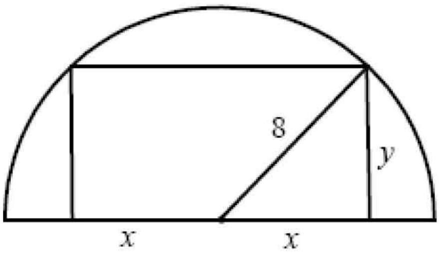
Putting in (i)

$$y = \frac{x^3/2}{x^2} = \frac{x}{2}$$

Hence height of the open tank is $\frac{x}{2}$.

Question # 10

Find the dimensions of the rectangular of maximum area which fits inside the semi-circle of radius $8cm$



Solution

Let $2x$ & y be dimension of rectangle.

Then from figure, using Pythagoras theorem

$$x^2 + y^2 = 8^2$$

$$\Rightarrow y^2 = 64 - x^2 \quad \text{.......... (i)}$$

Now Area of the rectangle is given by

$$A = 2x \cdot y$$

Squaring both sides

$$A^2 = 4x^2y^2$$
$$= 4x^2\left(64 - x^2\right)$$
$$= 256x^2 - 4x^4$$

Now suppose

$$f = A^2 = 256x^2 - 4x^4 \quad \text{.......... (ii)}$$

Diff. w.r.t x

$$\frac{df}{dx} = 512x - 16x^3 \quad \text{.......... (iii)}$$

Again diff. w.r.t x

$$\frac{d^2f}{dx^2} = 512 - 48x^2 \quad \text{......... (iv)}$$

For critical points, put $\frac{df}{dx} = 0$

$$\Rightarrow 512x - 16x^3 = 0$$
$$\Rightarrow 16x\left(32 - x^2\right) = 0$$
$$\Rightarrow 16x = 0 \quad \text{or} \quad 32 - x^2 = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 32$$
$$\Rightarrow x = \pm 4\sqrt{2}$$

Since x can not be zero or $-$ive, therefore

$$x = 4\sqrt{2}$$

Putting in (iv)

$$\left.\frac{d^2f}{dx^2}\right|_{x=4\sqrt{2}} = 512 - 48\left(4\sqrt{2}\right)^2$$
$$= 512 - 48(32) = 512 - 1536$$
$$= -1024 < 0$$

$\Rightarrow$ Area is max. for $x = 4\sqrt{2}$

Hence length $= 2x = 2\left(4\sqrt{2}\right)$

Breadth $= y = \sqrt{64 - \left(4\sqrt{2}\right)^2}$

$$= \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$$

Hence dimension is $8\sqrt{2}$ cm and $4\sqrt{2}$ cm.

Question # 11

Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3,-1)$

Solution

Let $P(x, y)$ be point and let $A(3,-1)$.

Then $d = |AP| = \sqrt{(x-3)^2 + (y+1)^2}$

$$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$$
$$= (x-3)^2 + \left(x^2 - 1 + 1\right)^2$$

$\because$ $y = x^2 - 1$ (given)

$$\Rightarrow d^2 = (x-3)^2 + x^4$$

Let $\quad f = d^2 = (x-3)^2 + x^4$.

Diff. w.r.t x

$$\frac{df}{dx} = 2(x-3) + 4x^3 \quad \text{........... (i)}$$

Again diff. w.r.t x

$$\frac{d^2f}{dx^2} = 2 + 12x^2 \quad \text{......... (ii)}$$

For stationary points, put $\frac{df}{dx} = 0$

$$2(x-3) + 4x^3 = 0$$

$\Rightarrow\ 2x-6+4x^3 = 0$

$\Rightarrow\ 4x^3+2x-6 = 0$

$\Rightarrow\ 2x^3+x-3 = 0 \qquad \div$ing by 2

By synthetic division

$$\begin{array}{c|cccc} 1 & 2 & 0 & 1 & -3 \\ & \downarrow & 2 & 2 & 3 \\ \hline & 2 & 2 & 3 & \underline{|0} \end{array}$$

$\Rightarrow\ x = 1 \quad$ or $\quad 2x^2+2x+3 = 0$

$$\Rightarrow\ x = \frac{-2\pm\sqrt{4-4(2)(3)}}{4}$$

$$= \frac{-2\pm\sqrt{-20}}{4}$$

This is complex and not acceptable.

Now put $x=1$ in (ii)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 2+12(1)^2 = 14 > 0$$

$\Rightarrow\ d$ is minimum at $x=1$.

Also $y = 1^2-1 = 0$.

Hence $(1,0)$ is the required point.

Question # 12

Find the point on the curve $y=x^2+1$ that is closest to the point $(18,1)$

Solution

Do yourself as Q # 11

Question # 1

Find δy and dy in the following cases:

(i) $y = x^2 - 1$ when x changes from 3 to 3.02

(ii) $y = x^2 + 2x$ when x changes from 2 to 1.8

(iii) $y = \sqrt{x}$ when x changes from 4 to 4.41

Solution

(i) $y = x^2 - 1$ …… (i)

$x = 3$ & $\delta x = 3.02 - 3 = 0.02$

$$y + \delta y = (x + \delta x)^2 - 1$$

$$\Rightarrow \delta y = (x + \delta x)^2 - 1 - x^2 + 1$$

$$= (x + \delta x)^2 - x^2$$

Put $x = 3$ & $\delta x = 0.02$

$$\delta y = (3 + 0.02)^2 - (3)^2$$

$$\Rightarrow \boxed{\delta y = 0.1204}$$

Taking differential of (i)

$$dy = d\left(x^2 - 1\right)$$

$$\Rightarrow dy = 2x\, dx$$

Put $x = 3$ & $dx = \delta x = 0.02$

$$dy = 2(3)(0.02)$$

$$\Rightarrow \boxed{dy = 0.12}$$

(ii) *Do yourself as above.*

(iii) $y = \sqrt{x} = x^{\frac{1}{2}}$ ……. (i)

$x = 4$ & $\delta x = 4.41 - 4 = 0.41$

$$y + \delta y = (x + \delta x)^{\frac{1}{2}}$$

$$\Rightarrow \delta y = (x + \delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

Put $x = 4$ & $\delta x = 0.41$

$$\delta y = (4 + 0.41)^{\frac{1}{2}} - (4)^{\frac{1}{2}}$$

$$= 2.1 - 2$$

$$\Rightarrow \boxed{\delta y = 0.1}$$

Taking differential of (i)

$$dy = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)dx$$

$$= \frac{1}{2}x^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{2x^{\frac{1}{2}}}\, dx$$

Put $x = 4$ & $dx = \delta x = 0.41$

$$dy = \frac{1}{2(4)^{\frac{1}{2}}}(0.41)$$

$$= \frac{0.41}{4}$$

$$\Rightarrow \boxed{dy = 0.1025}$$

Question # 2

Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

(i) $xy + x = 4$ (ii) $x^2 + 2y^2 = 16$

(iii) $x^4 + y^2 = xy^2$ (iv) $xy - \ln x = c$

Solution

(i) $xy + x = 4$

Taking differential on both sides

$$d(xy) + dx = d(4)$$

$$\Rightarrow xdy + ydx + dx = 0$$

$$\Rightarrow xdy + (y + 1)dx = 0$$

$$\Rightarrow xdy = -(y + 1)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y + 1}{x}$$

$$\& \quad \frac{dx}{dy} = -\frac{x}{y + 1}$$

(ii) *Do yourself as above*

(iii) $x^4 + y^2 = xy^2$

Taking differential

$$d\left(x^4\right) + d\left(y^2\right) = d\left(xy^2\right)$$

$$\Rightarrow 4x^3dx + 2ydy = x \cdot 2ydy + y^2dx$$

$$\Rightarrow 2ydy - 2xydy = y^2dx - 4x^3dx$$

$$\Rightarrow 2y(1 - x)dy = \left(y^2 - 4x^3\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 4x^3}{2y(1 - x)}$$

$$\& \quad \frac{dx}{dy} = \frac{2y(1 - x)}{y^2 - 4x^3}$$

(iv) $xy - \ln x = c$

Taking differential

$$d(xy) - d(\ln x) = d(c)$$

$$\Rightarrow xdy + ydx - \frac{1}{x}dx = 0$$

$$\Rightarrow xdy = \frac{1}{x}dx - ydx$$

$$= \left(\frac{1}{x} - y\right)dx$$

$$\Rightarrow xdy = \left(\frac{1-xy}{x}\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-xy}{x^2}$$

$$\frac{dx}{dy} = \frac{x^2}{1-xy}$$

Question # 3

Using differentials to approximate the values of

(i) $\sqrt[4]{17}$ (ii) $(31)^{\frac{1}{5}}$

(iii) $\cos 29^\circ$ (iv) $\sin 61^\circ$

Solution

(i) Let $y = f(x) = \sqrt[4]{x}$

where $x = 16$ and $\delta x = dx = 1$

Taking differential of above

$$dy = d\left(\sqrt[4]{x}\right)$$

$$= d(x)^{\frac{1}{4}}$$

$$= \frac{1}{4}x^{\frac{1}{4}-1}dx$$

$$= \frac{1}{4}x^{-\frac{3}{4}}dx$$

$$= \frac{1}{4x^{\frac{3}{4}}}dx$$

Put $x = 16$ and $dx = 1$

$$dy = \frac{1}{4(16)^{\frac{3}{4}}}(1)$$

$$= \frac{1}{4\left(2^4\right)^{\frac{3}{4}}}$$

$$= \frac{1}{4(8)} = 0.03125$$

Now $f(x+dx) \approx y + dy$

$$= f(x) + dy$$

$$\because\ y = f(x)$$

$$\Rightarrow \sqrt[4]{16+1} \approx \sqrt[4]{16} + 0.03125$$

$$\Rightarrow \sqrt[4]{17} \approx \left(2^4\right)^{\frac{1}{4}} + 0.03125$$

$$= 2 + 0.03125$$

$$= 2.03125$$

(ii) Let $y = f(x) = x^{\frac{1}{5}}$

Where $x = 32$ & $\delta x = dx = -1$

Try yourself as above.

(iii) Let $y = f(x) = \cos x$

Where $x = 30^\circ$ & $\delta x = -1^\circ = -\frac{\pi}{180}$ rad

$= -0.01745$ rad

Now $dy = d(\cos x)$

$$= -\sin x\, dx$$

Put $x = 30^\circ$ and $dx = \delta x = -0.01745$

$$dy = -\sin 30^\circ(-0.01745)$$

$$= -(0.5)(-0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y + dy$

$$= f(x) + dy$$

$$\Rightarrow \cos(30-1) = \cos 30^\circ + 0.008725$$

$$\Rightarrow \cos 29^\circ = 0.866 + 0.008725$$

$$= 0.8747$$

(iv) Let $y = f(x) = \sin x$

Where $x = 60^\circ$ & $\delta x = 1^\circ = \frac{\pi}{180}$ rad

$= 0.01745$ rad

Now $dy = d(\sin x)$

$$= \cos x\, dx$$

Put $x = 60^\circ$ and $dx = \delta x = 0.01745$

$$dy = \cos 60^\circ(0.01745)$$

$$= (0.5)(0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y + dy$

$$= f(x) + dy$$

$$\Rightarrow \sin(60+1) = \sin 60^\circ + 0.008725$$

$$\Rightarrow \sin 61^\circ = 0.866 + 0.008725$$

$$= 0.8747$$

Question # 4

Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02...

Solution

Let x be the length of side of cube where

$x = 5$ & $\delta x = 5.02 - 5 = 0.02$

Assume V denotes the volume of the cube.

Then $\quad V = x \cdot x \cdot x$

$\quad\quad = x^3$

Taking differential

$$dV = 3x^2 dx$$

Put $x = 5$ & $dx = \delta x = 0.02$

$$dV = 3(5)^2(0.02)$$
$$= 1.5$$

Hence increase in volume is 1.5 cubic unit.

Theorem on Anti-Derivatives

i) $\int cf(x)dx = c\int f(x)dx$ where c is constant.

ii) $\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$

Important Integral

Since $\dfrac{d}{dx}x^{n+1}=(n+1)x^n$

Taking integral w.r.t x

$$\int\frac{d}{dx}x^{n+1}dx=\int(n+1)x^n dx$$

$$\Rightarrow\ x^{n+1}=(n+1)\int x^n dx$$

$$\Rightarrow\ \boxed{\int x^n\,dx=\frac{x^{n+1}}{n+1}}\quad \text{where } n\neq -1$$

If $n=-1$ then

$$\int x^{-1}dx=\int\frac{1}{x}dx\quad(\text{here } x\neq 0)$$

Since $\dfrac{d}{dx}\ln x=\dfrac{1}{x}$

Therefore $\boxed{\int\frac{1}{x}dx=\ln|x|+c}$

Note: Since log of zero and negative numbers does not exist therefore in above formula mod assure that we are taking a log of +ive quantity.

Question # 1

Evaluate the following indefinite integrals.

(i) $\int(3x^2-2x+1)dx$

(ii) $\int\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)dx,(x>0)$

(iii) $\int x(\sqrt{x}+1)dx,(x>0)$

(iv) $\int(2x+3)^{\frac{1}{2}}dx$

(v) $\int(\sqrt{x}+1)^2dx,(x>0)$

(vi) $\int\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2dx,(x>0)$

(vii) $\int\dfrac{3x+2}{\sqrt{x}}dx,(x>0)$

(viii) $\int\dfrac{\sqrt{y}(y+1)}{y}dy,(y>0)$

(ix) $\int\dfrac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}d\theta,(\theta>0)$

(x) $\int\dfrac{(1-\sqrt{x})^2}{\sqrt{x}}dx,(x>0)$

(xi) $\int\dfrac{e^{2x}+e^x}{e^x}dx$

Solution

(i) $\int(3x^2-2x+1)dx=3\int x^2dx-2\int xdx+\int dx$

$$=3\cdot\frac{x^{2+1}}{2+1}-2\cdot\frac{x^{1+1}}{1+1}+x+c$$

$$=3\cdot\frac{x^3}{3}-2\cdot\frac{x^2}{2}+x+c$$

$$=x^3-x^2+x+c$$

(ii) $\int\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)dx=\int\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$

$$=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$$

(iii) $\int x(\sqrt{x}+1)dx=\int x\left(x^{\frac{1}{2}}+1\right)dx$

$$=\int\left(x^{\frac{3}{2}}+x\right)dx$$

$$=\int x^{\frac{3}{2}}dx+\int x\,dx$$

$$=\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{x^{1+1}}{1+1}+c$$

$$=\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^2}{2}+c$$

Important Integral

Since $\dfrac{d}{dx}(ax+b)^{n+1}=(n+1)(ax+b)^n\cdot a$

Taking integral

$$\int \frac{d}{dx}(ax+b)^{n+1}\,dx = \int (n+1)(ax+b)^n \cdot a\;dx$$
$$\Rightarrow\ (ax+b)^{n+1} = (n+1)\cdot a\int (ax+b)^n\;dx$$
$$\Rightarrow\ \boxed{\int (ax+b)^n\;dx = \frac{(ax+b)^{n+1}}{(n+1)\cdot a}}$$

(iv) $\int (2x+3)^{\frac{1}{2}}\,dx\ = \dfrac{(2x+3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 2}+c$

$$= \frac{(2x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\cdot 2}+c$$
$$= \frac{1}{3}(2x+3)^{\frac{3}{2}}+c$$

(v) $\int \left(\sqrt{x}+1\right)^2 dx\ = \int \left((\sqrt{x})^2+2\sqrt{x}+1\right)dx$

$$= \int \left(x+2(x)^{\frac{1}{2}}+1\right)dx$$
$$= \int x\,dx + 2\int (x)^{\frac{1}{2}}\,dx + \int dx$$
$$= \frac{x^{1+1}}{1+1}+2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+x+c$$
$$= \frac{x^2}{2}+2\frac{x^{\frac{3}{2}}}{3/2}+x+c$$
$$= \frac{x^2}{2}+\frac{4x^{\frac{3}{2}}}{3}+x+c$$

(vi) $\int \left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2 dx\ = \int \left(x+\dfrac{1}{x}-2\right)dx$

$$= \int x dx + \int \frac{1}{x}dx - 2\int dx$$
$$= \frac{x^2}{2}+\ln|x|-2x+c$$

(vii) $\int \dfrac{3x+2}{\sqrt{x}}dx\ = \int \dfrac{3x+2}{x^{1/2}}dx$

$$= \int \frac{3x}{x^{1/2}}+\frac{2}{x^{1/2}}dx$$
$$= \int \left(3x^{1/2}+2x^{-1/2}\right)dx$$
$$= 3\int x^{1/2}dx + 2\int x^{-1/2}dx$$

Now do yourself.

(viii) $\int \dfrac{\sqrt{y}\,(y+1)}{y}dy$

$$= \int \frac{\sqrt{y}\,(y+1)}{\left(\sqrt{y}\right)^2}dy\ = \int \frac{(y+1)}{\sqrt{y}}dy$$
$$= \int \left(\frac{y}{\sqrt{y}}+\frac{1}{\sqrt{y}}\right)dy\ = \int \left(y^{\frac{1}{2}}+y^{-\frac{1}{2}}\right)dy$$
$$= \int y^{\frac{1}{2}}dy + \int y^{-\frac{1}{2}}dy$$
$$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c\ = \frac{y^{\frac{3}{2}}}{\frac{3}{2}}+\frac{y^{\frac{1}{2}}}{\frac{1}{2}}+c$$
$$= \frac{2}{3}y^{\frac{3}{2}}+2y^{\frac{1}{2}}+c$$

(ix) $\int \dfrac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta = \int \dfrac{\theta-2\sqrt{\theta}+1}{\sqrt{\theta}}\,d\theta$

$$= \int \left(\frac{\theta}{\sqrt{\theta}}-\frac{2\sqrt{\theta}}{\sqrt{\theta}}+\frac{1}{\sqrt{\theta}}\right)d\theta$$
$$= \int \left(\theta^{\frac{1}{2}}-2+\theta^{-\frac{1}{2}}\right)d\theta$$
$$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1}-2\theta+\frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$
$$= \frac{\theta^{\frac{3}{2}}}{3/2}-2\theta+\frac{\theta^{\frac{1}{2}}}{1/2}+c$$
$$= \frac{2}{3}\theta^{\frac{3}{2}}-2\theta+2\theta^{\frac{1}{2}}+c\quad Ans$$

(x) $\int \dfrac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\,dx\ = \int \dfrac{1-2\sqrt{x}+x}{\sqrt{x}}\,dx$

$$= \int \left(\frac{1}{\sqrt{x}}-\frac{2\sqrt{x}}{\sqrt{x}}+\frac{x}{\sqrt{x}}\right)dx$$
$$= \int \left(x^{-\frac{1}{2}}-2+x^{\frac{1}{2}}\right)dx$$
$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-2x+\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$$
$$= \frac{x^{\frac{1}{2}}}{1/2}-3\theta+\frac{x^{\frac{3}{2}}}{3/2}+c$$

$$= 2x^{\frac{1}{2}} - 2x + \frac{2}{3}x^{\frac{3}{2}} + c \quad Ans$$

Important Integral

We know $\frac{d}{dx}e^{ax} = a \cdot e^{ax}$

Taking integral

$$\int \frac{d}{dx}e^{ax}\, dx = \int a \cdot e^{ax} dx$$

$$\Rightarrow \; e^{ax} = a\int e^{ax} dx$$

$$\Rightarrow \; \boxed{\int e^{ax} dx = \frac{e^{ax}}{a}}$$

Also note that $\int e^{(ax+b)} dx = \frac{e^{(ax+b)}}{a}$

(xi) $\int \frac{e^{2x} + e^{x}}{e^{x}}\, dx = \int \left(\frac{e^{2x}}{e^{x}} + \frac{e^{x}}{e^{x}} \right) dx$

$$= \int (e^{x} + 1)\, dx$$

$$= \int e^{x} dx + \int dx$$

$$= e^{x} + x + c \qquad \underline{Ans}$$

Question # 2

Evaluate

(i) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \begin{pmatrix} x+a>0 \\ x+b>0 \end{pmatrix}$

(ii) $\int \frac{1-x^2}{1+x^2} dx$

(iii) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x}}, (x>0, a>0)$

(iv) $\int (a-2x)^{\frac{3}{2}}\, dx$

(v) $\int \frac{(1+e^{x})^{3}}{e^{x}} dx$

(vi) $\int \sin(a+b)x\, dx$

(vii) $\int \sqrt{1-\cos 2x}\, dx, (1-\cos 2x>0)$

(viii) $\int \ln x \times \frac{1}{x}\, dx, (x>0)$

(ix) $\int \sin^2 x\; dx$

(x) $\int \frac{1}{1+\cos x} dx, \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \right)$

(xi) $\int \frac{ax+b}{ax^2+2bx+c} dx$

(xii) $\int \cos 3x \sin 2x\; dx$

(xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\, dx, (1+\cos 2x \neq 0)$

(xiv) $\int \tan^2 x\; dx$

Solution

(i) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$

$$= \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \cdot \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}}$$

$$= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{x+a-x-b}\, dx$$

$$= \int \frac{(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}}{a-b}\, dx$$

$$= \frac{1}{a-b}\left[\int (x+a)^{\frac{1}{2}}\, dx - \int (x+b)^{\frac{1}{2}} dx \right]$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+b)^{\frac{1}{2}}}{\frac{1}{2}+1} \right] + c$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right] + c$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + c \quad Ans.$$

Important Integral

Since $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$

Also $\frac{d}{dx}(-Cot^{-1}x) = \frac{1}{1+x^2}$

Therefore $\int \frac{1}{1+x^2} dx = Tan^{-1}x$ or $-Cot^{-1}x$

Similarly $\int \frac{1}{\sqrt{1-x^2}} dx = Sin^{-1}x$ or $-Cos^{-1}x$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = Sec^{-1}x \text{ or } -Csc^{-1}x$$

(ii) $\int \frac{1-x^2}{1+x^2} dx$

$$\begin{array}{r} -1 \\ 1+x^2 \overline{\big)\, 1-x^2} \\ \underset{+\quad +}{-1-x^2} \\ \hline 2 \end{array}$$

$$= \int \left(-1 + \frac{2}{1+x^2} \right) dx$$

$$= -\int dx + 2\int \frac{1}{1+x^2} dx$$

$$= -x + 2Tan^{-1}x + c$$

(iii) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x}}$

$$= \int \frac{dx}{\sqrt{x+a} + \sqrt{x}} \cdot \frac{\sqrt{x+a} - \sqrt{x}}{\sqrt{x+a} - \sqrt{x}}$$

$$=\int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\,dx$$

$$=\int \frac{(x+a)^{\frac{1}{2}}-(x)^{\frac{1}{2}}}{a}\,dx$$

$$=\frac{1}{a}\left[\int (x+a)^{\frac{1}{2}}\,dx-\int (x)^{\frac{1}{2}}dx\right]$$

$$=\frac{1}{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(x)^{\frac{1}{2}}}{\frac{1}{2}+1}\right]+c$$

$$=\frac{1}{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$

$$=\frac{2}{3a}\left[(x+a)^{\frac{3}{2}}-x^{\frac{3}{2}}\right]+c \quad \textit{Ans.}$$

(iv) $\int (a-2x)^{\frac{3}{2}}\,dx$

$$=\frac{(a-2x)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot(-2)}+c$$

$$=\frac{(a-2x)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot(-2)}+c$$

$$=-\frac{(a-2x)^{\frac{5}{2}}}{5}+c$$

(v) $\int \frac{(1+e^x)^3}{e^x}dx=\int \frac{(1+3e^x+3e^{2x}+e^{3x})}{e^x}dx$

$$=\int\left(\frac{1}{e^x}+\frac{3e^x}{e^x}+\frac{3e^{2x}}{e^x}+\frac{e^{3x}}{e^x}\right)dx$$

$$=\int\left(e^{-x}+3+3e^x+e^{2x}\right)dx$$

$$=\frac{e^{-x}}{-1}+3x+3e^x+\frac{e^{2x}}{2}+c$$

$$=-e^{-x}+3x+3e^x+\frac{1}{2}e^{2x}+c$$

Important Integrals

We know $\frac{d}{dx}\cos ax=-a\sin ax$

Taking integral

$$\int \frac{d}{dx}\cos ax\,dx=-\int a\sin ax\,dx$$

$$\Rightarrow\ \cos ax=-a\int \sin ax\ dx$$

$$\Rightarrow\ \boxed{\int \sin ax\,dx=-\frac{\cos ax}{a}}$$

Also $\frac{d}{dx}\sin ax=a\cdot\cos ax$

$$\therefore\ \boxed{\int \cos ax\,dx=\frac{\sin ax}{a}}$$

Similarly

$$\boxed{\int \sec^2 ax\ dx=\frac{\tan ax}{a}}$$

$$\boxed{\int \operatorname{cosec}^2 ax\ dx=-\frac{\cot ax}{a}}$$

$$\boxed{\int \sec ax\tan ax\ dx=\frac{\sec ax}{a}}$$

$$\boxed{\int \csc ax\cot ax\ dx=-\frac{\csc ax}{a}}$$

Also note that

$$\int \sin(ax+b)\,dx=-\frac{\cos(ax+b)}{a}$$

$$\int \cos(ax+b)\,dx=\frac{\sin(ax+b)}{a} \quad \text{and so on.}$$

(vi) $\int \sin(a+b)x\,dx=-\frac{\cos(a+b)x}{a+b}+c$

Do yourself

(vii) $\int \sqrt{1-\cos 2x}\,dx$

$$=\int \sqrt{2\sin^2 x}\,dx \quad \because\ \sin^2 x=\frac{1-\cos 2x}{2}$$

$$=\sqrt{2}\int \sin x\,dx \ =\sqrt{2}(-\cos x)+c$$

$$=-\sqrt{2}\cos x+c$$

Important Formula

$$\because\ \frac{d}{dx}[f(x)]^{n+1}=(n+1)[f(x)]^n\frac{d}{dx}f(x)$$

$$\Rightarrow\ \frac{d}{dx}[f(x)]^{n+1}=(n+1)[f(x)]^n f'(x)$$

Taking integral

$$\int \frac{d}{dx}[f(x)]^{n+1}dx=\int (n+1)[f(x)]^n f'(x)dx$$

$$\Rightarrow\ [f(x)]^{n+1}=(n+1)\int [f(x)]^n f'(x)dx$$

$$\Rightarrow\ \boxed{\int [f(x)]^n f'(x)dx=\frac{[f(x)]^{n+1}}{(n+1)}} \quad ;\ n\neq -1$$

Also $\frac{d}{dx}\ln|f(x)|=\frac{1}{f(x)}\cdot f'(x)$

Taking integral

$$\ln|f(x)|=\int \frac{f'(x)}{f(x)}dx$$

i.e. $\boxed{\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c}$

(viii) Let $I = \int \ln x \times \frac{1}{x}\, dx$

Put $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

So $I = \int [f(x)] f'(x)\, dx$

$= \frac{[f(x)]^{1+1}}{1+1} + c = \frac{[f(x)]^2}{2} + c$

$= \frac{(\ln x)^2}{2} + c$

(ix) $\int \sin^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

(x) $\int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx \qquad \because \cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$

$= \frac{1}{2}\int \sec^2 \frac{x}{2} dx = \frac{1}{2}\frac{\tan \frac{x}{2}}{\frac{1}{2}} + c = \tan \frac{x}{2} + c$

Alternative

$\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x} dx$

$= \int \frac{1-\cos x}{1-\cos^2 x} dx$

$= \int \frac{1-\cos x}{\sin^2 x} dx$

$= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$

$= \int \left(\operatorname{cosec}^2 x - \frac{\cos x}{\sin x \cdot \sin x}\right) dx$

$= \int \operatorname{cosec}^2 x\, dx - \int \operatorname{cosec} x \cot x\, dx$

$= -\cot x - (-\operatorname{cosec} x) + c$

$= \operatorname{cosec} x - \cot x + c$

(xi) Let $I = \int \frac{ax+b}{ax^2+2bx+c} dx$

Put $f(x) = ax^2 + 2bx + c$

$\Rightarrow f'(x) = 2ax + 2b$

$\Rightarrow f'(x) = 2(ax+b) \Rightarrow \frac{1}{2}f'(x) = ax + b$

So $I = \int \frac{\frac{1}{2}f'(x)}{f(x)} dx$

$= \frac{1}{2}\int \frac{f'(x)}{f(x)} dx = \frac{1}{2}\ln|f(x)| + c_1$

$= \frac{1}{2}\ln\left| ax^2 + 2bx + c \right| + c_1$

Review

- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
- $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

(xii) $\int \cos 3x \sin 2x\, dx$

$= \frac{1}{2}\int 2\cos 3x \sin 2x\, dx$

$= \frac{1}{2}\int [\sin(3x+2x) - \sin(3x-2x)]\, dx$

$= \frac{1}{2}\int [\sin 5x - \sin x]\, dx$

$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - (-\cos x)\right] + c$

$= -\frac{1}{2}\left[\frac{\cos 5x}{5} - \cos x\right] + c$

(xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x} dx$

$= -\int \frac{1-\cos 2x}{1+\cos 2x} dx \qquad \because \sin^2 x = \frac{1-\cos 2x}{2}$

$\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

$= -\int \frac{2\sin^2 x}{2\cos^2 x} dx$

$= -\int \tan^2 x\, dx = -\int (\sec^2 x - 1)\, dx$

$= -\int \sec^2 x\, dx + \int dx$

$= -\tan x + x + c$

(xiv) $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x\, dx - \int dx$

$= \tan x - x + c$

Important Integral

Since $\frac{d}{dx}\ln|ax+b| = \frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b)$

$\Rightarrow\ \frac{d}{dx}\ln|ax+b| = \frac{1}{ax+b}\cdot a$

On Integrating

$\Rightarrow\ \ln|ax+b| = a\int\frac{1}{ax+b}\,dx$

$\Rightarrow\ \boxed{\int\frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a}}$

Evaluate the following integrals:

Question # 1

$$\int \frac{-2x}{\sqrt{4-x^2}}dx$$

Solution

Let $I = \int \frac{-2x}{\sqrt{4-x^2}}dx$

Put $t = 4 - x^2 \quad \Rightarrow \quad dt = -2x\, dx$

So $I = \int \frac{dt}{\sqrt{t}} \quad = \int (t)^{-\frac{1}{2}}dt$

$$= \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad = \frac{(t)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{t} + c \quad = 2\sqrt{4-x^2} + c$$

Important Integrals

Since $\frac{d}{dx}Tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$

By Integrating, we have

$$Tan^{-1}\left(\frac{x}{a}\right) = \int \frac{a}{a^2+x^2}dx$$

$$= a\cdot\int \frac{1}{a^2+x^2}dx$$

$$\Rightarrow \boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}Tan^{-1}\left(\frac{x}{a}\right)}$$

Similarly

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}Sec^{-1}\frac{x}{a}}$$

Question # 2

$$\int \frac{dx}{x^2+4x+13}$$

Solution

Let $I = \int \frac{dx}{x^2+4x+13}$

$$= \int \frac{dx}{x^2+2(x)(2)+(2)^2-(2)^2+13}$$

$$= \int \frac{dx}{(x+2)^2-4+13}$$

$$= \int \frac{dx}{(x+2)^2+9} \quad = \int \frac{dx}{(x+2)^2+(3)^2}$$

Put $t = x+2 \quad \Rightarrow \quad dt = dx$

So $I = \int \frac{dt}{t^2+3^2}$

$$= \frac{1}{3}Tan^{-1}\frac{t}{3} + c$$

$$= \frac{1}{3}Tan^{-1}\frac{x+2}{3} + c$$

Question # 3

$$\int \frac{x^2}{4+x^2}dx$$

Solution

$$\int \frac{x^2}{4+x^2}dx$$

$$\begin{array}{r|l} & 1 \\ \hline 4+x^2 & x^2 \\ & \underline{-x^2 \mp 4} \\ & -4 \end{array}$$

$$= \int \left(1 - \frac{4}{4+x^2}\right)dx$$

$$= \int dx - 4\int \frac{dx}{4+x^2}$$

$$= x - 4\int \frac{dx}{(2)^2+x^2}$$

$$= x - 4\cdot\frac{1}{2}Tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x - 2\,Tan^{-1}\left(\frac{x}{2}\right) + c$$

Question # 4

$$\int \frac{1}{x\ln x}dx$$

Solution

Suppose $I = \int \frac{1}{x \ln x}dx$

$$= \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$$

Put $t = \ln x \quad \Rightarrow \quad dt = \frac{1}{x}dx$

So $I = \int \frac{1}{t}dt \quad = \ln|t| + c$

$$= \ln|\ln x| + c$$

Question # 5

$$\int \frac{e^x}{e^x+3}dx$$

Solution

Suppose $I=\int \frac{e^x}{e^x+3}dx$

Put $t=e^x+3 \quad \Rightarrow \quad dt=e^x dx$

So $I=\int \frac{dt}{t}=\ln|t|+c$

$=\ln\left|e^x+3\right|+c$

Question # 6

$$\int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$$

Solution

Let $I=\int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$

Put $t=x^2+2bx+c$

$\Rightarrow \quad dt=(2x+2b)\,dx \quad \Rightarrow \quad dt=2(x+b)\,dx$

$\Rightarrow \quad \frac{1}{2}dt=(x+b)\,dx$

So $I=\int \frac{\frac{1}{2}dt}{t^{\frac{1}{2}}} \quad =\frac{1}{2}\int t^{-\frac{1}{2}}\,dt$

$=\frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c_1 \quad =\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{1/2}+c_1$

$=\left(x^2+2bx+c\right)^{\frac{1}{2}}+c_1$

$=\sqrt{x^2+2bx+c}+c_1$

Question # 7

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Solution

Let $I=\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t=\tan x \quad \Rightarrow \quad dt=\sec^2 x\,dx$

So $I=\int \frac{dt}{\sqrt{t}} \quad =\int t^{-\frac{1}{2}}\,dt$

$=\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c \quad = \frac{t^{\frac{1}{2}}}{1/2}+c$

$=2(\tan x)^{\frac{1}{2}}+c \quad =2\sqrt{\tan x}+c$

Important Integral

$\int \sec\theta\,d\theta \;=\int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$

$=\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta$

$=\int \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}d\theta$

Take $t=\sec\theta+\tan\theta$

$\Rightarrow \quad dt=\left(\sec^2\theta+\sec\theta\tan\theta\right)d\theta$

So $\int \sec\theta\,d\theta=\int \frac{1}{t}dt$

$=\ln|t|+c$

$=\ln|\sec\theta+\tan\theta|+c$

$\Rightarrow \quad \boxed{\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+c}$

Similarly

$\boxed{\int \operatorname{cosec}\theta\,d\theta=\ln|\operatorname{cosec}\theta-\cot\theta|+c}$

See proof at page 133

Question # 8

(a) Show that

$$\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

(b) Show that

$$\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Solution

(a) Let $I=\frac{dx}{\sqrt{x^2-a^2}}$

Put $x=a\sec\theta \quad \Rightarrow \quad dx=a\sec\theta\tan\theta\,d\theta$

So $I=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{(a\sec\theta)^2-a^2}}$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\left(\sec^2\theta-1\right)}}$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\tan^2\theta}}$

$\because \; 1+\tan^2\theta=\sec^2\theta$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta} \quad =\int \sec\theta\,d\theta$

$=\ln|\sec\theta+\tan\theta|+c_1$

$=\ln\left|\sec\theta+\sqrt{\sec^2\theta-1}\right|+c_1$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 \qquad \begin{array}{|l} x=a\sec\theta \\ \frac{x}{a}=\sec\theta \end{array}$$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|+c_1$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|-\ln a+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+c$$

where $c=-\ln a+c_1$

(b) Let $I=\sqrt{a^2-x^2}\,dx$

Put $x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta$

So $I=\int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\;d\theta$

$$=\int\sqrt{a^2\left(1-\sin^2\theta\right)}\cdot a\cos\theta\;d\theta$$

$$=\int\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\;d\theta \qquad \because 1-\sin^2\theta=\cos^2\theta$$

$$=\int a\cos\theta\cdot a\cos\theta\;d\theta$$

$$=a^2\int\cos^2\theta\,d\theta \qquad \because \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=a^2\int\frac{1+\cos 2\theta}{2}d\theta \;=\frac{a^2}{2}\int\left(1+\cos 2\theta\right)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\sin\theta\sqrt{1-\sin^2\theta}\right)+c \qquad \begin{array}{|l} x=a\sin\theta \\ \frac{x}{a}=\sin\theta \\ \sin^{-1}\frac{x}{a}=\theta \end{array}$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2}\right)+c$$

$$=\frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Evaluate the following integrals:

Question # 9

$$\int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Solution

Let $I=\int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$

Put $x=\tan\theta \quad\Rightarrow\quad dx=\sec^2\theta\,d\theta$

$$I=\int\frac{\sec^2\theta\;d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$=\int\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{\frac{3}{2}}} \quad\because\; 1+\tan^2\theta=\sec^2\theta$$

$$=\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$=\int\frac{d\theta}{\sec\theta}\;=\int\cos\theta\,d\theta\;=\sin\theta+c$$

$$=\frac{\sin\theta}{\cos\theta}\cdot\cos\theta+c=\tan\theta\cdot\frac{1}{\sec\theta}+c$$

$$=\tan\theta\cdot\frac{1}{\sqrt{1+\tan^2\theta}}+c$$

$$=\frac{x}{\sqrt{1+x^2}}+c \qquad \because\; x=\tan\theta$$

Question # 10

$$\int\frac{1}{(1+x^2)\;Tan^{-1}x}dx$$

Solution

Let $I=\int\frac{1}{(1+x^2)\;Tan^{-1}x}dx$

$$=\int\frac{1}{Tan^{-1}x}\cdot\frac{1}{(1+x^2)}dx$$

Put $t=Tan^{-1}x \quad\Rightarrow\; dt=\frac{1}{1+x^2}dx$

So $I=\int\frac{1}{t}dt\;=\ln|t|+c$

$$=\ln\left|Tan^{-1}x\right|+c$$

Question # 11

$$\int\sqrt{\frac{1+x}{1-x}}\,dx$$

Solution

Let $I=\int\sqrt{\dfrac{1+x}{1-x}}\,dx$

Put $x=\sin\theta \quad\Rightarrow\quad dx=\cos\theta\,d\theta$

So $I=\int\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}\cdot\dfrac{1+\sin\theta}{1+\sin\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{(1+\sin\theta)^2}{\cos^2\theta}}\cdot\cos\theta\,d\theta$

$=\int\dfrac{1+\sin\theta}{\cos\theta}\cdot\cos\theta\,d\theta \quad =\int(1+\sin\theta)\,d\theta$

$=\theta-\cos\theta+c$

$=\theta-\sqrt{1-\sin^2\theta}+c \qquad \because\ x=\sin\theta$

$=Sin^{-1}x-\sqrt{1-x^2}+c \qquad \therefore\ \sin^{-1}x=\theta$

Question # 12

$\int\dfrac{\sin\theta}{1+\cos^2\theta}d\theta$

Solution

Let $I=\int\dfrac{\sin\theta}{1+\cos^2\theta}d\theta$

Put $t=\cos\theta$

$\Rightarrow\ dt=-\sin\theta\,d\theta \quad\Rightarrow\quad -dt=\sin\theta\,d\theta$

So $I=\int\dfrac{-dt}{1+t^2} \quad =-\int\dfrac{dt}{1+t^2}$

$=-\tan^{-1}t+c$

$=-\tan^{-1}(\cos\theta)+c$

Question # 13

$\int\dfrac{ax}{\sqrt{a^2-x^4}}\,dx$

Solution

Let $I=\int\dfrac{ax}{\sqrt{a^2-x^4}}\,dx$

$=a\int\dfrac{x}{\sqrt{a^2-x^4}}\,dx$

Put $t=x^2$ then $t^2=x^4$

$dt=2x\,dx \quad\Rightarrow\quad \dfrac{1}{2}dt=x\cdot dx$

So $I=a\int\dfrac{\frac{1}{2}dt}{\sqrt{a^2-t^2}}$

$=\dfrac{a}{2}\int\dfrac{dt}{\sqrt{a^2-t^2}}$

$=\dfrac{a}{2}Sin^{-1}\left(\dfrac{t}{a}\right)+c \qquad \because\ \int\dfrac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\dfrac{x}{a}$

$=\dfrac{a}{2}Sin^{-1}\left(\dfrac{x^2}{a}\right)+c$

Question # 14

$\int\dfrac{dx}{\sqrt{7-6x-x^2}}$

Solution

Let $I=\int\dfrac{dx}{\sqrt{7-6x-x^2}}$

$=\int\dfrac{dx}{\sqrt{-\left(x^2+6x-7\right)}}$

$=\int\dfrac{dx}{\sqrt{-\left(x^2+2(3)(x)+(3)^2-(3)^2-7\right)}}$

$=\int\dfrac{dx}{\sqrt{-\left((x+3)^2-16\right)}}$

$=\int\dfrac{dx}{\sqrt{16-(x+3)^2}}$

Put $t=x+3 \quad\Rightarrow\quad dx=dt$

So $I=\dfrac{dt}{\sqrt{16-t^2}} \quad =\int\dfrac{dx}{\sqrt{(4)^2-(t)^2}}$

$=Sin^{-1}\left(\dfrac{t}{4}\right)+c$

$=Sin^{-1}\left(\dfrac{x+3}{4}\right)+c$

Question # 15

$\int\dfrac{\cos x}{\sin x\cdot\ln\sin x}dx$

Solution

Let $I=\int\dfrac{\cos x}{\sin x\cdot\ln\sin x}dx$

$=\int\dfrac{1}{\ln\sin x}\cdot\dfrac{\cos x}{\sin x}dx$

Put $t=\ln\sin x \quad\Rightarrow\quad dt=\dfrac{1}{\sin x}\cdot\cos x\,dx$

So $I=\int\dfrac{1}{t}dt \quad =\ln|t|+c$

$=\ln|\ln\sin x|+c$

Question # 16

$$\int \cos x\left(\frac{\ln\sin x}{\sin x}\right)dx$$

Solution

Let $I = \int \cos x\left(\frac{\ln\sin x}{\sin x}\right)dx$

$= \int \ln\sin x\cdot\frac{\cos x}{\sin x}\,dx$

Put $t = \ln\sin x \quad \Rightarrow \quad dt = \frac{1}{\sin x}\cdot\cos x\,dx$

No do yourself

Question # 17

$$\int\frac{x\,dx}{4+2x+x^2}$$

Solution

Let $I = \int\frac{x\,dx}{4+2x+x^2}$

$= \frac{1}{2}\int\frac{2x\,dx}{x^2+2x+4}$

+ing and −ing 2 in numerator.

$\Rightarrow I = \frac{1}{2}\int\frac{(2x+2)-2}{x^2+2x+4}\,dx$

$= \frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+4}-\frac{2}{x^2+2x+4}\right)dx$

$= \frac{1}{2}\int\frac{2x+2}{x^2+2x+4}\,dx - \frac{1}{2}\int\frac{2}{x^2+2x+4}\,dx$

$= \frac{1}{2}\int\frac{\frac{d}{dx}\left(x^2+2x+4\right)}{x^2+2x+4}\,dx - \frac{2}{2}\int\frac{dx}{x^2+2x+1+3}$

$= \frac{1}{2}\ln\left|x^2+2x+4\right| - \int\frac{dx}{(x+1)^2+\left(\sqrt{3}\right)^2}$

$= \frac{1}{2}\ln\left|x^2+2x+4\right| - \frac{1}{\sqrt{3}}Tan^{-1}\frac{x+1}{\sqrt{3}}+c$

Question # 18

$$\int\frac{x}{x^4+2x^2+5}\,dx$$

Solution

Let $I = \int\frac{x}{x^4+2x^2+5}\,dx$

Put $t = x^2$ then $t^2 = x^4$

$dt = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}dt = x\,dx$

So $I = \int\frac{\frac{1}{2}dt}{t^2+2t+5} \quad = \frac{1}{2}\int\frac{dt}{t^2+2t+1+4}$

$= \frac{1}{2}\int\frac{dt}{(t+1)^2+(2)^2}$

$= \frac{1}{2}\cdot\frac{1}{2}Tan^{-1}\left(\frac{t+1}{2}\right)+c$

$= \frac{1}{4}Tan^{-1}\left(\frac{x^2+1}{2}\right)+c$

Question # 19

$$\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)\,dx$$

Solution

Let $I = \int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)\,dx$

Put $t = \sqrt{x}-\frac{x}{2}$

$\Rightarrow dt = \left(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}\right)dx \Rightarrow dt = \frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx$

$\Rightarrow 2\,dt = \left(\frac{1}{\sqrt{x}}-1\right)dx$

So $I = \int\cos t\cdot 2\,dt$

$= 2\int\cos t\,dt$

$= 2\sin t + c$

Question # 20

$$\int\frac{x+2}{\sqrt{x+3}}\,dx$$

Solution

Let $I = \int\frac{x+2}{\sqrt{x+3}}\,dx$

Put $t = x+3$ then $t-3 = x$

$\Rightarrow dt = dx$

So $I = \int\frac{t-3+2}{\sqrt{t}}dx$

$= \int\frac{t-1}{(t)^{\frac{1}{2}}}\,dx \quad = \int\left(\frac{t}{(t)^{\frac{1}{2}}}-\frac{1}{(t)^{\frac{1}{2}}}\right)dx$

$= \int\left((t)^{\frac{1}{2}}-(t)^{-\frac{1}{2}}\right)dx$

$= \frac{(t)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{(t)^{\frac{3}{2}}}{3/2}-\frac{(t)^{\frac{1}{2}}}{1/2}+c$

$= \frac{2(x+3)^{\frac{3}{2}}}{3}-2(x+3)^{\frac{1}{2}}+c$

$$= \frac{2(x+3)^{\frac{3}{2}}}{3} - 2\sqrt{x+3} + c$$

Question # 21

$$\int \frac{\sqrt{2}}{\sin x + \cos x}dx$$

Solution

Let $I = \int \frac{\sqrt{2}}{\sin x + \cos x}dx$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}dx$$

Put $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

So $I = \int \frac{1}{\sin\frac{\pi}{4}\cdot\sin x + \cos\frac{\pi}{4}\cdot\cos x}dx$

$$= \int \frac{1}{\cos\left(x - \frac{\pi}{4}\right)}dx \quad = \int \sec\left(x - \frac{\pi}{4}\right)dx$$

$$= \ln\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right| + c$$

Question # 22

$$\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

Solution

Let $I = \int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

$\because \cos\frac{\pi}{3} = \frac{1}{2}$ & $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore I = \int \frac{dx}{\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x}$

$$= \int \frac{dx}{\sin\left(x + \frac{\pi}{3}\right)} = \int \operatorname{cosec}\left(x + \frac{\pi}{3}\right)dx$$

$$= \ln\left|\operatorname{cosec}\left(x + \frac{\pi}{3}\right) - \cot\left(x + \frac{\pi}{3}\right)\right| + c$$

Integration by Parts

If u and v are function of x, then

$$\int uv\,dx = u\int v dx - \int\left(\left(\int v dx\right)\cdot u'\,dx\right)$$

Question # 1

Evaluate the following integrals by parts add a word representing all the functions are defined.

$u = x$
$v = \sin x$

(i) $\int x\sin x\,dx$ (ii) $\int \ln x\,dx$

(iii) $\int x\ln x\;dx$ (iv) $\int x^2\ln x\,dx$

(v) $\int x^3\ln x\,dx$ (vi) $\int x^4\ln x\,dx$

(vii) $\int \tan^{-1}x\;dx$ (viii) $\int x^2\sin x\;dx$

(ix) $\int x^2\tan^{-1}x\,dx$ (x) $\int x\tan^{-1}x\,dx$

(xi) $\int x^3\tan^{-1}x\;dx$ (xii) $\int x^3\cos x\,dx$

(xiii) $\int \sin^{-1}x\,dx$ (xiv) $\int x\sin^{-1}x\;dx$

(xv) $\int e^x\sin x\cos x\,dx$ (xvi) $\int x\sin x\cos x\,dx$

(xvii) $\int x\cos^2 x\,dx$ (xviii) $\int x\sin^2 x\;dx$

(xix) $\int (\ln x)^2\,dx$ (xx) $\int \ln(\tan x)\sec^2 x dx$

(xxi) $\int \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution

(i) Let $I = \int x\sin x\,dx$

Integration by parts

$$I = x\cdot(-\cos x) - \int(-\cos x)\cdot(1)\,dx$$
$$= -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + c$$

(ii) Let $I = \int \ln x\,dx$
$= \int \ln x\cdot 1\;dx$

$u = \ln x$
$v = 1$

Integrating by parts

$$I = \ln x\cdot x - \int x\cdot\frac{1}{x}\;dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

(iii) Let $I = \int x\ln x\;dx$

$u = \ln x$
$v = x$

Integrating by parts

$$I = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\;dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

(iv) *Do yourself*

(v) *Do yourself*

(vi) *Do yourself*

(vii) Let $I = \int\tan^{-1}x\;dx$
$= \int\tan^{-1}x\cdot 1\;dx$

$u = \tan^{-1}x$
$v = 1$

Integrating by parts

$$I = \tan^{-1}x\cdot x - \int x\cdot\frac{1}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

(viii) Let $I = \int x^2\sin x\;dx$

$u = x^2$
$v = \sin x$

Integrating by parts

$$I = x^2(-\cos x) - \int(-\cos x)\cdot 2x\;dx$$
$$= -x^2\cos x + 2\int x\cos x\,dx$$

$u = x$
$v = \cos x$

Again integrating by parts

$$I = -x^2\cos x + 2\left(x\sin x - \int\sin x\;(1)\;dx\right)$$
$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

(ix) Let $I = \int x^2\tan^{-1}x\,dx$

$u = \tan^{-1}x$
$v = x^2$

Integrating by parts

$$I = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+-\frac{1}{3}\int\frac{x}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\cdot\frac{x^2}{2}+-\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\ln\left|1+x^2\right|+c$$

(x) Let $I=\int x\tan^{-1}x\,dx$

Integrating by parts

$u=\tan^{-1}x$
$v=x$

$$I=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c\quad \textit{Ans.}$$

(xi) Let $I=\int x^3\tan^{-1}x\,dx$

Integrating by parts

$u=\tan^{-1}x$
$v=x^3$

$$I=\tan^{-1}x\cdot\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x--\frac{1}{4}\int\frac{x^4}{1+x^2}dx$$

$$\begin{array}{r|l} & x^2-1\\ \hline 1+x^2 & x^4\\ & \underline{\pm x^4\pm x^2}\\ & -x^2\\ & \underline{\mp 1\mp x^2}\\ & 1\end{array}$$

$$=\frac{x^4}{4}\tan^{-1}x$$
$$-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int x^2dx+\frac{1}{4}\int dx-\frac{1}{4}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\frac{x^3}{3}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{x^3}{12}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

(xii) *Do yourself as Question # 1(viii).*

(xiii) $I=\int\sin^{-1}x\,dx$
$=\int\sin^{-1}x\cdot 1\,dx$

$u=\sin^{-1}x$
$v=1$

Integrating by parts

$$I=\sin^{-1}x\cdot x-\int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\sin^{-1}x-\int(1-x^2)^{-\frac{1}{2}}(x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-x^2)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\cancel{1}/2}+c$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c$$

(xiv) Let $I=\int x\sin^{-1}x\,dx$

Integrating by parts

$$I=\sin^{-1}x\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}I_1-\frac{1}{2}\sin^{-1}x \ \ldots\ldots \text{(i)}$$

Where $I_1=\int\sqrt{1-x^2}\,dx$

Put $x=\sin\theta \quad \Rightarrow\ dx=\cos\theta\,d\theta$

$$\Rightarrow\ I_1=\int\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$
$$=\int\sqrt{\cos^2\theta}\ \cos\theta\ d\theta$$
$$=\int\cos^2\theta\ d\theta\ =\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$=\frac{1}{2}\int(1+\cos 2\theta)\ d\theta$$
$$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$$
$$=\frac{1}{2}\left[\theta+\frac{2\sin\theta\cos\theta}{2}\right]+c$$
$$=\frac{1}{2}\left[\ \theta+\sin\theta\sqrt{1-\sin^2\theta}\ \right]+c$$
$$=\frac{1}{2}\left[\ \sin^{-1}x+x\sqrt{1-x^2}\ \right]+c$$

Using value of I_1 in (i)

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}+c\right)\right]-\frac{1}{2}\sin^{-1}x$$
$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c-\frac{1}{2}\sin^{-1}x$$
$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c$$

(xv) Let $I=\int e^x\sin x\cos x\,dx$

$$=\frac{1}{2}\int e^x\cdot 2\sin x\cos x\ dx$$
$$=\frac{1}{2}\int e^x\sin 2x\ dx \quad \because\ \sin 2x=2\sin x\cos x$$

Integrating by parts

$$I=\frac{1}{2}\left[e^x\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^x dx\right]$$
$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\int e^x\cos 2x\ dx$$

Again integrating by parts

$$I=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\ e^x\right)$$
$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\frac{1}{2}\int\ e^x\sin 2x\right)$$
$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-I\right)+c$$
$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x-\frac{1}{4}I+c$$
$$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$$
$$\Rightarrow\ \frac{5}{4}I=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$$
$$\Rightarrow\ I=-\frac{1}{5}\ e^x\cos 2x+\frac{1}{10}e^x\sin 2x+\frac{4}{5}c$$

(xvi) Let $I=\int x\sin x\cos x\,dx$

$$=\frac{1}{2}\int x\cdot 2\sin x\cos x\ dx$$
$$=\frac{1}{2}\int x\cdot\sin 2x\ dx$$

| $u=x$
| $v=\sin 2x$

Integrating by parts

$$I=\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-\int\left(\frac{-\cos 2x}{2}\right)(1)dx\right]$$

(xvii)Let $I=\int x\cos^2 x\,dx$

$$=\int x\left(\frac{1+\cos 2x}{2}\right)dx$$
$$=\frac{1}{2}\int x(1+\cos 2x)\,dx$$
$$=\frac{1}{2}\int x\,dx+\frac{1}{2}\int x\cos 2x dx$$

| $u=x$
| $v=\cos 2x$

$$=\frac{1}{2}\cdot\frac{x^2}{2}+\frac{1}{2}\left[x\ \frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot(1)dx\right]$$
$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\int\sin 2x\ dx$$
$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$
$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x+\frac{1}{8}\cos 2x+c$$

(xviii) Let $I=\int x\sin^2 x\ dx$

$$=\int x\left(\frac{1-\cos 2x}{2}\right)dx$$
$$=\frac{1}{2}\int x(1-\cos 2x)\,dx$$
$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx \qquad \begin{array}{|l} u=x \\ v=\cos 2x \end{array}$$

Integrating by parts

$$I=\frac{1}{2}\frac{x^2}{2}-\frac{1}{2}\left[x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}.(1)\,dx\right]$$
$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\int\sin 2x\,dx$$
$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$
$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$$

(xix) Let $I=\int(\ln x)^2dx$

$$=\int(\ln x)^2\cdot 1\,dx \qquad \begin{array}{|l} u=(\ln x)^2 \\ v=1 \end{array}$$

Integrating by parts

$$I=(\ln x)^2\cdot x-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\,dx$$
$$=x(\ln x)^2-2\int(\ln x)\,dx$$

Again integrating by parts

$$I=x(\ln x)^2-2\left[\ln x\cdot x-\int x\cdot\frac{1}{x}\,dx\right]$$
$$=x(\ln x)^2-2x\ln x+2\int dx$$
$$=x(\ln x)^2-2x\ln x+2x+c$$

(xx) Let $I=\int\ln(\tan x)\sec^2 x\,dx$

Integrating by parts

$$I=\ln(\tan x)\cdot\tan x-\int\tan x\cdot\frac{1}{\tan x}\cdot\sec^2 x\,dx$$
$$=\tan x\ln(\tan x)-\int\sec^2 x\,dx$$
$$=\tan x\ln(\tan x)-\tan x+c$$

(xxi) Let $I=\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$$=\int\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}(x)\,dx \qquad \begin{array}{|l} u=\sin^{-1}x \\ v=\left(1-x^2\right)^{-\frac{1}{2}}(-2x) \end{array}$$
$$=-\frac{1}{2}\int\sin^{-1}x\cdot\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\,dx$$

Integrating by parts

$$I=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-\int\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\cdot\frac{1}{\sqrt{1-x^2}}dx\right]$$
$$=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}-\int\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$
$$=-\frac{1}{2}\left[2(1-x^2)^{\frac{1}{2}}\sin^{-1}x-2\int dx\right]$$
$$=-\sqrt{1-x^2}\sin^{-1}x+\int dx$$
$$=-\sqrt{1-x^2}\sin^{-1}x+x+c$$
$$=x-\sqrt{1-x^2}\sin^{-1}x+c$$

Question # 2

Evaluate the following integrals.

(i) $\int\tan^4 x dx$ (ii) $\int\sec^4 x\,dx$

(iii) $\int e^x\sin 2x\cos x\,dx$ (iv) $\int\tan^3 x\cdot\sec x\,dx$

(iv) $\int\tan^3 x\cdot\sec x\,dx$ (v) $\int x^3e^{5x}dx$

(vi) $\int e^{-x}\sin 2x\,dx$ (vii) $\int e^{2x}\cdot\cos 3x\,dx$

(viii) $\int\text{cosec}^3 x\,dx$

Solution

(i) Let $I=\int\tan^4 x dx$

$$=\int\tan^2 x\cdot\tan^2 x\,dx$$
$$=\int\tan^2 x\left(\sec^2 x-1\right)dx$$
$$=\int\left(\tan^2 x\sec^2 x-\tan^2 x\right)dx$$
$$=\int\tan^2 x\sec^2 x\,dx-\int\tan^2 x\,dx$$
$$=\int\tan^2 x\frac{d}{dx}(\tan x)\,dx-\int(\sec^2 x-1)\,dx$$
$$=\frac{\tan^{2+1}x}{2+1}-\int\sec^2 x\,dx+\int dx$$
$$=\frac{1}{3}\tan^3 x-\tan x+x+c$$

(ii) Let $I=\int \sec^4 x\, dx$

$=\int (\sec^2 x)\cdot(\sec^2 x)\, dx$

$=\int (1+\tan^2 x)\cdot(\sec^2 x)\, dx$

$=\int \sec^2 x\, dx+\int \tan^2 x\sec^2 x\, dx$

$=\tan x+\int (\tan x)^2 \dfrac{d}{dx}(\tan x)\, dx$

$=\tan x+\dfrac{\tan^3 x}{3}+c$

(iii) Let $I=\int e^x \sin 2x \cos x\, dx$

$=\dfrac{1}{2}\int e^x\left(2\sin 2x\cos x\right)dx$

$=\dfrac{1}{2}\int e^x\left(\sin(2x+x)+\sin(2x-x)\right)dx$

$=\dfrac{1}{2}\int e^x\left(\sin 3x+\sin x\right)dx$

$=\dfrac{1}{2}\int e^x \sin 3x\, dx+\dfrac{1}{2}\int e^x\sin x\, dx$

$=\dfrac{1}{2}I_1+\dfrac{1}{2}I_2$ .... (i)

Where $I_1=\int e^x\sin 3x\, dx$ and $I_2=\int e^x \sin x\, dx$

Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

(iv) $I=\int \tan^3 x\cdot\sec x\, dx$

$=\int \tan^2 x\cdot\tan x\cdot\sec x\, dx$

$=\int\left(\sec^2 x-1\right)\cdot\sec x\tan x\, dx$

Put $t=\sec x \quad\Rightarrow\quad dt=\sec x\tan x\, dx$

So $I=\int\left(t^2-1\right)dt$

$=\dfrac{t^3}{3}-t+c$

$=\dfrac{\sec^3 x}{3}-\sec+c$

(v) Let $I=\int x^3e^{5x}dx$ | $u=x^3$, $v=e^x$

Integrating by parts

$I=x^3\cdot\dfrac{e^{5x}}{5}-\int\dfrac{e^{5x}}{5}\cdot 3x^2dx$

$=\dfrac{1}{5}x^3e^{5x}-\dfrac{3}{5}\int x^2e^{5x}dx$ | $u=x^2$, $v=e^x$

Again integrating by parts

$$I=\frac{1}{5}x^3e^{5x}-\frac{3}{5}\left[x^2\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot 2x\,dx\right]$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{25}\int xe^{5x}\,dx$$

Again integrating by parts

$$I=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{25}\left[x\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot(1)dx\right]$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\int e^{5x}\,dx$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\cdot\frac{e^{5x}}{5}+c$$

$$=\frac{e^{5x}}{5}\left(x^3-\frac{3}{5}x^2+\frac{6}{25}x-\frac{6}{125}\right)+c$$

(vi) Let $I=\int e^{-x}\sin 2x\, dx$ | $u=e^{-x}$, $v=\sin 2x$

Integrating by parts

$$I=e^{-x}\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^{-x}(-1)\,dx$$

$$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\int e^{-x}\cos 2x\,dx$$

Again integrating by parts

$$I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\left[e^{-x}\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot e^{-x}(-1)\,dx\right]$$

$$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}\int e^{-x}\sin 2x\,dx$$

$$\Rightarrow\ I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}I+c$$

$$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$$

$$\Rightarrow\ \frac{5}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$$

$$\Rightarrow\ I=-\frac{2}{5}e^{-x}\cos 2x-\frac{1}{5}e^{-x}\sin 2x+\frac{4}{5}c$$

$$=-\frac{1}{5}e^{-x}\left(2\cos 2x+\sin 2x\right)+\frac{4}{5}c$$

(vii) *Do yourself as above*

(viii) $I=\int \operatorname{cosec}^3 x\, dx$ | $u=\operatorname{cosec} x$, $v=\operatorname{cosec}^2 x$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x \; dx$$

Integrating by parts

$$I = \csc x(-\cot x) i\int (-\cot x)(-\csc x \cot x)\, dx$$
$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \; dx$$
$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \left(\operatorname{cosec}^2 x - 1\right) \; dx$$
$$= -\operatorname{cosec} x \cot x - \int \left(\operatorname{cosec}^3 x - \operatorname{cosec} x\right) \; dx$$
$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x\, dx + \int \operatorname{cosec} x\, dx$$
$$= -\operatorname{cosec} x \cot x - I + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow \; I + I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow \; 2I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow \; I = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln\left|\csc x - \cot x\right| + \frac{1}{2}c$$

Question # 3

Show that

$$\int e^{ax} \sin bx \; dx = \frac{1}{\sqrt{a^2+b^2}}\; e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Solution

Let $I = \int e^{ax} \sin bx \; dx$ $\qquad u = e^{ax}$, $v = \sin bx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int \left(-\frac{\cos bx}{b}\right) \cdot e^{ax}(a)\, dx$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx \; dx$$

Again integrating by parts

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[e^{ax}\,\frac{\sin bx}{b} - \int \frac{\sin bx}{b}\cdot e^{ax} a\, dx\right]$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax} \sin bx\, dx$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2} I + c_1$$
$$\Rightarrow \; I + \frac{a^2}{b^2} I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx + c_1$$
$$\Rightarrow \; \left(\frac{b^2+a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(-b\cos bx + a\sin bx) + c_1$$
$$\Rightarrow \; \left(b^2+a^2\right) I = e^{ax}\left(a\sin bx - b\cos bx\right) + b^2 c_1$$

Put $a = r\cos\theta$ & $b = r\sin\theta$

Squaring and adding

$$a^2 + b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$\Rightarrow \; a^2 + b^2 = r^2(1) \qquad \Rightarrow \; r = \sqrt{a^2+b^2}$$

Also

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \qquad \Rightarrow \; \frac{b}{a} = \tan\theta$$
$$\Rightarrow \; \theta = \tan^{-1}\frac{b}{a}$$

So

$$\left(b^2+a^2\right) I = e^{ax}\left(r\cos\theta\sin bx - r\sin\theta\cos bx\right) + b^2 c_1$$
$$\left(b^2+a^2\right) I = e^{ax} r\left(\sin bx\cos\theta - \cos bx\sin\theta\right) + b^2 c_1$$
$$\Rightarrow \; \left(a^2+b^2\right) I = e^{ax} r \sin\left(bx - \theta\right) + b^2 c_1$$

Putting value of r and θ

$$\left(a^2+b^2\right) I = e^{ax}\sqrt{a^2+b^2}\; \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + b^2 c_1$$
$$\Rightarrow I = \frac{\sqrt{a^2+b^2}}{(a^2+b^2)}\; e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + \frac{b^2}{a^2+b^2}c_1$$
$$\Rightarrow \; I = \frac{1}{\sqrt{a^2+b^2}}\; e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Where $c = \dfrac{b^2}{a^2+b^2}c_1$

Question # 4

Evaluate the following indefinite integrals.

(i) $\int \sqrt{a^2 - x^2}\; dx$ $\qquad$ (ii) $\int \sqrt{x^2 - a^2}\; dx$

(iii) $\int \sqrt{4 - 5x^2}\, dx$ $\qquad$ (iv) $\int \sqrt{3 - 4x^2}\, dx$

(v) $\int \sqrt{x^2 + 4}\, dx$ $\qquad$ (vi) $\int x^2 e^{ax} dx$

Solution

(i) Let $I = \int \sqrt{a^2 - x^2}\; dx$ $\qquad u = \sqrt{a^2 - x^2}$, $v = 1$

$$= \int \sqrt{a^2 - x^2}\cdot 1\; dx$$

Integrating by parts

$$I = \sqrt{a^2 - x^2}\cdot x - \int x \cdot \frac{1}{2}\left(a^2 - x^2\right)^{-\frac{1}{2}} \cdot (-2x)\; dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\; dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\; dx$$

$$= x\sqrt{a^2-x^2} - \int \left(\frac{a^2-x^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} - \frac{a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\ dx + \int \frac{a^2}{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow\ I = x\sqrt{a^2-x^2} - I + a^2 \int \frac{1}{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow\ I + I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ 2I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 Sin^{-1}\frac{x}{a} + \frac{1}{2}c$$

Review

- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left| x + \sqrt{x^2+a^2} \right| + c$

(ii)Let $I = \int \sqrt{x^2-a^2}\ dx$ | $u = \sqrt{x^2-a^2}$, $v = 1$

$$= \int \sqrt{x^2-a^2} \cdot 1\ dx$$

Integrating by parts

$$I = \sqrt{x^2-a^2} \cdot x - \int x \cdot \frac{1}{2}\left(x^2-a^2\right)^{-\frac{1}{2}} \cdot (2x)\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \left(\frac{x^2-a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} + \frac{a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\ dx - \int \frac{a^2}{\sqrt{x^2-a^2}} dx$$

$$\Rightarrow\ I = x\sqrt{x^2-a^2} - I - a^2 \int \frac{1}{\sqrt{x^2-a^2}} dx$$

$$\Rightarrow\ I + I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\because\ \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ 2I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left| x + \sqrt{x^2-a^2} \right| + \frac{1}{2}c$$

(iii) Let $I = \int \sqrt{4-5x^2}\ dx$

$$= \int \sqrt{4-5x^2} \cdot 1\ dx$$

Integrating by parts

$$I = \sqrt{4-5x^2} \cdot x - \int x \cdot \frac{1}{2}(4-5x^2)^{-\frac{1}{2}} \cdot (-10x)\ dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{-5x^2}{(4-5x^2)} dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{4-5x^2-4}{(4-5x^2)} dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \sqrt{4-5x^2} dx + 4\int \frac{1}{\sqrt{4-5x^2}}\ dx$$

$$\Rightarrow\ I = \sqrt{4-5x^2} \cdot x - I + 4\int \frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}\ dx$$

$$\Rightarrow\ I + I = \sqrt{4-5x^2} \cdot x + 4\int \frac{1}{\sqrt{5}\sqrt{\frac{4}{5}-x^2}}\ dx$$

$$\Rightarrow\ 2I = \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}\ dx$$

$$= \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} Sin^{-1}\left(\frac{x}{2/\sqrt{5}} \right) + c_1$$

$$\because\ \int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}$$

$$\Rightarrow\ I = \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}} Sin^{-1}\left(\frac{\sqrt{5}\,x}{2} \right) + \frac{1}{2}c_1$$

$$= \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt{5}} Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + c$$

Where $c = \frac{1}{2}c_1$

(iv) *Same as above.*

(v) *Same as Q # 4(ii)*

Use $\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|x+\sqrt{x^2+4}\right| + c$

(vi) *Do yourself as Question # 2(v)*

Important Formula

Since $\frac{d}{dx}\left(e^{ax}f(x)\right) = e^{ax}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}e^{ax}$

$$= e^{ax}f'(x) + f(x)\cdot e^{ax}(a)$$

$$= e^{ax}\left[\,a\,f(x) + f'(x)\,\right]$$

On integrating

$$\int \frac{d}{dx}\left(e^{ax}f(x)\right)\,dx = \int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]dx$$

$$\Rightarrow\ e^{ax}f(x) = \int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]dx$$

$$\Rightarrow\ \boxed{\int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]dx = e^{ax}f(x)+c}$$

Question # 5

Evaluate the following integrals.

(i) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$ (ii) $\int e^x\left(\cos x+\sin x\right)dx$

(iii) $\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

(iv) $\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

(v) $\int \frac{xe^x}{(1+x)^2}dx$ (vi) $\int \frac{xe^x}{(1+x)^2}dx$

(vii) $\int e^{-x}\left(\cos x-\sin x\right)dx$

(viii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$ (ix) $\int \frac{2x}{1-\sin x}dx$

(x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$ (xi) $\int \left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

Solution

(i) Let $I = \int e^x\left(\frac{1}{x}+\ln x\right)dx$

$$= \int e^x\left(\ln x+\frac{1}{x}\right)dx$$

Put $f(x) = \ln x \ \Rightarrow\ f'(x) = \frac{1}{x}$

So $I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c \quad = e^x\ln x + c$$

(ii) Let $I = \int e^x\left(\cos x+\sin x\right)dx$

$$= \int e^x\left(\sin x+\cos x\right)dx$$

Put $f(x) = \sin x \ \Rightarrow\ f'(x) = \cos x$

So $I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c$$

$$= e^x\sin x + c$$

(iii) Let $I = \int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

Put $f(x) = \sec^{-1}x \ \Rightarrow\ f'(x) = \frac{1}{x\sqrt{x^2-1}}$

So $I = \int e^{ax}\left[a\,f(x)+f'(x)\right]dx$

$$= e^{ax}f(x)+c$$

$$= e^{ax}\sec^{-1}x + c$$

(iv) Let $I = \int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

$$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$$

$$= \int e^{3x}\left(3\frac{1}{\sin x}-\frac{\cos x}{\sin x\cdot\sin x}\right)dx$$

$$= \int e^{3x}\left(3\csc x-\csc x\cot x\right)dx$$

Put $f(x) = \csc x \ \Rightarrow\ f'(x) = -\csc x\cot x$

$\Rightarrow\ I = \int e^{3x}\left(3f(x)+f'(x)\right)dx$

$$= e^{3x}f(x)+c$$

$$= e^{3x}\csc x + c$$

(v) Let $I = \int e^{2x}\left(-\sin x+2\cos x\right)dx$

$$= \int e^{2x}\left(2\cos x-\sin x\right)dx$$

Put $f(x) = \cos x \ \Rightarrow\ f'(x) = -\sin x$

So $I = \int e^{2x}\left(2f(x)+f'(x)\right)dx$

$$= e^{2x} f(x) + c$$
$$= e^{2x} \cos x + c$$

(vi) Let $I = \int \frac{xe^x}{(1+x)^2} dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2} dx$$
$$= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx$$
$$= \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

So $I = \int e^x \left(f(x) + f'(x) \right) dx$

$$= e^x f(x) + c$$
$$= e^x \left(\frac{1}{1+x} \right) + c$$

(vii) Let $I = \int e^{-x} \left(\cos x - \sin x \right) dx$

$$= \int e^{-x} \left((-1)\sin x + \cos x \right) dx$$

Put $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

So $I = \int e^{-x} \left((-1) f(x) + f'(x) \right) dx$

$$= e^{-x} f(x) + c$$
$$= e^{-x} \sin x + c$$

(viii) Let $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2} dx$

$$= \int e^{m\tan^{-1}x} \cdot \frac{1}{1+x^2} dx$$

Put $t = \tan^{-1} x \quad \Rightarrow \quad dt = \frac{1}{1+x^2} dx$

So $I = \int e^{mt} dt$

$$= \frac{e^{mt}}{m} + c$$
$$= \frac{1}{m} e^{m\tan^{-1}x} + c$$

Important Integral

Let $I = \int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

Put $t = \cos x \quad \Rightarrow \quad dt = -\sin x \, dx$

$\Rightarrow \quad -dt = \sin x \, dx$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$$= -\ln|t| + c$$
$$= -\ln|\cos x| + c$$
$$= \ln|\cos x|^{-1} + c \qquad \because \; m\ln x = \ln x^m$$
$$= \ln\left|\frac{1}{\cos x}\right| + c \quad = \ln|\sec x| + c$$

$$\Rightarrow \boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

Similarly, we have

$$\boxed{\int \cot x \, dx = \ln|\sin x| + c}$$

(ix) Let $I = \int \frac{2x}{1-\sin x} dx$

$$= \int \frac{2x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx$$
$$= \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx$$
$$= \int \frac{2x + 2x\sin x)}{\cos^2 x} dx$$
$$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x} \right) dx$$
$$= \int \frac{2x}{\cos^2 x} dx + \int \frac{2x\sin x}{\cos x \cdot \cos x} dx$$
$$= 2\int x\sec^2 x \, dx + 2\int x\sec x \tan x \, dx$$

Integrating by parts

$$I = 2\left[x \cdot \tan x - \int \tan x \cdot 1 \, dx \right] + 2\left[x \cdot \sec x - \int \sec x(1) \, dx \right]$$
$$= 2\left[x \cdot \tan x - \ln|\sec x| \right] + 2\left[x \cdot \sec x - \ln|\sec x + \tan x| \right] + c$$
$$= 2x\tan x - 2\ln|\sec x| + 2x\sec x - 2\ln|\sec x + \tan x| + c$$

(x) Let $I = \int \frac{e^x (1+x)}{(2+x)^2} dx$

$$= \int \frac{e^x (2+x-1)}{(2+x)^2} dx$$

$$= \int e^x \left(\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right) dx$$

$$= \int e^x \left((2+x)^{-1} - (2+x)^{-2} \right) dx$$

Put $f(x) = (2+x)^{-1} \quad \Rightarrow \quad f'(x) = -(2+x)^{-2}$

So $I = \int e^x \left(f(x) + f'(x) \right) dx$

$$= e^x f(x) + c$$

$$= e^x (2+x)^{-1} + c$$

$$= \frac{e^x}{2+x} + c$$

(xi) Let $I = \int \left(\frac{1-\sin x}{1-\cos x} \right) e^x \, dx$

$$= \int \left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) e^x \, dx$$

$$= \int \left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) e^x \, dx$$

$$= \int \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2} \right) e^x \, dx$$

$$= \int e^x \left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2} \right) dx$$

Put $f(x) = -\cot\frac{x}{2} \quad \Rightarrow \quad f'(x) = \operatorname{cosec}^2\frac{x}{2} \cdot \frac{1}{2}$

$$\Rightarrow \quad f'(x) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

So $I = \int e^x \left(f(x) + f'(x) \right)$

$$= e^x f(x) + c$$

$$= e^x \left(-\cot\frac{x}{2} \right) + c$$

$$= -e^x \cot\frac{x}{2} + c.$$

EXERCISE 3.5

Q.1 $\displaystyle\int \frac{3x+1}{x^2-x-6}\,dx$

Solution:

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

$$= \int \frac{3x+1}{x^2-3x+2x-6}\,dx$$

$$= \int \frac{3x+1}{x(x-3)+2(x-3)}\,dx$$

$$= \int \frac{3x+1}{(x+2)(x-3)}\,dx$$

Let

$$\frac{3x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \quad \text{——— (1)}$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x+2)(x-3)$ on both sides in (1)

$$3x+1 = A(x-3) + B(x+2) \quad \text{——— (2)}$$

<u>To find A</u>

Put $x+2=0$

$x=-2$ in (2)

$$3(-2)+1 = A(-2-3)$$

$$-6+1 = A(-5)$$

$$-5 = -5A$$

$$A = \frac{-5}{-5}$$

$$\boxed{A = 1}$$

<u>To find B</u>

Put $x-3=0$

$$x = 3 \text{ in } (2)$$

$$3(3) + 1 = B(3 + 2)$$

$$9 + 1 = 5B$$

$$5B = 10$$

$$B = \frac{10}{5}$$

$\therefore$ From eq. (1)

$$\frac{3x + 1}{(x + 2)(x - 3)} = \frac{1}{x + 2} + \frac{2}{x - 3}$$

$$\int \frac{3x + 1}{(x + 2)(x - 3)} dx = \int \frac{dx}{x + 2} + 2\int \frac{dx}{x - 3}$$

$$= \boxed{ln|x + 2| + 2ln|x - 3| + c} \quad \text{Ans.}$$

Q.2 $\displaystyle \int \frac{\mathbf{5x + 8}}{\mathbf{(x + 3)(2x - 1)}} \mathbf{dx}$

Solution:

$$\int \frac{5x + 8}{(x + 3)(2x - 1)} dx$$

Let

$$\frac{5x + 8}{(x + 3)(2x - 1)} = \frac{A}{x + 3} + \frac{B}{2x - 1} \quad \text{———} \quad (1)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x + 3)(2x - 1)$ on both sides in (1)

$$5x + 8 = A(2x - 1) + B(x + 3) \quad \text{———} \quad (2)$$

To find A

Put $x + 3 = 0$

$$x = -3 \text{ in } (2)$$

$$5(-3) + 8 = A[2(-3) - 1]$$

$$-15 + 8 = A(-6 - 1)$$

$$-7 = -7A$$

$$A = \frac{-7}{-7}$$

$$\boxed{A = 1}$$

To find B

Put $2x - 1 = 0$

$$2x = 1$$

$$x = \frac{1}{2} \text{ in (2)}$$

$$5\left(\frac{1}{2}\right) + 8 = B\left(\frac{1}{2} + 3\right)$$

$$\frac{5}{2} + 8 = B\left(\frac{1+6}{2}\right)$$

$$\frac{5+16}{2} = \left(\frac{7}{2}\right) B$$

$$\frac{21}{2} = \frac{7}{2} B$$

$$\frac{21}{2} \times \frac{2}{7} = B$$

$$\boxed{B = 3}$$

$\therefore$ From eq. (1)

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$$

Integrate

$$\int \frac{5x+8}{(x+3)(2x-1)} dx = \int \frac{dx}{x+3} + \frac{3}{2}\int \frac{2dx}{2x-1}$$

$$= \boxed{ln|x+3| + \frac{3}{2} ln|2x-1| + c} \qquad \text{Ans.}$$

Q.3 $\displaystyle\int \frac{x^2+3x-34}{x^2+2x-15} dx$ **(Guj. Board 2006)**

Solution:

$$\int \frac{x^2+3x-34}{x^2+2x-15} dx$$

$$= \int \left(1 + \frac{x-19}{x^2+2x-15}\right) dx$$

$$\begin{array}{r|l} & 1 \\ x^2+2x-15 & x^2+3x-34 \\ & \pm x^2 \pm 2x \mp 15 \\ \hline & x-19 \end{array}$$

$$= \int dx + \int \frac{x-19}{x^2+5x-3x-15} dx$$

$$= x + \int \frac{x-19}{x(x+5)-3(x+5)} dx$$

$$= x + \int \frac{x-19}{(x-3)(x+5)} dx$$

$= \quad x + I \quad \text{———} \quad (1)$

$I = \int \frac{x - 19}{(x - 3)(x + 5)}\, dx$

Let

$\frac{x - 19}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5} \quad \text{———} \quad (2)$

Where A & B are constant of potential fractions which are to be determined.

Multiplying (x – 3) (x + 5) on both sides in (2)

$X - 19 = A(x + 5) + B(x - 3) \quad \text{———} \quad (3)$

<u>To find A</u>

Put $x - 3 = 0$

$x = 3$ in (3)

$3 - 19 = A(3 + 5)$

$-16 = 8A$

$A = \frac{-16}{8}$

$\boxed{A = -2}$

<u>To find B</u>

Put $x + 5 = 0$

$x = -5$ in (3)

$-5 - 19 = B(-5 - 3)$

$-24 = -8B$

$B = \frac{-24}{-8}$

$\boxed{B = 3}$

$\therefore$ From eq. (2)

$\frac{x - 19}{(x - 3)(x + 5)} = \frac{-2}{x - 3} + \frac{3}{x + 5}$

Integrate

$\int \frac{x - 19}{(x - 3)(x + 5)}\, dx = 2\int \frac{dx}{x - 3} + 3\int \frac{dx}{x + 5}$

$I = -2\, ln\,|x - 3| + 3\, ln\,|x + 5| + c$

$\therefore$ From eq. (1)

$= \boxed{x - 2ln\,|x - 3| + 3ln\,|x + 5| + c}$ Ans.

Q.4 $\displaystyle\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx$

Solution:

$$\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx$$

Let

$$\frac{(a-b)x}{(x-a)\,(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \quad \text{———} \quad (1)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x-a)\,(x-b)$ on both sides in (1)

$$(a-b)\,x = A\,(x-b) + B\,(x-a) \quad \text{———} \quad (2)$$

<u>To find A</u>

Put $x - a = 0$

$x = a$ in (2)

$$(a-b)\,a = A\,(a-b)$$

$$\frac{(a-b)a}{a-b} = A$$

$$\boxed{A = a}$$

<u>To find B</u>

Put $x - b = 0$

$x = b$ in eq. (2)

$$(a-b)\,b = B\,(b-a)$$

$$\frac{(a-b)\,b}{b-a} = B$$

$$B = -\frac{(b-a)\,b}{b-a}$$

$$\boxed{B = -b}$$

From eq. (1)

$$\frac{(a-b)\,x}{(x-a)\,(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$$

Integrate

$$\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx = a\int\frac{dx}{x-a} - b\int\frac{dx}{x-b}$$

$$= \boxed{a\,ln\,|x-a| - b\,ln\,|x-b| + c} \qquad \text{Ans.}$$

Q.5 $\displaystyle\int \frac{3 - x}{1 - x - 6x^2}\,dx$

Solution:

$$\int \frac{3 - x}{1 - x - 6x^2}\,dx$$

$$= \int \frac{3 - x}{1 - 3x + 2x - 6x^2}\,dx$$

$$= \int \frac{3 - x}{(1 - 3x) + 2x\,(1 - 3x)}\,dx$$

$$= \int \frac{3 - x}{(1 + 2x)\,(1 - 3x)}\,dx$$

Let

$$\frac{3 - x}{(1 + 2x)\,(1 - 3x)} = \frac{A}{1 + 2x} + \frac{B}{1 - 3x} \quad \text{———} \quad (1)$$

Where A and B are constant of partial fractions which are to be determined

Multiplying $(1 + 2x)\,(1 - 3x)$ on both sides in (1)

$$3 - x = A\,(1 - 3x) + B\,(1 + 2x) \quad \text{———} \quad (2)$$

<u>To find A</u>

Put $1 + 2x = 0$

$$2x = -1$$

$$x = \frac{-1}{2} \text{ in (2)}$$

$$3 - \frac{-1}{2} = A\left[1 - 3\left(\frac{-1}{2}\right)\right]$$

$$3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right)$$

$$\frac{6 + 1}{2} = A\left(\frac{2 + 3}{2}\right)$$

$$\frac{7}{2} = \left(\frac{5}{2}\right) A$$

$$\frac{7 \times 2}{2 \times 5} = A$$

$$\boxed{A = \frac{7}{5}}$$

To find B

Put $1 - 3x = 0$

$$3x = 1$$

$$x = \frac{1}{3} \text{ in } (2)$$

$$3 - \frac{1}{3} = B\left(1 + 2\left(\frac{1}{3}\right)\right)$$

$$\frac{9-1}{3} = B\left(1 + \frac{2}{3}\right)$$

$$\frac{8}{3} = B\left(\frac{3+2}{3}\right)$$

$$\frac{8}{3} = \left(\frac{5}{3}\right)B$$

$$\frac{8}{3} \times \frac{3}{5} = B$$

$$\boxed{B = \frac{8}{5}}$$

$\therefore$ From eq. (1)

$$\frac{3-x}{(1+2x)(1-3x)} = \frac{\frac{7}{5}}{1+2x} + \frac{\frac{8}{5}}{1-3x}$$

Integrate

$$\int \frac{3-x}{(1+2x)(1-3x)}\,dx = \frac{7}{5}\int\frac{dx}{1+2x} + \frac{8}{5}\int\frac{dx}{1-3x}$$

$$= \frac{7}{5.2}\int\frac{2dx}{1+2x} - \frac{8}{5\times 3}\int\frac{-3}{1-3x}\,dx$$

$$= \boxed{\frac{7}{10}\ln|1+2x| - \frac{8}{5}\ln|1-3x| + c} \qquad \text{Ans.}$$

Q.6 $\displaystyle\int \frac{2x}{x^2 - a^2}\,dx$

Solution:

$$\int \frac{2x}{x^2-a^2}\,dx$$

$$= \int \frac{2x}{(x+a)(x-a)}\,dx$$

Let

$$\frac{2x}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} \text{ ——————— } (1)$$

Where A and B are constant of partial fractions which are to be determined

Multiplying $(x + a)(x - a)$ on both sides in (1)

$$2x = A(x-a) + B(x+a) \text{ ——————— } (2)$$

To find A

Put $x + a = 0$

$x = -a$ in (2)

$$2(-a) = A(-a-a)$$

$$-2a = 2a\,A$$

$$A = \frac{-2a}{-2a}$$

$$\boxed{A = 1}$$

To find B

Put $x - a = 0$

$x = a$ in (2)

$$2a = B(a+a)$$

$$2a = 2a\,B$$

$$B = \frac{2a}{2a}$$

$$\boxed{B = 1}$$

$\therefore$ From eq. (1)

$$\frac{2x}{(x+a)(x-a)} = \frac{1}{x+a} + \frac{1}{x-a}$$

Integrate

$$\int \frac{2x}{(x+a)(x-a)}\,dx = \int \frac{dx}{x+a} + \int \frac{dx}{x-a}$$

$$= \boxed{\ln|x+a| + \ln|x-a| + c} \qquad \text{Ans.}$$

Q.7 $\displaystyle\int \frac{1}{6x^2 - 5x - 4}\,dx$

Solution:

$$\int \frac{1}{6x^2 - 5x - 4}\,dx$$

$$= \int \frac{1}{x^2 + 8x - 3x - 4}\, dx$$

$$= \int \frac{1}{2x\,(3x + 4) - 1\,(3x + 4)}\, dx$$

$$= \int \frac{1}{(2x - 1)\,(3x + 4\,)}\, dx$$

Let

$$\frac{1}{(2x - 1)\,(3x + 4)} = \frac{A}{2x - 1} + \frac{B}{3x + 4} \quad \text{———} \quad (1)$$

Where A and B are constant of partial fractions which are to be determined.

Multiplying $(2x - 1)\,(3x + 4)$ on both sides in (1)

$$1 \quad = \quad A\,(3x + 4) + B\,(2x - 1) \text{———} \quad (2)$$

<u>To find A</u>

Put $\quad 2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$ in eq. (2)

$$1 = A\left(\frac{3}{2} + 4\right)$$

$$1 = A\left(\frac{3 + 8}{2}\right)$$

$$2 = A\,(11)$$

$$A = \frac{2}{11}$$

<u>To find B</u>

Put $\quad 3x + 4 = 0$

$3x = -4$

$x = \frac{-4}{3}$ in eq. (2)

$$1 = B\left[\,2\left(\frac{-4}{3}\right) - 1\,\right]$$

$$1 = B\left(\frac{-8}{3} - 1\right)$$

$$1 = B\left(\frac{-8-3}{3}\right)$$

$$3 = -11B$$

$$\boxed{B = \frac{-3}{11}}$$

$\therefore$ From eq. (1)

$$\frac{1}{(2x-1)(3x+4)} = \frac{\frac{2}{11}}{2x-1} + \frac{\frac{-3}{11}}{3x+4}$$

$$\int \frac{dx}{(2x-1)(3x+4)} = \frac{1}{11}\int \frac{2dx}{2x-1} - \frac{1}{11}\int \frac{3}{3x+4}\, dx$$

$$= \boxed{\frac{1}{11} ln\,|2x-1| - \frac{1}{11} ln\,|3x+4| + c} \qquad \text{Ans.}$$

Q.8 $\displaystyle \int \frac{\mathbf{2x^3 - 3x^2 - x - 7}}{\mathbf{2x^2 - 3x - 2}}\, \mathbf{dx}$

Solution:

$$\int \frac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2}\, dx$$

$$\because \begin{array}{r|l} & \quad x \\ \hline 2x^2 - 3x - 2 & 2x^3 - 3x^2 - x - 7 \\ & \underline{\mp 2x^3 \pm 3x^2 \pm 2x} \\ & \qquad\qquad x - 7 \end{array}$$

$$= \int \left(x + \frac{x-7}{2x^2 - 3x - 2}\right) dx$$

$$= \int x dx + \int \frac{x-7}{2x^2 - 4x + x - 2}\, dx$$

$$= \frac{x^2}{2} + \int \frac{x-7}{2x(x-2) + 1(x-2)}\, dx$$

$$= \frac{x^2}{2} + \int \frac{x-7}{(2x+1)(x-2)}\, dx$$

$$= \frac{x^2}{2} + I \quad \text{——} \quad (1)$$

Where

$$I = \int \frac{x-7}{(2x+1)(x-2)}\, dx$$

$$\frac{x-7}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2} \quad \text{——} \quad (2)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(2x + 1)(x - 2)$ on both sides in (2)

$$x - 7 = A(x-2) + B(2x+1) \quad \text{———} \quad (3)$$

<u>To find A</u>

Put

$$2x + 1 = 0$$
$$2x = -1$$
$$x = \frac{-1}{2} \text{ in } (3)$$

$$\frac{-1}{2} - 7 = A\left(\frac{-1}{2} - 2\right)$$

$$\frac{-1-14}{2} = A\left(\frac{-1-4}{2}\right)$$

$$\frac{-15}{2} = A\left(\frac{-5}{2}\right)$$

$$\frac{-15}{2} \times \frac{-2}{2} = A$$

$$\boxed{A = 3}$$

<u>To find B</u>

Put

$$x - 2 = 0$$
$$x = 2 \text{ in } (3)$$

$$2 - 7 = B[2(2) + 1]$$
$$-5 = B(4+1)$$
$$-5 = 5B$$
$$B = \frac{-5}{5}$$

$$\boxed{B = -1}$$

$\therefore$ From eq. (2)

$$\frac{x - 7}{(2x+1)(x-2)} = \frac{3}{2x+1} + \frac{-1}{x-2}$$

$$\int \frac{x-7 \quad dx}{(2x+1)(x-2)} = \frac{3}{2}\int \frac{2dx}{2x+1} - \int \frac{dx}{x-2}$$

$$= \boxed{\frac{3}{2} \ln|2x+1| - \ln|x-2| + c} \quad \text{Ans.}$$

Q.9 $\displaystyle \int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} dx$

Solution:

$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\,dx$$

Let $\quad \dfrac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3} \quad \text{———} \quad (1)$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x-1)(x-2)(x-3)$ on both sides in (1)

$3x^2 - 12x + 11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ ——(2)

<u>To find A</u>

Put $\quad x - 1 = 0$

$\quad x = 1$ in (2)

$3(1)^2 - 12(1) + 11 = A(1-2)(1-3)$

$3 - 12 + 11 = A(-1)(-2)$

$2 = 2A$

$A = \dfrac{2}{2}$

$\boxed{A = 1}$

<u>To find B</u>

Put $\quad x - 2 = 0$

$\quad x = 2$ in (2)

$3(2)^2 - 12(2) + 11 = B(2-1)(2-3)$

$3(4) - 24 + 11 = B(1)(-1)$

$12 - 24 + 11 = -B$

$-1 = -B$

$\boxed{B = 1}$

<u>To find C</u>

$\therefore$ From eq. (1)

$$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Integrate

$$\int \frac{3x^2 - 12x + 11\quad dx}{(x-1)(x-2)(x-3)} = \int \frac{dx}{x-1} + \int \frac{dx}{x-2} + \int \frac{dx}{x-3}$$

$$= \boxed{\ln|x-1| + \ln|x-2| + \ln|x-3| + c} \quad \text{Ans.}$$

Q.10 $\displaystyle\int \frac{2x-1}{x(x-1)(x-3)}\,dx$

Solution:

$$\int \frac{2x-1}{x(x-1)(x-3)}\,dx$$

Let

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3} \quad \text{———} \quad (1)$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $x(x-1)(x-3)$ on both sides in (1)

$$2x - 1 = A(x-1)(x-3) + Bx(x-3) + Cx(x-1) \text{ ——(2)}$$

To find A

Put $x = 0$ in eq (2)

$$2(0) - 1 = A(0-1)(0-3)$$

$$-1 = A(-1)(-3)$$

$$-1 = 3A$$

$$\boxed{A = \frac{-1}{3}}$$

To find B

Put $x - 1 = 0$

$x = 1$ in (2)

$$2(1) - 1 = B(1)(1-3)$$

$$2 - 1 = B(-2)$$

$$1 = -2B$$

$$\boxed{B = \frac{-1}{2}}$$

To find C

Put $x - 3 = 0$

$x = 3$ in (2)

$$2(3) - 1 = C(3)(3-1)$$

$$6 - 1 = C(3)(2)$$

$$5 = 6C$$

$$\boxed{C = \frac{5}{6}}$$

$\therefore$ From eq. (1)

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{\frac{-1}{3}}{x} + \frac{\frac{-1}{2}}{x-1} + \frac{\frac{5}{6}}{x-3}$$

Integrate $\displaystyle\int \frac{2x-1}{x(x-1)(x-3)}\,dx = \frac{-1}{3}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x-1} + \frac{5}{6}\int \frac{dx}{x-3}$

$$= \boxed{\frac{-1}{3}\,ln\,|x| - \frac{1}{2}\,ln\,|x-1| + \frac{5}{6}\,ln\,|x-3| + c} \quad \text{Ans.}$$

Q.11 $\displaystyle\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}\,dx$

Solution:

$$\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}\,dx$$

$$= \int \frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}\,dx$$

Let

$$\frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3} \quad \text{——— (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x+1)(x-1)(2x+3)$ on both sides in (1)

$$5x^2+9x+6 = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x-1)(x-1) \text{— (2)}$$

To find A

Put $x + 1 = 0$

$x = 1$ in (2)

$5(-1)^2 + 9(-1) + 6 = A(-1-1)(-2+3)$

$5 - 9 + 6 = A(-2)(1)$

$2 = -2A$

$A = \frac{2}{-2}$

$\boxed{A = -1}$

To find B

Put $x - 1 = 0$

$x = 1$ in (2)

$$5(1)^2 + 9(1) + 6 = B\,(1 + 1)\,(2 + 3)$$
$$5 + 9 + 6 = B\,(2)\,(5)$$
$$20 = 10\,B$$
$$B = \frac{20}{10}$$
$$\boxed{B = 2}$$

To find C

Put $2x + 3 = 0$
$$2x = -3$$
$$x = \frac{-3}{2} \text{ in (2)}$$
$$5\left(\frac{-3}{2}\right)^2 + 9\left(\frac{-3}{2}\right) + 6 = C\left(\frac{-3}{2} + 1\right)\left(\frac{-3}{2} - 1\right)$$
$$5\left(\frac{9}{4}\right) - \frac{27}{2} + 6 = C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$
$$\frac{45}{4} - \frac{27}{2} + 6 = C\left(\frac{-1}{2}\right)\left(\frac{-5}{2}\right)$$
$$\frac{45 - 54 + 24}{4} = \frac{5}{4}\,C$$
$$\frac{15}{4} \times \frac{4}{5} = C$$
$$\boxed{C = 3}$$

$\therefore$ From eq. (1)
$$\frac{5x^2 + 9x + 6}{(x + 1)\,(x - 1)\,(2x + 3)} = \frac{-1}{x + 1} + \frac{2}{x - 1} + \frac{3}{2x + 3}$$

Integrate
$$\int \frac{5x^2 + 9x + 6}{(x + 1)\,(x - 1)\,(2x + 3)}\,dx = -\int \frac{dx}{x + 1} + 1\int \frac{dx}{x - 1} + \frac{3}{2}\int \frac{2}{2x + 3}\,dx$$
$$= \boxed{ln\,|x+1| + 2ln\,|x - 1| + \frac{3}{2}\,ln\,|2x + 3| + c} \quad \text{Ans.}$$

Q.12 $\displaystyle\int \frac{\mathbf{4 + 7x}}{\mathbf{(1 + x)^2\,(2 + 3x)}}\,\mathbf{dx}$

Solution:

$$\int \frac{4+7x}{(1+x)^2\,(2+3x)}\,dx$$

Let

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2} \quad \text{———} \quad (1)$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(2 + 3x)\,(1 + x)^2$ on both sides in (1)

$$4 + 7x = A\,(1 + x)^2 + B(2 + 3x)\,(1 + x) + C\,(2 + 3x) \text{ ——} (2)$$

$$4 + 7x = A\,(1 + x^2 + 2x) + B(2 + 5x + 3x^2) + 2C + 3Cx$$

$$4 + 7x = (A + 3B)\,x^2 + (2A + 5B + 3C)\,x + (A + 2B + 2C) \text{ ——} \quad (3)$$

<u>To find A</u>

Put $2 + 3x = 0$

$3x = -2$

$x = \frac{-2}{3}$ in (3)

$$4 + 7\left(\frac{-2}{3}\right) = A\,(1 - \frac{2}{3})^2$$

$$4 - \frac{14}{3} = A\left(\frac{3-2}{3}\right)^2$$

$$\frac{12-14}{3} = A\left(\frac{1}{3}\right)^2$$

$$\frac{-2}{3} = \frac{1}{9}A$$

$$\frac{-2\times 9}{3} = A$$

$$\boxed{A = -6}$$

<u>To find C</u>

Put $1 + x = 0$

$x = -1$ in (2)

$$4 + 7\,(-1) = C\,(2 - 3)$$

$$4 - 7 = C\,(-1)$$

$$-3 = -C$$

$$\boxed{C = 3}$$

To find B comparing the coefficient of x^2 in (3)

$$A + 3B = 0$$

Definite Integral

Let $\int f(x)dx = \varphi(x)+c$

Then $\int_a^b f(x)\,dx = \left|\varphi(x)\right|_a^b$ or $\left[\varphi(x)\right]_a^b$

$= \varphi(b)-\varphi(a)$

Also

- $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

where $a < c < b$

Evaluate the following definite integrals:

Question # 1

$$\int_1^2 \left(x^2+1\right)dx$$

Solution

$$\int_1^2 \left(x^2+1\right)dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + \left|x\right|_1^2 = \left(\frac{2^3}{3}-\frac{1^3}{3}\right)+\left(2-1\right)$$

$$= \frac{8}{3}-\frac{1}{3}+1 \quad = \frac{10}{3}$$

Question # 2

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$$

Solution

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$$

$$= \int_{-1}^1 x^{\frac{1}{3}}dx + \int_{-1}^1 dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_{-1}^1 + \left|x\right|_{-1}^1$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_{-1}^1 + \left(1-(-1)\right)$$

$$= \frac{3}{4}\left((1)^{\frac{4}{3}}-(-1)^{\frac{4}{3}}\right)+\left(1+1\right)$$

$$= \frac{3}{4}\left(1-1\right)+2 \quad = 2$$

Question # 3

$$\int_{-2}^0 \frac{1}{\left(2x-1\right)^2}dx$$

Solution

$$\int_{-2}^0 \frac{1}{\left(2x-1\right)^2}dx$$

$$= \int_{-2}^0 \left(2x-1\right)^{-2}dx$$

$$= \left|\frac{\left(2x-1\right)^{-2+1}}{\left(-2+1\right)\cdot 2}\right|_{-2}^0$$

$$= \left|\frac{\left(2x-1\right)^{-1}}{(-1)\ 2}\right|_{-2}^0$$

$$= \frac{\left(2(0)-1\right)^{-1}}{-2}-\frac{\left(2(-2)-1\right)^{-1}}{-2}$$

$$= \frac{\left(0-1\right)^{-1}}{-2}-\frac{\left(-4-1\right)^{-1}}{-2}$$

$$= \frac{\left(-1\right)^{-1}}{-2}-\frac{\left(-5\right)^{-1}}{-2}$$

$$= \frac{1}{(-2)(-1)}-\frac{1}{(-2)(-5)}$$

$$= \frac{1}{2}-\frac{1}{10} \quad = \frac{2}{5}$$

Question # 4

$$\int_{-6}^2 \sqrt{3-x}\,dx$$

Solution

$$\int_{-6}^{2}\sqrt{3-x}\,dx$$

$$=\int_{-6}^{2}(3-x)^{\frac{1}{2}}\,dx$$

$$=\left|\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)}\right|_{-6}^{2}\quad=\left|\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left((3-2)^{\frac{3}{2}}-(3+6)^{\frac{3}{2}}\right)$$

$$=-\frac{2}{3}\left((1)^{\frac{3}{2}}-(9)^{\frac{3}{2}}\right)\quad=-\frac{2}{3}(1-27)\quad=\frac{52}{3}.$$

Question # 5

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt$$

Solution

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt$$

$$=\int_{1}^{\sqrt{5}}(2t-1)^{\frac{3}{2}}\,dt$$

$$=\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot 2}\right|_{1}^{\sqrt{5}}\quad=\left|\frac{(2t-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot 2}\right|_{1}^{\sqrt{5}}$$

$$=\left|\frac{(2t-1)^{\frac{5}{2}}}{5}\right|_{1}^{\sqrt{5}}\quad=\frac{\left(2\sqrt{5}-1\right)^{\frac{5}{2}}}{5}-\frac{(2(1)-1)^{\frac{5}{2}}}{5}$$

$$=\frac{\left(2\sqrt{5}-1\right)^{\frac{5}{2}}}{5}-\frac{1}{5}$$

$$=\frac{\sqrt{\left(2\sqrt{5}-1\right)^5}}{5}-\frac{1}{5}\qquad \textit{Ans.}$$

Question # 6

$$\int_{2}^{\sqrt{5}}x\sqrt{x^2-1}\,dx$$

Solution

$$\int_{2}^{\sqrt{5}}x\sqrt{x^2-1}\,dx$$

$$=\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot 2x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(x^2-1\right)\,dx$$

$$=\frac{1}{2}\left|\frac{\left(x^2-1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{\sqrt{5}}\quad=\frac{1}{2}\left|\frac{\left(x^2-1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{\sqrt{5}}$$

$$=\frac{1}{2}\cdot\frac{2}{3}\left[\left(\left(\sqrt{5}\right)^2-1\right)^{\frac{3}{2}}-\left((2)^2-1\right)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[(5-1)^{\frac{3}{2}}-(4-1)^{\frac{3}{2}}\right]\quad=\frac{1}{3}\left[(4)^{\frac{3}{2}}-(3)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[\left(2^2\right)^{\frac{3}{2}}-(3)^{1+\frac{1}{2}}\right]\quad=\frac{1}{3}\left[(2)^3-3(3)^{\frac{1}{2}}\right]$$

$$=\frac{1}{3}\left[8-3\sqrt{3}\right]$$

Question # 7

$$\int_{1}^{2}\frac{x}{x^2+2}\,dx$$

Solution

$$\int_{1}^{2}\frac{x}{x^2+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{2x}{x^2+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{\frac{d}{dx}\left(x^2+2\right)}{x^2+2}\,dx\quad=\frac{1}{2}\left|\ln\left|x^2+2\right|\right|_{1}^{2}$$

$$=\frac{1}{2}\left(\ln\left|2^2+2\right|-\ln\left|1^2+2\right|\right)$$

$$=\frac{1}{2}\left(\ln 6-\ln 3\right)$$

$$=\frac{1}{2}\ln\left(\frac{6}{3}\right)\quad=\frac{1}{2}\ln 2$$

Question # 8

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 dx$$

Solution

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 dx \quad = \int_2^3 \left(x^2+\frac{1}{x^2}-2\right) dx$$

$$= \int_2^3 x^2\,dx + \int_2^3 x^{-2}\,dx - 2\int_2^3 dx$$

Now do yourself

Question # 9

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\;dx$$

Solution

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\;dx$$

$$= \int_{-1}^{1} \left(\frac{2x+1}{2}\right)\left(x^2+x+1\right)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}(2x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}\frac{d}{dx}(2x+1)\,dx$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{1}$$

NOTE

$$(3)^{\frac{3}{2}} = (3)^{1+\frac{1}{2}}$$

$$= 3^1\cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left|\left(x^2+x+1\right)^{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left[\left((1)^2+(1)+1\right)^{\frac{3}{2}} - \left((-1)^2+(-1)+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(1+1+1)^{\frac{3}{2}} - (1-1+1)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(3)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] \quad = \frac{1}{3}\left[3\sqrt{3}-1\right]$$

$$= \sqrt{3}-\frac{1}{3}$$

Question # 10

$$\int_0^3 \frac{dx}{x^2+9}$$

Solution

$$\int_0^3 \frac{dx}{x^2+9} \quad = \int_0^3 \frac{dx}{x^2+3^2}$$

$$= \left|\frac{1}{3}Tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}Tan^{-1}\left(\frac{3}{3}\right) - \frac{1}{3}Tan^{-1}\left(\frac{0}{3}\right)$$

$$= \frac{1}{3}Tan^{-1}(1) - \frac{1}{3}Tan^{-1}(0)$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right) - \frac{1}{3}(0) \quad = \frac{\pi}{12}$$

Question # 11

$$\int_{\pi/6}^{\pi/3} \cos t\,dt$$

Solution *Do yourself*

Question # 12

$$\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)dx$$

Solution *Do yourself*

Question # 13

$$\int_1^2 \ln x\;dx$$

Solution

Let $I = \int_1^2 \ln x\;dx \quad = \int_1^2 \ln x\cdot 1\;dx$

Integrating by parts

$$I = \left|\ln x\cdot x\right|_1^2 - \int_1^2 x\cdot\frac{1}{x}dx$$

$$= \left|x\ln x\right|_1^2 - \int_1^2 dx$$

$$= (2\cdot\ln 2 - 1\cdot\ln 1) - \left|x\right|_1^2$$

$$= (2\cdot\ln 2 - 1\cdot(0)) - (2-1)$$

$$=(2\cdot\ln 2-0)-1=2\ln 2-1$$

Question # 14

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)dx$$

Solution

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)dx$$

$$=\int_0^2 e^{\frac{x}{2}}\,dx-\int_0^2 e^{-\frac{x}{2}}\,dx$$

$$=\left|\frac{e^{\frac{x}{2}}}{1/2}\right|_0^2-\left|\frac{e^{-\frac{x}{2}}}{-1/2}\right|_0^2 \quad = 2\left|e^{\frac{x}{2}}\right|_0^2+2\left|e^{-\frac{x}{2}}\right|_0^2$$

$$=2\left(e^{\frac{2}{2}}-e^{\frac{0}{2}}\right)+2\left(e^{-\frac{2}{2}}-e^{-\frac{0}{2}}\right)$$

$$=2\left(e^1-e^0\right)+2\left(e^{-1}-e^0\right)$$

$$=2\left(e-1+\frac{1}{e}-1\right)\quad=2\left(e+\frac{1}{e}-2\right)$$

$$=2\left(\frac{e^2+1-2e}{e}\right)\quad=2\frac{(e-1)^2}{e}$$

Question # 15

$$\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$$

Solution

Let $I=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$

$$=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}\,d\theta$$

$$\because\ \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=\int_0^{\pi/4}\left(\frac{\cos\theta}{2\cos^2\theta}++\frac{\sin\theta}{2\cos^2\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\frac{1}{2\cos\theta}\,d\theta+\int_0^{\pi/4}\frac{\sin\theta}{2\cos\theta\cdot\cos\theta}\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\sec\theta\,d\theta+\frac{1}{2}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta$$

$$=\frac{1}{2}\left|\ln\left|\sec\theta+\tan\theta\right|\right|_0^{\pi/4}+\frac{1}{2}\left|\sec\theta\right|_0^{\pi/4}$$

$$=\frac{1}{2}\left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln\left|\sec(0)+\tan(0)\right|\right)+\frac{1}{2}\left(\sec\frac{\pi}{4}-\sec(0)\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-\ln\left|1+0\right|\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-0\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|+\sqrt{2}-1\right)\qquad \textit{Ans.}$$

Question # 16

$$\int_0^{\pi/6}\cos^3\theta\,d\theta$$

Solution

$$\int_0^{\pi/6}\cos^3\theta\,d\theta\ =\int_0^{\pi/6}\cos^2\theta\cdot\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\cos\theta\,d\theta-\int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta$$

$$=\left|\sin\theta\right|_0^{\pi/6}-\int_0^{\pi/6}\sin^2\theta\frac{d}{d\theta}\sin\theta\,d\theta$$

$$=\left(\sin\frac{\pi}{6}-\sin(0)\right)-\left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$$

$$=\left(\frac{1}{2}-0\right)-\frac{1}{3}\left(\sin^3\frac{\pi}{6}-\sin^3(0)\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\left(\frac{1}{2}\right)^2-(0)^3\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\frac{1}{8}\right)\quad=\frac{1}{2}-\frac{1}{24}\quad=\frac{11}{24}$$

Question # 17

$$\int_{\pi/6}^{\pi/4}\cos^2\theta\cdot\cot^2\theta\,d\theta$$

Solution

$$\int_{\pi/6}^{\pi/4} \cos^2\theta\cdot\cot^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta\left(\operatorname{cosec}^2\theta - 1\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\operatorname{cosec}^2\theta - \cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\frac{1}{\sin^2\theta} - \cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cot^2\theta\, d\theta - \int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\operatorname{cosec}^2\theta - 1\right) d\theta - \int_{\pi/6}^{\pi/4} \left(\frac{1+\cos\theta}{2}\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta - \int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta\, d\theta$$

$$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4} - \frac{3}{2}\int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/4}$$

$$= \left(-\cot\frac{\pi}{4} + \cot\frac{\pi}{6}\right) - \frac{3}{2}\left|\theta\right|_{\pi/6}^{\pi/4} - \frac{1}{2}\left(\frac{\sin 2\left(\pi/4\right)}{2} - \frac{\sin 2\left(\pi/6\right)}{2}\right)$$

$$= \left(-1+\sqrt{3}\right) - \frac{3}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}/2}{2}\right)$$

$$= \left(-1+\sqrt{3}\right) - \frac{3}{2}\left(\frac{\pi}{12}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$$

$$= -1+\sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8} \quad = -\frac{5}{4} + \frac{9}{8}\sqrt{3} - \frac{\pi}{8}$$

$$= \frac{9\sqrt{3} - 10 - \pi}{8}$$

Question # 18

$$\int_0^{\pi/4} \cos^4 t\, dt$$

Solution

$$\int_0^{\pi/4} \cos^4 t\, dt \quad = \int_0^{\pi/4} \left(\cos^2 t\right)^2 dt$$

$$= \int_0^{\pi/4} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \int_0^{\pi/4} \left(\frac{1+2\cos 2t + \cos^2 2t}{4}\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1+2\cos 2t + \cos^2 2t\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1+2\cos 2t + \frac{1+\cos 4t}{2}\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(\frac{2+4\cos 2t + 1 + \cos 4t}{2}\right) dt$$

$$= \frac{1}{8}\int_0^{\pi/4} \left(3+4\cos 2t + \cos 4t\right) dt$$

$$= \frac{1}{8}\left|3t + 4\frac{\sin 2t}{2} + \frac{\sin 4t}{4}\right|_0^{\pi/4}$$

$$= \frac{1}{8}\left(3\left(\frac{\pi}{4}\right) + 2\sin 2\left(\pi/4\right) + \frac{\sin 4\left(\pi/4\right)}{4} - 3(0) - 2\sin 2(0) - \frac{\sin 4(0)}{4}\right)$$

$$= \frac{1}{8}\left(\frac{3\pi}{4} + 2 + \frac{0}{4} - 0 - 0 - \frac{0}{4}\right) \quad = \frac{1}{8}\left(\frac{3\pi}{4} + 2\right)$$

$$= \frac{1}{8}\left(\frac{3\pi+8}{4}\right) \quad = \frac{3\pi+8}{32}$$

Question # 19

$$\int_0^{\pi/3} \cos^2\theta\sin\theta\, d\theta$$

Solution

Let $I = \int_0^{\pi/3} \cos^2\theta\sin\theta\, d\theta$

Put $t = \cos\theta \quad \Rightarrow \quad dt = -\sin\theta\, d\theta$

$\Rightarrow \; -dt = \sin\theta\, d\theta$

When $\theta = 0$ then $t = 1$

And when $\theta = \pi/3$ then $t = 1/2$

So $I = \int_1^{1/2} t^2(-dt)$

$= -\int_1^{1/2} t^2\,dt \quad = -\left|\frac{t^3}{3}\right|_1^{1/2}$

$= -\left(\frac{\left(1/2\right)^3}{3} - \frac{(1)^3}{3}\right) \quad = -\left(\frac{1/8}{3} - \frac{1}{3}\right)$

$= -\left(\frac{1}{24} - \frac{1}{3}\right) \quad = -\left(-\frac{7}{24}\right) \quad = \frac{7}{24}$

Question # 20

$\int_0^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\;d\theta$

Solution

$\int_0^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\;d\theta$

$= \int_0^{\pi/4} \left(1+\cos^2\theta\right)\frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

$= \int_0^{\pi/4} \left(\frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta\right)d\theta$

$= \int_0^{\pi/4} \left(\tan^2\theta + \sin^2\theta\right)d\theta$

$= \int_0^{\pi/4} \left(\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right)d\theta$

$= \int_0^{\pi/4} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2}\right)d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \left(2\sec^2\theta - 1 - \cos 2\theta\right)d\theta$

$= \frac{1}{2}\left|2\tan\theta - \theta - \frac{\sin 2\theta}{2}\right|_0^{\pi/4}$

$= \frac{1}{2}\left(2\tan\frac{\pi}{4} - \frac{\pi}{4} - \frac{\sin 2\left(\pi/4\right)}{2} - 2\tan(0) + 0 + \frac{\sin 2(0)}{2}\right)$

$= \frac{1}{2}\left(2(1) - \frac{\pi}{4} - \frac{1}{2} - 2(0) + 0 + 0\right)$

$= \frac{1}{2}\left(\frac{3}{2} - \frac{\pi}{4}\right) \quad = \frac{1}{2}\left(\frac{6-\pi}{4}\right) \quad = \frac{6-\pi}{8}$

Question # 21

$\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\,d\theta$

Solution

Let $I = \int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\,d\theta$

$= \int_0^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + 1\right)}\,d\theta$

$= \int_0^{\pi/4} \frac{\sec^2\theta}{\left(\tan\theta + 1\right)}\,d\theta$

Put $t = \tan\theta + 1 \quad \Rightarrow \quad dt = \sec^2\theta\,d\theta$

When $x = 0$ then $t = 1$

Also when $x = \frac{\pi}{4}$ then $t = 2$

So $I = \int_1^2 \frac{dt}{t}$

$= \left|\ln t\right|_1^2$

$= \ln 2 - \ln 1 \quad = \ln 2 - 0 \quad = \ln 2$

Review

If $f(x) = \begin{cases} g(x) & : \quad a \le x \le b \\ h(x) & : \quad b \le x \le c \end{cases}$

Then

$$\int_a^c f(x)\,dx = \int_a^b g(x) + \int_b^c h(x)$$

Question # 22

$\int_{-1}^{5} \left|x-3\right|dx$

Solution

Let $I=\int_{-1}^{5}|x-3|\,dx$

Since

$$|x-3|=\begin{cases} x-3 & \text{if } x-3\geq 0 \Rightarrow x\geq 3 \\ -(x-3) & \text{if } x-3<0 \Rightarrow x<3 \end{cases}$$

So $\int_{-1}^{5}|x-3|\,dx=\int_{-1}^{3}[-(x-3)]\,dx+\int_{3}^{5}(x-3)\,dx$

$$=-\int_{-1}^{3}(x-3)\,dx+\int_{3}^{5}(x-3)\,dx$$

$$=-\left|\frac{(x-3)^2}{2}\right|_{-1}^{3}+\left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$=-\left(\frac{(3-3)^2}{2}-\frac{(-1-3)^2}{2}\right)+\left(\frac{(5-3)^2}{2}-\frac{(3-3)^2}{2}\right)$$

$$=-\left(\frac{0}{2}-\frac{16}{2}\right)+\left(\frac{4}{2}-\frac{0}{2}\right)\quad=8+2\quad=10$$

Question # 23

$$\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$$

Solution

Let $I=\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$

$$=\int_{1/8}^{1}\left(x^{1/3}+2\right)^2 x^{-2/3}\,dx$$

Put $t=x^{\frac{1}{3}}+2$

$\Rightarrow\ dt=\frac{1}{3}x^{-\frac{2}{3}}dx\quad\Rightarrow\ 3dt=x^{-\frac{2}{3}}dx$

When $x=\frac{1}{8}$ then $t=\frac{5}{2}$

And when $x=1$ then $t=3$

So $I=\int_{5/2}^{3}(t)^2\,3dt\quad=3\left|\frac{t^3}{3}\right|_{5/2}^{3}$

$$=3\left(\frac{3^3}{3}-\frac{\left(5/2\right)^3}{3}\right)\quad=3\left(\frac{27}{3}-\frac{125/8}{3}\right)$$

$$=3\left(\frac{27}{3}-\frac{125}{24}\right)\quad=3\left(\frac{91}{24}\right)\quad=\frac{91}{8}$$

Question # 24

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

Solution

$$\begin{array}{r|l} & \ \ x-1 \\ \hline x+1 & x^2-2 \\ & \underline{\mp x^2 \mp x} \\ & -x-2 \\ & \underline{\mp x \mp 1} \\ & -1 \end{array}$$

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

$$=\int_{1}^{3}\left(x-1-\frac{1}{x+1}\right)dx$$

$$=\int_{1}^{3}x\,dx-\int_{1}^{3}dx-\int_{1}^{3}\frac{dx}{x+1}$$

$$=\left|\frac{x^2}{2}\right|_{1}^{3}-|x|_{1}^{3}-\big|\ln|x+1|\big|_{1}^{3}$$

$$=\left(\frac{3^2}{2}-\frac{1^2}{2}\right)-(3-1)-\left(\ln|3+1|-\ln|1+1|\right)$$

$$=\left(\frac{9}{2}-\frac{1}{2}\right)-(2)-(\ln 4-\ln 2)$$

$$=4-2-\ln\frac{4}{2}\quad=2-\ln 2$$

Question # 25

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

Solution

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

$$=\int_{2}^{3}\frac{3x^2-2x+1}{x^3-x^2+x-1}\,dx$$

$$=\int_{2}^{3}\frac{\frac{d}{dx}\left(x^3-x^2+x-1\right)}{x^3-x^2+x-1}\,dx$$

$$=\left|\ln\left|x^3-x^2+x-1\right|\right|_{2}^{3}$$

$$=\ln\left|3^3-3^2+3-1\right|-\ln\left|2^3-2^2+2-1\right|$$

$$=\ln|27-9+3-1|-\ln|8-4+2-1|$$

$$=\ln 20-\ln 5\quad=\ln\frac{20}{5}\quad=\ln 4$$

Question # 26

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx$$

Solution

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx = \int_0^{\pi/4} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\pi/4} \left(\frac{\sin x}{\cos x \cdot \cos x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\pi/4} \left(\sec x \tan x - \sec^2 x \right) dx$$

$$= \left| \sec x - \tan x \right|_0^{\pi/4}$$

$$= \left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4} \right) - \left(\sec(0) - \tan(0) \right)$$

$$= \sqrt{2} - 1 - 1 + 0 \quad = \sqrt{2} - 2$$

Question # 27

$$\int_0^{\pi/4} \frac{1}{1+\sin x}\, dx$$

Solution

Let $I = \displaystyle\int_0^{\pi/4} \frac{1}{1+\sin x}\, dx$

$$= \int_0^{\pi/4} \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int_0^{\pi/4} \frac{1-\sin x}{1-\sin^2 x}\, dx \quad = \int_0^{\pi/4} \frac{1-\sin x}{\cos^2 x}\, dx$$

Now same as Question # 24

Question # 28

$$\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx$$

Solution

Let $I = \displaystyle\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx$

Put $t = 4 - 3x \quad \Rightarrow \quad 3x = 4 - t$

Also $dt = -3dx \quad \Rightarrow \quad -\frac{1}{3}dt = dx$

When $x = 0$ then $t = 4$

And when $x = 1$ then $t = 1$

So $I = \displaystyle\int_4^1 \frac{4-t}{\sqrt{t}} \left(-\frac{1}{3} dt \right)$

$$= -\frac{1}{3} \int_4^1 \left(\frac{4}{t^{1/2}} - \frac{t}{t^{1/2}} \right) dt$$

$$= +\frac{1}{3} \int_1^4 \left(4t^{-\frac{1}{2}} - t^{\frac{1}{2}} \right) dt$$

Now do yourself

Question # 29

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x (2+\sin x)}\, dx$$

Solution

Let $I = \displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x (2+\sin x)}\, dx$

Put $t = \sin x \quad \Rightarrow \quad dt = \cos x\, dx$

When $x = \dfrac{\pi}{6}$ then $t = \dfrac{1}{2}$

When $x = \dfrac{\pi}{2}$ then $t = 1$

So $I = \displaystyle\int_{1/2}^{1} \frac{dt}{t(2+t)}$

Now consider

$$\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t}$$

$$\Rightarrow \quad 1 = A(2+t) + Bt \quad \ldots\ldots \text{(i)}$$

Put $t = 0$ in (i)

$$1 = A(2+0) + B(0) \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$

Put $2 + t = 0 \Rightarrow \quad t = -2$ in eq. (i)

$$1 = 0 + B(-2) \quad \Rightarrow \quad 1 = -2B \quad \Rightarrow \quad B = -\frac{1}{2}$$

So $\quad \dfrac{1}{t(2+t)} = \dfrac{1/2}{t} + \dfrac{-1/2}{2+t}$

$$\Rightarrow \quad \int_{1/2}^{1} \frac{1}{t(2+t)} dt = \int_{1/2}^{1} \frac{1/2}{t}\, dt + \int_{1/2}^{1} \frac{-1/2}{2+t}\, dt$$

$$= \frac{1}{2} \int_{1/2}^{1} \frac{1}{t}\, dt - \frac{1}{2} \int_{1/2}^{1} \frac{1}{2+t}\, dt$$

$$= \frac{1}{2} \left| \ln|t| \right|_{1/2}^{1} - \frac{1}{2} \left| \ln|2+t| \right|_{1/2}^{1}$$

$$= \frac{1}{2}\left[\ln|1| - \ln\left|\frac{1}{2}\right| \right]$$
$$- \frac{1}{2}\left[\ln|2+1| - \ln\left|2+\frac{1}{2}\right| \right]$$
$$= \frac{1}{2}\left[0 - \ln\frac{1}{2} \right] - \frac{1}{2}\left[\ln 3 - \ln\frac{5}{2} \right]$$
$$= \frac{1}{2}\left[-\ln\frac{1}{2} - \ln 3 + \ln\frac{5}{2} \right]$$
$$= \frac{1}{2}\ln\left(\frac{5/2}{1/2 \times 3} \right) \quad = \frac{1}{2}\ln\left(\frac{5}{3}\right)$$

Question # 30

$$I = \int_0^{\pi/2} \frac{\sin x\, dx}{(1+\cos x)(2+\cos x)}$$

Solution

Let $I = \int_0^{\pi/2} \frac{\sin x\, dx}{(1+\cos x)(2+\cos x)}$

Put $t = \cos x \quad \Rightarrow \quad dt = -\sin x\, dx$

$\Rightarrow \; -dt = \sin x\, dx$

When $x = 0$ then $t = 1$

And when $x = \frac{\pi}{2}$ then $t = 0$

So $I = \int_1^0 \frac{-dt}{(1+t)(2+t)}$

$$= -\int_1^0 \frac{dt}{(1+t)(2+t)} \quad = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

Now consider

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$$

$\Rightarrow \; 1 = A(2+t) + B(1+t)$ …. (i)

Put $1+t = 0 \quad \Rightarrow \quad t = -1$ in (i)

$1 = A(2-1) + 0 \quad \Rightarrow \quad A = 1$

Put $2+t = 0 \quad \Rightarrow \quad t = -2$ in (i)

$1 = 0 + B(1-2) \quad \Rightarrow \quad 1 = -B \quad$ i.e. $B = -1$

So

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$$

$$\int_0^1 \frac{1}{(1+t)(2+t)}\, dt = \int_0^1 \frac{1}{1+t}\, dt - \int_0^1 \frac{1}{2+t}\, dt$$
$$= \left| \ln|1+t| \right|_0^1 - \left| \ln|2+t| \right|_0^1$$
$$= \left(\ln|1+1| - \ln|1+0| \right)$$
$$- \left(\ln|2+1| - \ln|2+0| \right)$$
$$= \ln 2 - 0 - \ln 3 + \ln 2$$
$$= \ln\left(\frac{2\times 2}{3}\right) \quad = \ln\left(\frac{4}{3}\right)$$

Example 4

Find the area bounded by the curve

$$f(x) = x^3 - 2x^2 + 1$$

and the x-axis in the first quadrant.

Solution

Put $f(x) = 0$

$\Rightarrow x^3 - 2x + 1 = 0$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ & \downarrow & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$\Rightarrow (x-1)(x^2 - x - 1) = 0$

$\Rightarrow x - 1 = 0$ or $x^2 - x - 1 = 0$

$\Rightarrow x = 1$ or $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Thus the curve cuts the x-axis at $x = 1$, $\dfrac{1 \pm \sqrt{5}}{2}$

Since we are taking area in the first quad. only

$\therefore x = 1, \dfrac{1+\sqrt{5}}{2}$ ignoring $\dfrac{1-\sqrt{5}}{2}$ as it is $-$ive.

Intervals in 1st quad. are $[0,1]$ & $\left[1, \dfrac{1+\sqrt{5}}{2}\right]$

Since $f(x) \geq 0$ whenever $x \in [0,1]$

and $f(x) \leq 0$ whenever $x \in \left[1, \dfrac{1+\sqrt{5}}{2}\right]$

$\therefore$ Area in 1st quad. $= \int_0^1 \left(x^3 - 2x^2 + 1\right) dx$

$$= \left| \frac{x^4}{4} - 2\frac{x^3}{3} + x \right|_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} + 1\right) - 0$$

$$= \frac{7}{12} \text{ sq. unit}$$

Question # 1

Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$.

Solution

$y = x^2 + 1$; $x = 1$ to $x = 2$

$\therefore y \geq 0$ whenever $x \in [1,2]$

$\therefore$ Area $= \int_1^2 (x^2 + 1) dx$

$$= \int_1^2 x^2 dx + \int_1^2 dx$$

$$= \left| \frac{x^3}{3} \right|_1^2 + \left| x \right|_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(1)^3}{3} \right) + (2-1)$$

$$= \left(\frac{8}{3} - \frac{1}{3} \right) + 1$$

$$= \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. unit.}$$

Question # 2

Find the area above the x-asis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Solution

$y = 5 - x^2$; $x = -1$ to $x = 2$

$\therefore y > 0$ whenever $x \in (-1, 2)$

$\therefore$ Area $= \int_{-1}^2 (5 - x^2) dx$

$$= \left| 5x - \frac{x^3}{3} \right|_{-1}^2$$

$$= \left(5(2) - \frac{(2)^3}{3} \right) - \left(5(-1) - \frac{(-1)^3}{3} \right)$$

$$= \left(10 - \frac{8}{3} \right) - \left(-5 + \frac{1}{3} \right)$$

$$= \frac{22}{3} - \left(-\frac{14}{3} \right) = \frac{22}{3} + \frac{14}{3}$$

$$= \frac{36}{3} = 12 \text{ sq. unit}$$

Question # 3

Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = 1$ to $x = 4$.

Solution

$$y = 3\sqrt{x} \quad ; \qquad x=1 \text{ to } x=4$$

Since $y \geq 0$ when $x \in [1,4]$

$$\therefore \text{ Area } = \int_1^4 3\sqrt{x}\, dx$$

$$= \int_1^4 3x^{\frac{1}{2}}\, dx = 3\int_1^4 x^{\frac{1}{2}}\, dx$$

$$= 3\left|\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4 = 3\left|\frac{x^{\frac{3}{2}}}{3/2}\right|_1^4$$

$$= 3\times\frac{2}{3}\left|x^{\frac{3}{2}}\right|_1^4 = 2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)$$

$$= \frac{3}{4}\left((4)^{\frac{4}{3}}-(1)^{\frac{4}{3}}\right) = 2\left((2^2)^{\frac{3}{2}}-1\right)$$

$$= 2(8-1) = 14 \text{ sq. unit}$$

Question # 4

Find the area bounded by cos function from $x=-\dfrac{\pi}{2}$ to $x=\dfrac{\pi}{2}$

Solution

$$y = \cos x \quad ; \qquad x=-\frac{\pi}{2} \text{ to } x=\frac{\pi}{2}$$

$$\because \quad y>0 \quad \text{whenever } x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \text{ Area } = \int_{-\pi/2}^{\pi/2}\cos x\, dx$$

$$= \left|\sin x\right|_{-\pi/2}^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)$$

$$= 1+1 = 2 \text{ sq. unit}$$

Question # 5

Find the area between the x-asis and the curve $y = 4x-x^2$

Solution

$$y = 4x-x^2$$

Putting $y=0$, we have

$$4x-x^2 = 0$$

$$\Rightarrow \; x(4-x) = 0$$

$$\Rightarrow \; x = 0 \text{ or } x = 4$$

Now $y>0$ when $x\in(0,4)$

$$\therefore \text{ Area } = \int_0^4\left(4x-x^2\right)dx$$

$$= \left|\frac{4x^2}{2}-\frac{x^3}{3}\right|_0^4 = \left|2x^2-\frac{x^3}{3}\right|_0^4$$

$$= \left(2(4)^2-\frac{(4)^3}{3}\right)-\left(2(0)^2-\frac{(0)^3}{3}\right)$$

$$= \left(32-\frac{64}{3}\right)-(0-0)$$

$$= \frac{32}{2} \text{ sq. unit.}$$

Question # 6

Determine the area bounded by the parabola $y = x^2+2x-3$ and the x-axis.

Solution

$$y = x^2+2x-3$$

Putting $y = 0$, we have

$$x^2+2x-3 = 0$$

$$\Rightarrow \; x^2+3x-x-2 = 0$$

$$\Rightarrow \; x(x+3)-1(x+3) = 0$$

$$\Rightarrow \; (x+3)(x-1) = 0$$

$$\Rightarrow \; x = -3 \text{ or } x = 1$$

Now $y\leq 0$ whenever $x\in[-3,1]$

$$\therefore \text{ Area } = -\int_{-3}^{1}\left(x^2+2x-3\right)dx$$

$$= -\left|\frac{x^3}{3}+\frac{2x^2}{2}-3x\right|_{-3}^{1}$$

$$= -\left|\frac{x^3}{3}+x^2-3x\right|_{-3}^{1}$$

$$= -\left(\frac{(1)^3}{3}+(1)^2-3(1)\right) + \left(\frac{(-3)^3}{3}+(-3)^2-3(-3)\right)$$

$$= -\left(\frac{1}{3}+1-3\right)+\left(\frac{-27}{3}+9+9\right)$$

$$= -\left(-\frac{5}{3}\right)+(-9+18)$$

$$= \frac{5}{3}+9 = \frac{32}{3} \text{ sq. unit}$$

Question # 7

Find the area bounded by the curve $y = x^3+1$, the x-axis and line $x=2$.

Solution

$y = x^3+1$

Putting $y=0$, we have

$x^3+1 = 0$

$\Rightarrow (x+1)(x^2-x+1) = 0$

$\Rightarrow x+1=0$ or $x^2-x+1=0$

$\Rightarrow x=-1$ or $x=\dfrac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$

$=\dfrac{1\pm\sqrt{1-4}}{2}$

$\Rightarrow x=\dfrac{1\pm\sqrt{-3}}{2}$

Which is not possible.

Now $y\ge 0$ when $x\in[-1,2]$

$\therefore$ Area $= \int_{-1}^{2}(x^3+1)\,dx$

$= \left|\dfrac{x^4}{4}+x\right|_{-1}^{2}$

$= \left(\dfrac{(2)^4}{4}+2\right)-\left(\dfrac{(-1)^4}{4}-1\right)$

$= \left(\dfrac{16}{4}+2\right)-\left(\dfrac{1}{4}-1\right)$

$= 6-\dfrac{3}{4} = \dfrac{27}{4}$ sq. unit

Question # 8

Find the area bounded by the curve $y = x^3-2x+4$ and the x-axis.

Solution

$y = x^3-2x+4 \quad ; \quad x=1$

Putting $y=0$, we have

$x^3-2x+4 = 0$

By synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & \downarrow & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\Rightarrow (x+2)(x^2-2x+2) = 0$

$\Rightarrow x+2=0$ or $x^2-2x+2 = 0$

$\Rightarrow x=-2$ or $x=\dfrac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2}$

$=\dfrac{2\pm\sqrt{4-8}}{2}$

$=\dfrac{2\pm\sqrt{-4}}{2}$

This is imaginary.

Now $y\ge 0$ when $x\in[-2,1]$

$\therefore$ Area $= \int_{-2}^{1}(x^3-2x+4)\,dx$

$= \int_{-2}^{1}x^3dx-2\int_{-2}^{1}x\,dx+4\int_{-2}^{1}dx$

$= \left|\dfrac{x^4}{4}\right|_{-2}^{1}-2\left|\dfrac{x^2}{2}\right|_{-2}^{1}+4\left|x\right|_{-2}^{1}$

$= \left(\dfrac{(1)^4}{4}-\dfrac{(-2)^4}{4}\right)-2\left(\dfrac{(1)^2}{2}-\dfrac{(-2)^2}{2}\right)+4(1-(-2))$

$= \left(\dfrac{1}{4}-\dfrac{16}{4}\right)-2\left(\dfrac{1}{2}-\dfrac{4}{2}\right)+4(1+2)$

$= \left(\dfrac{1}{4}-4\right)-2\left(\dfrac{1}{2}-2\right)+4(3)$

$= \left(-\dfrac{15}{4}\right)-2\left(-\dfrac{3}{2}\right)+12$

$= -\dfrac{15}{4}+3+12 = \dfrac{45}{4}$ sq. unit

Question # 9

Find the area between the curve

Solution

$y = x^3-4x$

Putting $y=0$, we have

$x^3-4x = 0$

$\Rightarrow x(x^2-4) = 0$

$\Rightarrow x(x+2)(x-2) = 0$

$\Rightarrow x = 0$ or $x=-2$ or $x=2$

Now $y\ge 0$ whenever $x\in[-2,0]$

And $y\le 0$ whenever $x\in[0,2]$

$\therefore$ Area $= \int_{-2}^{0}y\,dx-\int_{0}^{2}y\,dx$

$$= \int_{-2}^{0}\left(x^3-4x\right)dx-\int_{0}^{2}\left(x^3-4x\right)dx$$

$$= \left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{-2}^{0}-\left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left|\frac{x^4}{4}-2x^2\right|_{-2}^{0}-\left|\frac{x^4}{4}-2x^2\right|_{0}^{2}$$

$$= \left(\frac{(0)^4}{4}-2(0)^2\right)-\left(\frac{(-2)^4}{4}-2(-2)^2\right)-\left(\frac{(2)^4}{4}-2(2)^2\right)+\left(\frac{(0)^4}{4}-2(0)^2\right)$$

$$= (0-0)-\left(\frac{16}{4}-8\right)-\left(\frac{16}{4}-8\right)+(0-0)$$

$$= -(4-8)-(4-8) \quad = -(-4)-(-4)$$

$$= 4+4 \quad = 8 \text{ sq. unit.}$$

Question # 9
Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution

$$y = x(x-1)(x+1)$$

Putting $y = 0$, we have

$$x(x-1)(x+1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

Now $y \geq 0$ whenever $x \in [-1,0]$

And $y \leq 0$ whenever $x \in [0,1]$

$$\therefore \text{ Area} = \int_{-1}^{0} y\,dx - \int_{0}^{1} y\,dx$$

$$= \int_{-1}^{0} x(x-1)(x+1)\,dx - \int_{0}^{1} x(x-1)(x+1)\,dx$$

$$= \int_{-1}^{0}\left(x^3-x\right)dx-\int_{0}^{1}\left(x^3-x\right)dx$$

$$= \left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{-1}^{0}-\left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{0}^{1}$$

$$= \left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)-\left(\frac{(1)^4}{4}-\frac{(1)^2}{2}\right)+\left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)$$

$$= (0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)+(0-0)$$

$$= 0-\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+0$$

$$= \frac{1}{4}+\frac{1}{4} \quad = \frac{1}{2} \text{ sq. unit}$$

Question # 11
Find the area between the x-asis and the curve $y = \cos\frac{1}{2}x$ from $x=-\pi$ to $x=\pi$

Solution

$$g(x) = \cos\frac{1}{2}x \quad ; \quad x=-\pi \text{ to } x=\pi$$

$$\because\ g(x) \geq 0 \quad \text{when} \quad x\in[-\pi,\pi]$$

$$\therefore \text{ Area} = \int_{-\pi}^{\pi}\cos\frac{1}{2}x\,dx$$

$$= \left|\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right|_{-\pi}^{\pi} \quad = 2\left|\sin\frac{x}{2}\right|_{-\pi}^{\pi}$$

$$= 2\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{-\pi}{2}\right)\right)$$

$$= 2(1-(-1)) \quad = 2(1+1)$$

$$= 2(2) \quad = 4 \text{ sq. unit.}$$

Question # 12
Find the area between the x-asis and the curve $y = \sin 2x$ from $x=0$ to $x=\frac{\pi}{3}$

Solution

$$y = \sin 2x \quad ; \quad x=0 \text{ to } x=\frac{\pi}{3}$$

$$\because\ y\geq 0 \quad \text{when} \quad x\in\left[0,\frac{\pi}{3}\right]$$

$$\therefore \text{ Area} = \int_{0}^{\pi/3}\sin 2x\,dx$$

$$= \left|-\frac{\cos 2x}{2}\right|_{0}^{\pi/3} \quad = -\frac{1}{2}\left(\cos\frac{2\pi}{3}-\cos(0)\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2}-1\right) = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4} \text{ sq. unit.}$$

Question # 13
Find the area between the x-asis and the curve
$y = \sqrt{2ax - x^2}$ when $a > 0$

Solution

$$y = \sqrt{2ax - x^2}$$

Putting $y = 0$, we have

$$\sqrt{2ax - x^2} = 0$$

On squaring

$$2ax - x^2 = 0$$

$$\Rightarrow \quad x(2a - x) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad 2a - x = 0 \Rightarrow x = 2a$$

$$\because \quad y \geq 0 \text{ when } x \in [0, 2a]$$

$$\therefore \quad \text{Area} = \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - \left(a^2 - 2ax + x^2\right)}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - (a - x)^2}\, dx$$

Put $a - x = a\sin\theta$

$$\Rightarrow \quad -dx = a\cos\theta\, d\theta$$

$$\Rightarrow \quad dx = -a\cos\theta\, d\theta$$

When $x = 0$

$$a - 0 = a\sin\theta \Rightarrow a\sin\theta = a$$

$$\Rightarrow \sin\theta = 1 \quad \Rightarrow \theta = \frac{\pi}{2}$$

When $x = 2a$

$$a - 2a = a\sin\theta \quad \Rightarrow \quad -a = a\sin\theta$$

$$\Rightarrow \quad -1 = \sin\theta \quad \Rightarrow \quad \theta = -\frac{\pi}{2}$$

$$\text{So area} = \int_{\pi/2}^{-\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\, (-a\cos\theta d\theta)$$

$$= -a\int_{\pi/2}^{-\pi/2} \sqrt{a^2\left(1 - \sin^2\theta\right)}\, \cos\theta\, d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2} \sqrt{a^2\cos^2\theta}\, \cos\theta\, d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2} a\cos\theta \cdot \cos\theta\, d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2} \cos^2\theta\, d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -\frac{a^2}{2}\int_{\pi/2}^{-\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= -\frac{a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\pi/2}^{-\pi/2}$$

$$= -\frac{a^2}{2}\left(-\frac{\pi}{2} + \sin(-\pi) - \frac{\pi}{2} - \sin\pi\right)$$

$$= -\frac{a^2}{2}(-\pi - 0 - 0)$$

$$= -\frac{a^2}{2}(-\pi) = \frac{a^2\pi}{2} \text{ sq. unit}$$

Question # 1
Check each of the following equations written against the differential equation is its solution.

(i) $x\dfrac{dy}{dx}=1+y$, $y=cx-1$

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0$, $y^2+y=c-\dfrac{1}{x}$

(iii) $y\dfrac{dy}{dx}-e^{2x}=1$, $y^2 = 2x+e^{2x}+c$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0$, $y = ce^{x^2}$

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}}$, $y = Tan\left(e^x+c\right)$

Solution

(i) $x\dfrac{dy}{dx}=1+y$

$\Rightarrow x\,dy=(1+y)\,dx \Rightarrow \dfrac{dy}{1+y}=\dfrac{dx}{x}$

Integrating both sides

$$\int\frac{dy}{1+y}=\int\frac{dx}{x}$$

$\Rightarrow \ln(1+y)=\ln x+\ln c$

$=\ln cx$

$\Rightarrow 1+y=cx$

$\Rightarrow y=cx-1$ *Proved*

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0$

$\Rightarrow x^2(2y+1)\dfrac{dy}{dx}=1 \Rightarrow x^2(2y+1)\,dy=dx$

$\Rightarrow (2y+1)\,dy=\dfrac{1}{x^2}dx$

On integrating

$$\int(2y+1)\,dy=\int\frac{1}{x^2}dx$$

$\Rightarrow 2\int y\,dy+\int dy=\int x^{-2}\,dx$

$\Rightarrow 2\cdot\dfrac{y^2}{2}+y=\dfrac{x^{-2+1}}{-2+1}+c$

$\Rightarrow y^2+y=\dfrac{x^{-1}}{-1}+c$

$\Rightarrow y^2+y=c-\dfrac{1}{x}$ *Proved*

(iii) $y\dfrac{dy}{dx}-e^{2x}=1$

$\Rightarrow y\dfrac{dy}{dx} = 1+e^{2x} \Rightarrow ydy = \left(1+e^{2x}\right)dx$

On integrating

$$\int ydy = \int\left(1+e^{2x}\right)dx$$

$\Rightarrow \dfrac{y^2}{2} = x+\dfrac{e^{2x}}{2}+\dfrac{c}{2} \Rightarrow y^2 = 2x+e^{2x}+c$

$\Rightarrow y^2 = 2x+e^{2x}+c$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0$

$\Rightarrow \dfrac{1}{x}\dfrac{dy}{dx} = 2y \Rightarrow \dfrac{dy}{dx} = 2xy$

$\Rightarrow \dfrac{dy}{y} = 2xdx$

On integrating

$$\int\frac{dy}{y} = 2\int xdx$$

$\Rightarrow \ln y = 2\cdot\dfrac{x^2}{2}+\ln c$

$= x^2+\ln c$

$= x^2\ln e+\ln c$ $\quad\because \ln e=1$

$= \ln e^{x^2}+\ln c$

$\Rightarrow \ln y = \ln ce^{x^2}$

$\Rightarrow y = ce^{x^2}$ *Proved*

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}} \Rightarrow \dfrac{dy}{y^2+1} = e^xdx$

Integrating both sides

$\Rightarrow \displaystyle\int\frac{dy}{y^2+1} = \int e^xdx$

$\Rightarrow Tan^{-1}y = e^x+c$

$\Rightarrow y = Tan\left(e^x+c\right)$

Solve the following differential equations:

Question # 2

$$\frac{dy}{dx} = -y$$

Solution

$\dfrac{dy}{dx} = -y \Rightarrow \dfrac{dy}{y} = -dx$

On integrating

$$\int\frac{dy}{y} = -\int dx$$

$$\ln y = -x + \ln c$$
$$= -x\ln e + \ln c \qquad \because \ln e = 1$$
$$= \ln e^{-x} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-x} \qquad \Rightarrow y = ce^{-x}$$

Question # 3

$$ydx + xdy = 0$$

Solution

$$ydx + xdy = 0 \quad \Rightarrow ydx = -xdy$$
$$\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

On integrating

$$\ln x = -\ln y + \ln c$$
$$\Rightarrow \ln x = \ln\frac{c}{y}$$
$$\Rightarrow x = \frac{c}{y} \qquad \Rightarrow xy = c$$

Question # 4

$$\frac{dy}{dx} = \frac{1-x}{y}$$

Solution *Do yourself*

Question # 5

$$\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$$

Solution

$$\frac{dy}{dx} = \frac{y}{x^2} \qquad \Rightarrow \frac{dy}{y} = x^{-2}dx$$

Integrating

$$\int\frac{dy}{y} = \int x^{-2}dx$$
$$\Rightarrow \ln y = \frac{x^{-2+1}}{-2+1} + \ln c$$
$$\Rightarrow \ln y = \frac{x^{-1}}{-1} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x}\ln e + \ln c$$
$$= \ln e^{-\frac{1}{x}} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-\frac{1}{x}} \qquad \Rightarrow y = ce^{-\frac{1}{x}}$$

Question # 6

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$

Solution

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$
$$\Rightarrow \sin y\, dy = \frac{dx}{\operatorname{cosec} x}$$
$$\Rightarrow \sin y\, dy = \sin x\, dx$$

Integrating

$$\int \sin y\, dy = \int \sin x\, dx$$
$$\Rightarrow -\cos y = -\cos x - c$$
$$\Rightarrow \cos y = \cos x + c$$

Question # 7

$$xdy + y(x-1)dx = 0$$

Solution

$$xdy + y(x-1)dx = 0$$
$$\Rightarrow xdy = -y(x-1)dx$$
$$\Rightarrow \frac{dy}{y} = -\frac{x-1}{x}dx$$
$$\Rightarrow \frac{dy}{y} = -\left(\frac{x}{x} - \frac{1}{x}\right)dx$$
$$\Rightarrow \frac{dy}{y} = -\left(1 - \frac{1}{x}\right)dx$$

On integrating

$$\int\frac{dy}{y} = -\int\left(1 - \frac{1}{x}\right)dx$$
$$\Rightarrow \ln y = -x + \ln x + \ln c$$
$$= -x\ln e + \ln x + \ln c$$
$$= \ln e^{-x} + \ln x + \ln c$$
$$\Rightarrow \ln y = \ln cxe^{-x} \Rightarrow y = cxe^{-x}$$

Question # 8

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}, (x, y > 0)$$

Solution

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$
$$\Rightarrow \frac{x^2+1}{x}dx = \frac{y+1}{y}dy$$

On integrating

$$\int\frac{x^2+1}{x}dx = \int\frac{y+1}{y}dy$$
$$\Rightarrow \int\left(\frac{x^2}{x} + \frac{1}{x}\right)dx = \int\left(\frac{y}{y} + \frac{1}{y}\right)dy$$

$$\Rightarrow \int\left(x+\frac{1}{x}\right)dx = \int\left(1+\frac{1}{y}\right)dy$$

$$\Rightarrow \int x\,dx+\int\frac{1}{x}dx = \int dy+\int\frac{1}{y}dy$$

$$\Rightarrow \frac{x^2}{2}+\ln x = y+\ln y-\ln c$$

$$\Rightarrow \frac{x^2}{2}\ln e+\ln x+\ln c = y\ln e+\ln y$$

$$\Rightarrow \ln e^{\frac{x^2}{2}}+\ln x+\ln c = \ln e^y+\ln y$$

$$\Rightarrow \ln cxe^{\frac{x^2}{2}} = \ln ye^y$$

$$\Rightarrow cxe^{\frac{x^2}{2}} = ye^y \quad \text{i.e. } ye^y = cxe^{\frac{x^2}{2}}$$

Question # 9

$$\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\left(1+y^2\right)$$

Solution *Do yourself*

Question # 10

$$2x^2y\frac{dy}{dx}=x^2-1$$

Solution *Do yourself*

Question # 11

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

Solution

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

$$\Rightarrow \frac{dy}{dx} = x-\frac{2xy}{2y+1}$$

$$= x\left(1-\frac{2y}{2y+1}\right)$$

$$= x\left(\frac{2y+1-2y}{2y+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = x\left(\frac{1}{2y+1}\right) \Rightarrow (2y+1)dy = x\,dx$$

Now do yourself

Question # 12

$$\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2 = 0$$

Solution

$$\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2 = 0$$

$$\Rightarrow \left(x^2-yx^2\right)\frac{dy}{dx} = -y^2-xy^2$$

$$\Rightarrow x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$\Rightarrow \frac{1-y}{y^2}dy = -\frac{1+x}{x^2}dx$$

Now do yourself

Question # 13

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

Solution

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

$$\Rightarrow \sec^2 x\tan y\,dx = -\sec^2 y\tan x\,dy$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$$

On integrating

$$\int\frac{\sec^2 x}{\tan x}dx = -\int\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \int\frac{\frac{d}{dx}(\tan x)}{\tan x}dx = -\int\frac{\frac{d}{dy}(\tan y)}{\tan y}dy$$

$$\Rightarrow \ln\tan x = -\ln\tan y+\ln c$$

$$\Rightarrow \ln\tan x+\ln\tan y = \ln c$$

$$\Rightarrow \ln(\tan x\tan y) = \ln c$$

$$\Rightarrow \tan x\tan y = c$$

Question # 14

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

Solution

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

$$\Rightarrow y-x\frac{dy}{dx} = 2y^2+2\frac{dy}{dx}$$

$$\Rightarrow y-2y^2 = 2\frac{dy}{dx}+x\frac{dy}{dx}$$

$$\Rightarrow y(1-2y) = (2+x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{2+x} = \frac{dy}{y(1-2y)}$$

On integrating

$$\int\frac{dx}{2+x} = \int\frac{dy}{y(1-2y)} \quad \text{………. (i)}$$

Now consider

$$\frac{1}{y(1-2y)} = \frac{A}{y}+\frac{B}{1-2y}$$

$\Rightarrow\ 1 = A(1-2y)+By$ ………. (ii)

Put $y=0$ in (ii)

$$1 = A(1-2(0))+0 \Rightarrow A=1$$

Put $1-2y=0 \Rightarrow 2y=1 \Rightarrow y=\frac{1}{2}$ in (ii)

$$1 = 0+B\left(\frac{1}{2}\right) \Rightarrow B=2$$

So $\dfrac{1}{y(1-2y)} = \dfrac{1}{y}+\dfrac{2}{1-2y}$

Using in (i) $\leftarrow *$

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dy$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{-2}{1-2y}dy$$

$$\Rightarrow \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(1-2y)}{1-2y}dy$$

$$\Rightarrow \ln(2+x) = \ln y-\ln(1-2y)-\ln c$$

$$\Rightarrow \ln(2+x)+\ln c = \ln y-\ln(1-2y)$$

$$\Rightarrow \ln c(2+x) = \ln\frac{y}{(1-2y)}$$

$$\Rightarrow c(2+x) = \frac{y}{(1-2y)}$$

$$\Rightarrow y = c(2+x)(1-2y)$$

Alternative ($\leftarrow *$)

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dx$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{2}{2y-1}dy$$

$$\Rightarrow \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(2y-1)}{2y-1}dy$$

$$\Rightarrow \ln(2+x) = \ln y-\ln(2y-1)-\ln c$$

$$\Rightarrow \ln(2+x)+\ln c = \ln y-\ln(2y-1)$$

$$\Rightarrow \ln c(2+x) = \ln\frac{y}{(2y-1)}$$

$$\Rightarrow c(2+x) = \frac{y}{(2y-1)}$$

i.e. $\dfrac{y}{(2y-1)} = c(2+x)$

Review

- $\int \tan x\,dx = \ln|\sec x| = -\ln|\cos x|$
- $\int \cot x\,dx = \ln|\sin x| = -\ln|\csc x|$
- $\int \sec x\,dx = \ln|\sec x+\tan x|$
- $\int \csc x\,dx = \ln|\csc x-\cot x|$

Question # 15

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

Solution

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

$$\Rightarrow \cos x\tan y\frac{dy}{dx} = -1$$

$$\Rightarrow \tan y\,dy = -\frac{1}{\cos x}dx$$

$$\Rightarrow \tan y\,dy = -\sec x\,dx$$

On integrating

$$\int\tan y\,dy = -\int\sec x\,dx$$

$$\Rightarrow -\ln|\cos y| = -\ln|\sec x+\tan x|-\ln c$$

$$\Rightarrow \ln|\cos y| = +\ln|\sec x+\tan x|+\ln c$$

$$\Rightarrow \ln|\cos y| = \ln|c(\sec x+\tan x)|$$

$$\Rightarrow \cos y = c(\sec x+\tan x)$$

Question # 16

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

Solution

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

$$\Rightarrow y-x\frac{dy}{dx} = 3+3x\frac{dy}{dx}$$

$$\Rightarrow y-3 = 3x\frac{dy}{dx}+x\frac{dy}{dx}$$

$$= (3x+x)\frac{dy}{dx}$$

$$\Rightarrow y-3 = 4x\frac{dy}{dx} \Rightarrow \frac{dx}{x} = 4\frac{dy}{y-3}$$

Now do yourself

Question # 17

$$\sec x+\tan y\frac{dy}{dx} = 0$$

Solution

$$\sec x+\tan y\frac{dy}{dx} = 0$$

$$\Rightarrow \tan y\frac{dy}{dx} = -\sec x$$

$$\Rightarrow \tan y\,dy = -\sec x\,dx$$

Now do yourself as Question # 15

Question # 18

$$\left(e^x+e^{-x}\right)\frac{dy}{dx} = e^x-e^{-x}$$

Solution

$$\left(e^x+e^{-x}\right)\frac{dy}{dx} = e^x-e^{-x}$$

$$\Rightarrow dy = \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

On integrating

$$\int dy = \int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$\Rightarrow y = \int\frac{\frac{d}{dx}\left(e^x+e^{-x}\right)}{e^x+e^{-x}}dx$$

$$\Rightarrow y = \ln\left(e^x+e^{-x}\right)+c$$

Question # 19

Find the general solution of the equation $\frac{dy}{dx}-x = xy^2$. Also find the perpendicular solution if $y=1$ when $x=0$.

Solution

$$\frac{dy}{dx}-x = xy^2 \quad\Rightarrow\quad \frac{dy}{dx} = x+xy^2$$

$$\Rightarrow \frac{dy}{dx} = x\left(1+y^2\right) \quad\Rightarrow\quad \frac{dy}{1+y^2} = x\,dx$$

$$\Rightarrow \int\frac{dy}{1+y^2} = \int x\,dx$$

$$\Rightarrow Tan^{-1}y = \frac{x^2}{2}+c$$

$$\Rightarrow y = Tan\left(\frac{x^2}{2}+c\right)$$

Question # 20

Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x=4$ when $t=0$

Solution

$$\frac{dx}{dt} = 2x \quad\Rightarrow\quad \frac{dx}{x} = 2dt$$

$$\Rightarrow \int\frac{dx}{x} = 2\int dt$$

$$\Rightarrow \ln x = 2t+\ln c$$

$$= \ln e^{2t}+\ln c \qquad \because \ln e^x = x$$

$$\Rightarrow \ln x = \ln ce^{2t}$$

$$\Rightarrow x = ce^{2t} \;\ldots\ldots\; \text{(i)}$$

When $t=0$ then $x=4$, putting in (i)

$$4 = ce^{2(0)} \quad\Rightarrow\quad 4 = c\,e^0$$

$$\Rightarrow 4 = c(1) \quad\Rightarrow\quad c = 4$$

Putting in (i)

$$\Rightarrow x = 4e^{2t}$$

Question # 21

Solve the differential equation $\frac{ds}{dt}+2st = 0$. Also find the perpendicular solution if $s=4e$, when $t=0$

Solution

$$\frac{ds}{dt}+2st = 0$$

$$\Rightarrow \frac{ds}{dt} = -2st \quad\Rightarrow\quad \frac{ds}{s} = -2t\,dt$$

On integrating

$$\int\frac{ds}{s} = -2\int t\,dt$$

$$\Rightarrow \ln s = -2\frac{t^2}{2}+\ln c$$

$$= -t^2+\ln c$$

$$= \ln e^{-t^2}+\ln c \qquad \because \ln e^x = x$$

$$\Rightarrow \ln s = \ln ce^{-t^2}$$

$$\Rightarrow s = ce^{-t^2} \;\ldots\ldots\; \text{(i)}$$

When $t=0$ then $s=4e$, using in (i)

$$4e = ce^{-(0)^2} \quad\Rightarrow\quad 4e = c(1)$$

$$\Rightarrow c = 4e$$

Putting in (i)

$$s = 4e\cdot e^{-t^2}$$

$$\Rightarrow s = 4e^{1-t^2}$$

Question # 22
In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution

Number of bacteria initially = 200
No. of bacteria after two hours = 2(200)
= 400
No. of bacteria after four hours = 2(400)
= 800 *Ans.*

Question # 23
A ball is thrown vertically upward with a velocity of $2450 cm/\sec$. Neglecting air resistance, find

(i) velocity of ball at any time t
(ii) distance travelled in any time t
(iii) maximum height attained by the ball.

Solution

i) When a body is projected upward its acceleration is $-g$. (where $g = 980$ cm/sec^2)

i.e. acceleration $= \dfrac{dv}{dt} = -g$,
where v is velocity of ball.

$$\Rightarrow \frac{dv}{dt} = -980$$
$$\Rightarrow dv = -980\,dt$$

On integrating
$$\int dv = -980\int dt$$
$$\Rightarrow v = -980t + c_1 \text{ .... (i)}$$

Initially, when $t = 0$ then $v = 2450$ cm/sec
$$2450 = -980(0) + c_1$$
$$\Rightarrow c_1 = 2450$$

Putting in (i)
$$\boxed{v = -980t + 2450}$$

ii) Since velocity $= v = \dfrac{dx}{dt}$
where x is height of ball.

$$\Rightarrow \frac{dx}{dt} = -980t + 2450$$
$$\Rightarrow dx = (-980t + 2450)\,dt$$

Integrating
$$\int dx = \int(-980t + 2450)\,dt$$
$$\Rightarrow x = -980\frac{t^2}{2} + 2450t + c_2$$
$$\Rightarrow x = -490t^2 + 2450t + c_2 \text{ .... (ii)}$$

Initially, when $t = 0$ then $x = 0$
$$0 = -490(0) + 2450(0) + c_2$$
$$\Rightarrow c_2 = 0$$

Putting value of c_2 in (ii)
$$\Rightarrow x = -490t^2 + 2450t + 0$$
$$\Rightarrow \boxed{x = 2450t - 490t^2}$$

iii)
$$\because v = -980t + 2450$$

When body is at max. height then $v = 0$
$$\Rightarrow -980t + 2450 = 0$$
$$\Rightarrow 980t = 2450 \quad \Rightarrow t = \frac{2450}{980}$$
$$\Rightarrow t = 2.5 \text{ sec}$$

Since $x = 2450t - 4980t^2$
When $t = 2.5$ sec
$$x = 2450(2.5) - 490(2.5)^2$$
$$= 6125 - 3062.5$$
$$= 3062.5$$

Hence ball attains max. height of 3062.5 cm.

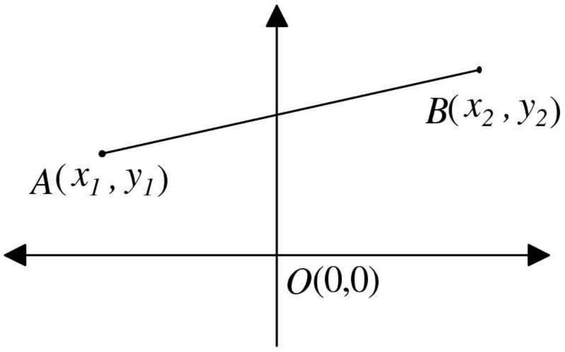
Distance Formula

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in a plane and d be a distance between A and B then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

See proof on book at page181



Ratio Formula

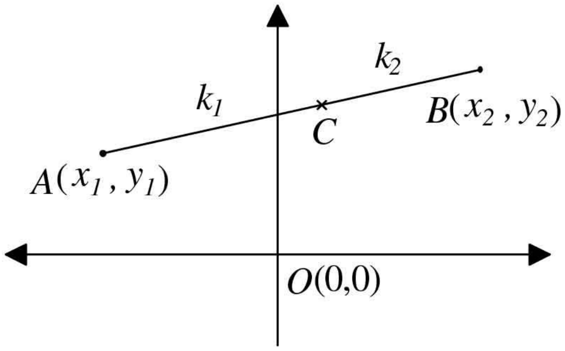
Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in a plane. The coordinates of the point C dividing the line segment AB in the ratio $k_1 : k_2$ are

$$\left(\frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}, \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}\right)$$

See proof on book at page 182

If C be the midpoint of AB i.e. $k_1 : k_2 = 1:1$ then coordinate of C becomes

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Question # 1

Describe the location in the plane of the point $P(x, y)$ for which

(i) $x > 0$ (ii) $x > 0$ and $y > 0$ (iii) $x = 0$
(iv) $y = 0$ (v) $x < 0$ and $y \geq 0$ (vi) $x = y$
(vii) $|x| = -|y|$ (viii) $|x| \geq 3$ (ix) $x > 2$ and $y = 2$
(x) x and y have opposite signs.

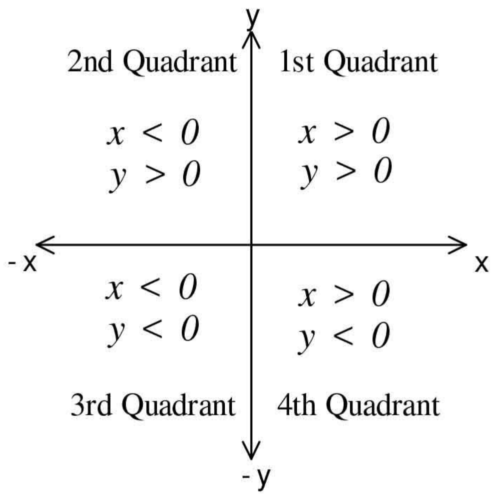
Solution

(i) $x > 0$
Right half plane

(ii) $x > 0$ and $y > 0$
The 1st quadrant.

(iii) $x = 0$
y-axis

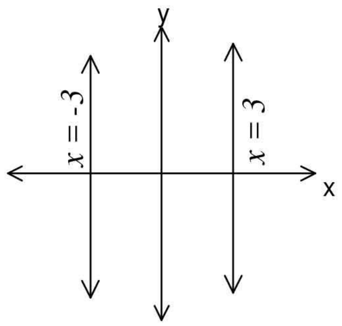
(iv) $y = 0$
x-axis



(v) $x<0$ and $y\geq 0$
2nd quadrant & negative x-axis

(vi) $x=y$
It is a line bisecting 1st and 3rd quadrant.

(vii) $|x|=-|y|$
A positive value can't equal to a negative value, except number zero, so origin,
$(0,0)$, is the only point which satisfies $|x|=-|y|$



(viii) $|x|\geq 3$
$\Rightarrow \pm x\geq 3 \quad \Rightarrow x\geq 3$ or $-x\geq 3$
$\Rightarrow x\geq 3$ or $x\leq -3$
which is the set of points lying on right side of the line $x=3$ and the points lying on left side of the line $x=-3$.

(ix) $x>2$ and $y=2$
The set of all points on the line $y=2$ for which $x>2$.

(x) x and y have opposite signs.
It is the set of points lying in 2nd and 4th quadrant.

Question # 2
Find each of the following
(i)the distance between the two given points
(ii)Midpoint of the line segment joining the two points

(a) $A(3,1):B(-2,-4)$ (b) $A(-8,3);B(2,-1)$ (c) $A\left(-\sqrt{5},-\frac{1}{3}\right);B\left(-3\sqrt{5},5\right)$

Solution
(a) $A(3,1)$; $B(-2,-4)$

(i) $|AB|=\sqrt{(-2-3)^2+(-4-1)^2}=\sqrt{(-5)^2+(-5)^2}$
$=\sqrt{25+25}=\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$

Review:
The midpoint of $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

(ii) Midpoint of AB = $\left(\frac{3-2}{2},\frac{1-4}{2}\right)=\left(\frac{1}{2},\frac{-3}{2}\right)$

(b) $A(-8,3)$; $B(2,-1)$
Do yourself as above.

(c) $A\left(-\sqrt{5},-\dfrac{1}{3}\right)$; $B\left(-3\sqrt{5},5\right)$

(i)
$$|AB|=\sqrt{\left(-3\sqrt{5}+\sqrt{5}\right)^2+\left(5+\frac{1}{3}\right)^2}=\sqrt{\left(2\sqrt{5}\right)^2+\left(\frac{16}{3}\right)^2}$$
$$=\sqrt{20+\frac{256}{9}}=\sqrt{\frac{436}{9}}=\sqrt{\frac{4\times109}{9}}=\frac{2\sqrt{109}}{3}$$

(ii) Midpoint of $AB=\left(\dfrac{-\sqrt{5}-3\sqrt{5}}{2},\dfrac{-\frac{1}{3}+5}{2}\right)=\left(\dfrac{-4\sqrt{5}}{2},\dfrac{\frac{14}{3}}{2}\right)=\left(-2\sqrt{5},\dfrac{7}{3}\right)$

Question # 3

Which of the following points are at a distance of 15 units from the origin?

(a) $\left(\sqrt{176},7\right)$ (b) $(10,-10)$ (c) $(1,15)$ (d) $\left(\dfrac{15}{2},\dfrac{15}{2}\right)$

Solution

(a) Distance of $\left(\sqrt{176},7\right)$ from origin $=\sqrt{\left(\sqrt{176}-0\right)^2+(7-0)^2}$
$$=\sqrt{(176)+(49)}$$
$$=\sqrt{(176)+(49)}=\sqrt{225}=15$$

$\Rightarrow$ the point $\left(\sqrt{176},7\right)$ is at 15 unit away from origin.

(b) Distance of $(10,-10)$ from origin $=\sqrt{(10-0)^2+(-10-0)^2}$
$$=\sqrt{100+100}=\sqrt{200}$$
$$=\sqrt{100\times2}=10\sqrt{2}\neq15$$

$\Rightarrow$ the point $(10,-10)$ is not at distance of 15 unit from origin.

(c) *Do yourself as above*

(d) Distance of $\left(\dfrac{15}{2},\dfrac{15}{2}\right)$ from origin $=\sqrt{\left(\dfrac{15}{2}-0\right)^2+\left(\dfrac{15}{2}-0\right)^2}$
$$=\sqrt{\frac{225}{4}+\frac{225}{4}}=\sqrt{\frac{225}{2}}=\frac{15}{\sqrt{2}}\neq15$$

Hence the point $\left(\dfrac{15}{2},\dfrac{15}{2}\right)$ is not at distance of 15 unit from origin.

Question # 4

Show that

(i) the point $A(0,2)$, $B(\sqrt{3},-1)$ and $C(0,-2)$ are vertices of a right triangle.

(ii) the point $A(3,1)$, $B(-2,-3)$ and $C(2,2)$ are vertices of an isosceles triangle.

(iii) the point $A(3,1)$, $B(-2,-3)$ and $C(2,2)$ and $D(4,-5)$ are vertices of a parallelogram. Is the parallelogram a square?

Solution

(i) Given: $A(0,2)$, $B(\sqrt{3},-1)$ and $C(0,-2)$

$$|AB| = \sqrt{\left(\sqrt{3}-0\right)^2 + (-1-2)^2} = \sqrt{\left(\sqrt{3}\right)^2 + (-3)^2}$$
$$= \sqrt{3+9} = \sqrt{12} \qquad \Rightarrow |AB|^2 = 12$$
$$|BC| = \sqrt{\left(0-\sqrt{3}\right)^2 + (-2+1)^2} = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$$
$$= \sqrt{3+1} = \sqrt{4} = 2 \qquad \Rightarrow |BC|^2 = 4$$
$$|CA| = \sqrt{(0-0)^2 + (2+2)^2} = \sqrt{0+(4)^2}$$
$$= \sqrt{16} = 4 \qquad \Rightarrow |CA|^2 = 16$$

$\because |AB|^2 + |BC|^2 = 12 + 4 = 16 = |CA|^2$

$\therefore$ by Pythagoras theorem A, B & C are vertices of a right triangle.

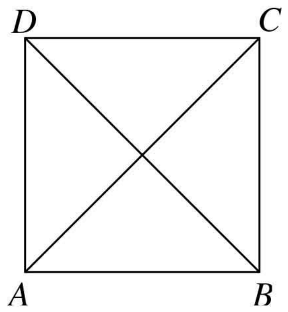
(ii) Given: $A(3,1)$, $B(-2,-3)$ and $C(2,2)$

$$|AB| = \sqrt{(-2-3))^2 + (-3-1)^2} = \sqrt{(-5)^2 + (-4)^2} = \sqrt{25+16} = \sqrt{41}$$
$$|BC| = \sqrt{(2-(-2))^2 + (2-(-3))^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{16+25} = \sqrt{41}$$
$$|CA| = \sqrt{(3-2)^2 + (1-2)^2} = \sqrt{(1)^2 + (-1)^2}$$
$$= \sqrt{1+1} = \sqrt{2}$$

$\because |AB| = |BC| \quad \Rightarrow A, B$ & C are vertices of an isosceles triangle.

(iii) Given: $A(5,2)$, $B(-2,3)$ & $C(-3,-4)$ and $D(4,-5)$

$$|AB| = \sqrt{(-2-5)^2 + (3-2)^2} = \sqrt{(-7)^2 + (1)^2}$$
$$= \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$
$$|BC| = \sqrt{(-3+2)^2 + (-4-3)^2} = \sqrt{(-1)^2 + (-7)^2}$$
$$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$
$$|CD| = \sqrt{(4+3)^2 + (-5+4)^2} = \sqrt{(7)^2 + (-1)^2}$$
$$= \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$
$$|DA| = \sqrt{(5-4)^2 + (2+5)^2} = \sqrt{(1)^2 + (7)^2}$$
$$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$



$\because |AB| = |CD|$ and $|BC| = |DA| \Rightarrow A, B, C$ and D are vertices of parallelogram.

Now $|AC|=\sqrt{(-3-5)^2+(-4-2)^2}=\sqrt{(-8)^2+(-6)^2}$

$=\sqrt{64+36}=\sqrt{100}=10$

$|BD|=\sqrt{(4+2)^2+(-5-3)^2}=\sqrt{(6)^2+(-8)^2}$

$=\sqrt{36+64}=\sqrt{100}=10$

Since all sides are equals and also both diagonals are equal therefore A,B,C,D are vertices of a square.

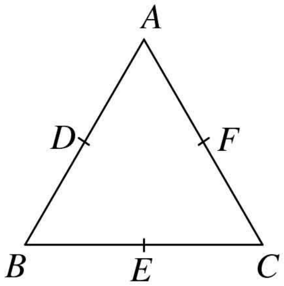
Question # 5

The midpoints of the sides of a triangle are $(1,-1)$,$(-4,-3)$ and $(-1,1)$. Find coordinates of the vertices of the triangle.

Solution

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of triangle ABC, and let $D(1,-1)$, $E(-4,-3)$ and $F(-1,1)$ are midpoints of sides AB, BC and CA respectively.

Then



$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) = (1,-1)$$

$\Rightarrow\ x_1+x_2=2\ \ldots$ (i) and $y_1+y_2=-2\ \ldots$ (ii)

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right) = (-4,-3)$$

$\Rightarrow\ x_2+x_3=-8\ldots$ (iii) and $y_2+y_3=-6\ \ldots$ (iv)

$$\left(\frac{x_3+x_1}{2},\frac{y_3+y_1}{2}\right) = (-1,1)$$

$\Rightarrow\ x_1+x_3=-2\ \ldots$ (v), and $y_1+y_3=2\ \ldots$ (vi)

Subtracting (i) and (iii)

$$\begin{array}{rcl} x_1+x_2 & = & 2 \\ \underset{-}{} \ x_2 \underset{-}{+} x_3 & = & -8\ \ldots \quad \text{(vii)} \\ \hline x_1 \quad\ \ -x_3 & = & 10 \end{array}$$

Adding (v) and (vii)

$$\begin{array}{rcl} x_1+x_3 & = & -2 \\ x_1-x_3 & = & 10 \\ \hline 2x_1 & = & 8 \end{array} \Rightarrow \boxed{x_1=4}$$

Putting value of x_1 in (i)

$4+x_2=2$

$\Rightarrow\ x_2=2-4 \Rightarrow \boxed{x_2=-2}$

Putting value of x_1 in (v)

$4+x_3=-2$

$\Rightarrow\ x_3=-2-4 \Rightarrow \boxed{x_3=-6}$

Subtracting (ii) and (iv)

$$\begin{array}{rcl} y_1+y_2 & = & -2 \\ \underset{-}{} \ y_2 \underset{-}{+} y_3 & = & \underset{+}{-}6\ \ldots \quad \text{(viii)} \\ \hline y_1 \quad\ \ -y_3 & = & 4 \end{array}$$

Adding (vi) and (viii)

$$\begin{array}{rcl} y_1+y_3 & = & 2 \\ y_1-y_3 & = & 4 \\ \hline 2y_1 & = & 6 \end{array} \Rightarrow \boxed{y_1=3}$$

Putting value of y_1 in (ii)

$3+y_2=-2$

$\Rightarrow\ y_2=-2-3 \Rightarrow \boxed{y_2=-5}$

Putting value of y_1 in (v)

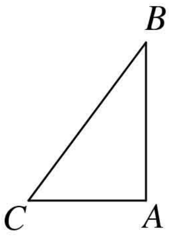
$3+y_3=2$

$\Rightarrow\ y_3=2-3 \Rightarrow \boxed{y_3=-1}$

Hence vertices of triangle are $(4,3),(-2,-5)$ & $(-6,-1)$.

Question # 6

Find h such that the point $A\left(\sqrt{3},-1\right)$, $B(0,2)$ and $C(h,-2)$ are vertices of a right angle with right angle at the vertex A.



Solution

Since ABC is a right triangle therefore by Pythagoras theorem

$$|AB|^2+|CA|^2=|BC|^2$$

$$\Rightarrow \left[\left(0-\sqrt{3}\right)^2+(2+1)^2\right]+\left[\left(\sqrt{3}-h\right)^2+(-1+2)^2\right]=(h-0)^2+(-2-2)^2$$

$$\Rightarrow [3+9]+\left[3-2\sqrt{3}h+h^2+1\right]=h^2+16$$

$$\Rightarrow 12+4-2\sqrt{3}\,h+h^2=h^2+16$$

$$\Rightarrow -2\sqrt{3}\,h=h^2+16-12-4-h^2 \quad \Rightarrow -2\sqrt{3}\,h=0 \quad \Rightarrow \boxed{h=0}.$$

Question # 7

Find h such that $A(-1,h),B(3,2)$ and $C(7,3)$ are collinear.

Solution

Points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Since given points are collinear therefore

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}=0$$

$$\Rightarrow -1(2-3)-h(3-7)+1(9-14)=0 \quad \Rightarrow -1(-1)-h(-4)+1(-5)=0$$

$$\Rightarrow 1+4h-5=0 \quad \Rightarrow 4h-4=0 \quad \Rightarrow 4h=4 \quad \Rightarrow \boxed{h=1}$$

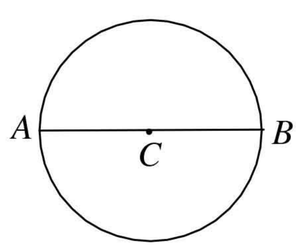
Question # 8

The points $A(-5,-2)$ and $B(5,-4)$ are end of a diameter of a circle. Find the centre and radius of the circle.

Solution

The centre of the circle is mid point of AB

i.e. centre 'C' $=\left(\dfrac{-5+5}{2},\dfrac{-2-4}{2}\right)=\left(\dfrac{0}{2},\dfrac{-6}{2}\right)=(0,-3)$



Now radius $=|AC|$

$$=\sqrt{(0+5)^2+(-3+2)^2}$$

$$=\sqrt{25+1} \quad =\sqrt{26}$$

Question # 9

Find h such that the points $A(h,1)$, $B(2,7)$ and $C(-6,-7)$ are vertices of a right triangle with right angle at the vertex A

Solution

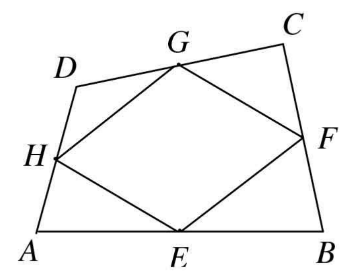
Do yourself as Question # 6

Hint: you will get a equation $h^2+4h-60=0$

Solve this quadratic equation to get two values of h.

Question # 10

A quadrilateral has the points $A(9,3), B(-7,7), C(-3,-7)$ and $D(-5,5)$ as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.



Solution

Given: $A(9,3)$, $B(-7,7)$, $C(-3,-7)$ and $D(5,-5)$

Let E, F, G and H be the mid-points of sides of quadrilateral

Coordinate of E $= \left(\frac{9-7}{2},\frac{3+7}{2}\right) = \left(\frac{2}{2},\frac{10}{2}\right) = (1,5)$

Coordinate of F $= \left(\frac{-7-3}{2},\frac{7-7}{2}\right) = \left(\frac{-10}{2},\frac{0}{2}\right) = (-5,0)$

Coordinate of G $= \left(\frac{-3+5}{2},\frac{-7-5}{2}\right) = \left(\frac{2}{2},\frac{-12}{2}\right) = (1,-6)$

Coordinate of H $= \left(\frac{9+5}{2},\frac{3-5}{2}\right) = \left(\frac{14}{2},\frac{-2}{2}\right) = (7,-1)$

Now $|EF| = \sqrt{(-5-1)^2+(0-5)^2} = \sqrt{36+25} = \sqrt{61}$

$|FG| = \sqrt{(1+5)^2+(-6-0)^2} = \sqrt{36+36} = \sqrt{72}=6\sqrt{2}$

$|GH| = \sqrt{(7-1)^2+(-1+6)^2} = \sqrt{36+25} = \sqrt{61}$

$|HE| = \sqrt{(1-7)^2+(5+1)^2} = \sqrt{36+36} = \sqrt{72}=6\sqrt{2}$

Since $|EF| = |GH|$ and $|FG| = |HE|$

Therefore $EFGH$ is a parallelogram.

Question # 11

Find h such that the quadrilateral with vertices $A(-3,0), B(1,-2,) C(5,0)$ and $D(1,h)$ is parallelogram. Is it a square?

Solution

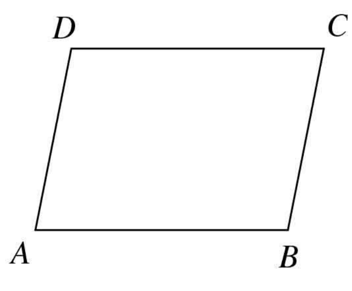
Given: $A(-3,0)$, $B(1,-2)$, $C(5,0)$, $D(1,h)$

Quadrilateral $ABCD$ is a parallelogram if

$|AB| = |CD|$ & $|BC| = |AD|$

when $|AB|=|CD|$

$\Rightarrow \sqrt{(1+3)^2+(-2-0)^2} = \sqrt{(1-5)^2+(h-0)^2}$



$$\Rightarrow \sqrt{16+4}=\sqrt{16+h^2} \quad \Rightarrow \sqrt{20}=\sqrt{16+h^2}$$

On squaring

$$20=16+h^2 \quad \Rightarrow h^2=20-16 \quad \Rightarrow h^2=4 \quad \Rightarrow h=\pm 2$$

When $h=2$, then $D(1,h)=D(1,2)$

Then $|AB| = \sqrt{(1+3)^2+(-2-0)^2} = \sqrt{16+4} = \sqrt{20}$

$|BC| = \sqrt{(5-1)^2+(0+2)^2} = \sqrt{16+4} = \sqrt{20}$

$|CA| = \sqrt{(1-5)^2+(2-0)^2} = \sqrt{16+4} = \sqrt{20}$

$|DA| = \sqrt{(-3-1)^2+(-0-2)^2} = \sqrt{16+4} = \sqrt{20}$

Now for diagonals

$|AC| = \sqrt{(5+3)^2+(0-0)^2} = \sqrt{64+0} = 8$

$|BD| = \sqrt{(1-1)^2+(2+2)^2} = \sqrt{0+16} = 4$

Since all sides are equal but diagonals $|AC| \neq |BD|$

Therefore $ABCD$ is not a square.

Now when $h=-2$, then $D(1,h)=D(1,-2)$ but we also have $B(1,-2)$

i.e. B and D represents the same point, which can not happened in quadrilateral so we can not take $h=-2$.

Question # 12

If two vertices of an equilateral triangle are $A(-3,0)$ and $B(3,0)$, find the third vertex. How many of these triangles are possible?

Solution

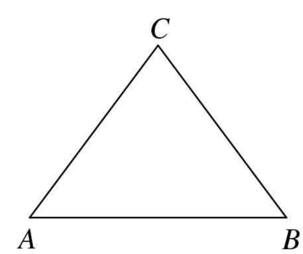
Given: $A(-3,0)$, $B(3,0)$

Let $C(x,y)$ be a third vertex of an equilateral triangle ABC.

Then $|AB| = |BC| = |CA|$

$$\Rightarrow \sqrt{(3+3)^2+(0-0)^2} = \sqrt{(x-3)^2+(y-0)^2} = \sqrt{(x+3)^2+(y-0)^2}$$

$$\Rightarrow \sqrt{36+0} = \sqrt{x^2-6x+9+y^2} = \sqrt{x^2+6x+9+y^2}$$



On squaring

$$36 = x^2+y^2-6x+9 = x^2+y^2+6x+9 \quad \ldots\ldots\ldots\ldots(i)$$

From equation (i)

$$x^2+y^2-6x+9 = x^2+y^2+6x+9$$

$$\Rightarrow x^2+y^2-6x+9-x^2-y^2-6x-9 = 0$$

$$\Rightarrow -12x = 0 \quad \Rightarrow x=0$$

Again from equation (i)

$$36 = x^2+y^2-6x+9$$

$$\Rightarrow 36 = (0)^2+y^2-6(0)+9 \qquad \because x=0$$

$$\Rightarrow 36 = y^2+9 \quad \Rightarrow y^2 = 36-9 = 27 \quad \Rightarrow y = \pm 3\sqrt{3}$$

so coordinate of C is $\left(0,3\sqrt{3}\right)$ or $\left(0,-3\sqrt{3}\right)$.

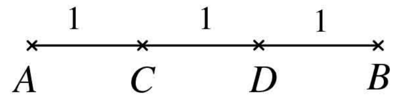
And hence two triangle can be formed with vertices $A(-3,0), B(3,0), C(0,3\sqrt{3})$ and $A(-3,0), B(3,0), C(0,-3\sqrt{3})$.

Question # 13

Find the points trisecting the join of $A(-1,4)$ and $B(6,2)$.

Solution

Given: $A(-1,4)$, $B(6,2)$



Let C and D be points trisecting A and B

Then $AC:CB = 1:2$

So coordinate of $C = \left(\dfrac{1(6)+2(-1)}{1+2}, \dfrac{1(2)+2(4)}{1+2}\right)$

$= \left(\dfrac{6-2}{3}, \dfrac{2+8}{3}\right) = \left(\dfrac{4}{3}, \dfrac{10}{3}\right)$

Also $AD:DB = 2:1$

So coordinate of $D = \left(\dfrac{2(6)+1(-1)}{2+1}, \dfrac{2(2)+1(4)}{2+1}\right)$

$= \left(\dfrac{12-1}{3}, \dfrac{4+4}{3}\right) = \left(\dfrac{11}{3}, \dfrac{8}{3}\right)$

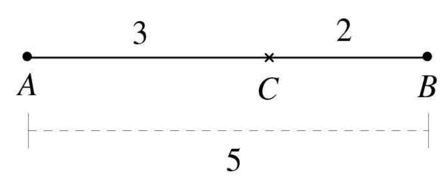
Hence $\left(\dfrac{4}{3}, \dfrac{10}{3}\right)$ and $\left(\dfrac{11}{3}, \dfrac{8}{3}\right)$ are points trisecting A and B.

Question # 14

Find the point three-fifth of the way along the line segment from $A(-5,8)$ to $B(5,3)$.

Solution

Given: $A(-5,8)$, $B(5,3)$



Let $C(x, y)$ be a required point

$\because$ $AC:CB = 3:2$

$\therefore$ Co-ordinate of $C = \left(\dfrac{3(5)+2(-5)}{3+2}, \dfrac{3(3)+2(8)}{3+2}\right)$

$= \left(\dfrac{15-10}{5}, \dfrac{9+16}{5}\right) = \left(\dfrac{5}{5}, \dfrac{25}{5}\right) = (1, 5)$

Question # 15

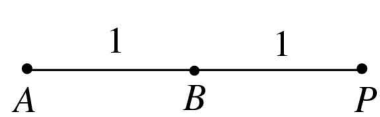
Find the point P on the joint of $A(1,4)$ and $B(5,6)$ that is twice as far from A as B is from A and lies

(i) on the same side of A as B does. (ii) on the opposite side of A as B does.

Solution

Given: $A(1,4)$, $B(5,6)$

(i) Let $P(x, y)$ be required point, then



$AB : AP = 1:2$

$\Rightarrow AB : BP = 1:1$ i.e. B is midpoint of AP

Then $B(5,6) = \left(\dfrac{1+x}{2}, \dfrac{4+y}{2}\right)$

$\Rightarrow 5 = \dfrac{1+x}{2}$ and $6 = \dfrac{4+y}{2}$

$\Rightarrow 10 = 1+x$ and $12 = 4+y$

$\Rightarrow x = 10-1$, $y = 12-4$

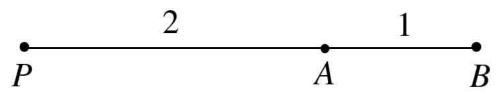
$= 9$, $= 8$

Hence $P(9,8)$ is required point.

(ii) Since $PA : AB = 2:1$

$\Rightarrow A(1,4) = \left(\dfrac{2(5)+1(x)}{2+1}, \dfrac{2(6)+1(y)}{2+1}\right)$

$= \left(\dfrac{10+x}{3}, \dfrac{12+y}{3}\right)$



$\Rightarrow 1 = \dfrac{10+x}{3}$ and $4 = \dfrac{12+y}{3}$

$\Rightarrow 3 = 10+x$ and $12 = 12+y$

$\Rightarrow x = 3-10$ and $y = 12-12$

$= -7$, $= 0$

Hence $P(-7,0)$ is required point.

Question # 16

Find the point which is equidistant from the points $A(5,3), B(2,-2)$ and $C(4,2)$. What is the radius of the circumcircle of the ΔABC ?

Solution

Given: $A(5,3)$, $B(-2,2)$ and $C(4,2)$

Let $D(x,y)$ be a point equidistance from A , B and C then

$$\left|\overline{DA}\right| = \left|\overline{DB}\right| = \left|\overline{DC}\right|$$

$$\Rightarrow \left|\overline{DA}\right|^2 = \left|\overline{DB}\right|^2 = \left|\overline{DC}\right|^2$$

$$\Rightarrow (x-5)^2 + (y-3)^2 = (x+2)^2 + (y-2)^2 = (x-4)^2 + (y-2)^2 \quad \text{........ (i)}$$

From eq. (i)

$$(x-5)^2 + (y-3)^2 = (x+2)^2 + (y-2)^2$$

$$\Rightarrow x^2 - 10x + 25 + y^2 - 6y + 9 = x^2 + 4x + 4 + y^2 - 4y + 4$$

$$\Rightarrow x^2 - 10x + 25 + y^2 - 6y + 9 - x^2 - 4x - 4 - y^2 + 4y - 4 = 0$$

$$\Rightarrow -14x - 2y + 26 = 0 \quad \Rightarrow 7x + y - 13 = 0 \quad \text{......... (ii)}$$

Again from equation (i)

$$(x+2)^2 + (y-2)^2 = (x-4)^2 + (y-2)^2$$

$\Rightarrow\ x^2+4x+4+y^2-4y+4 = x^2-8x+16+y^2-4y+4$

$\Rightarrow\ 12x-12 = 0 \quad \Rightarrow\ 12x = 12 \quad \Rightarrow\ x = 1$

Put $x=1$ in eq. (ii)

$7(1)+y-13 = 0 \quad \Rightarrow\ y-6 = 0 \quad \Rightarrow\ y = 6$

Hence $(1,6)$ is required point.

Now radius of circumcircle $= \left|\overline{DA}\right|$

$= \sqrt{(5-1)^2+(3-6)^2} = \sqrt{16+9} = \sqrt{25} = 5$ units

Intersection of Median

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of triangle.

Intersection of median is called centroid of triangle and can be determined as

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

See proof at page 184

Centre of In-Circle (In-Centre)

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of triangle.

And $|AB| = c$, $|BC| = a$, $|CA| = b$

Then in-centre of triangle $= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

See proof at page 184

Question # 17

The points $(4,-2),(-2,4)$ and $(5,5)$ are the vertices of a triangle. Find in-centre of the triangle.

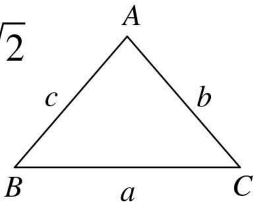
Solution

Let $A(4,-2)$, $B(-2,4)$, $C(5,5)$ are vertices of triangle then

$a = |BC| = \sqrt{(5+2)^2+(5-4)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$

$b = |CA| = \sqrt{(4-5)^2+(-2-5)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$

$c = |AB| = \sqrt{(-2-4)^2+(4+2)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$



Now

In-centre $= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

$= \left(\dfrac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}},\dfrac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right)$

$= \left(\dfrac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}},\dfrac{-10\sqrt{2}+20\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}\right)$

$= \left(\dfrac{40\sqrt{2}}{16\sqrt{2}},\dfrac{40\sqrt{2}}{16\sqrt{2}}\right) = \left(\dfrac{5}{2},\dfrac{5}{2}\right)$

Question # 18

Find the points that divide the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.

Solution

Given: $A(x_1, y_1)$, $B(x_2, y_2)$

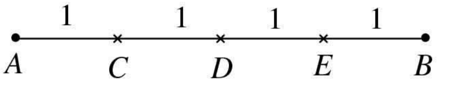
Let C, D and E are points dividing AB into four equal parts.

$\because\ AC:CB = 1:3$

$$\Rightarrow \text{ Co-ordinates of } C = \left(\frac{1(x_2)+3(x_1)}{1+3}, \frac{1(y_2)+3(y_1)}{1+3}\right) = \left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$$

Now $AD:DB = 2:2$

$= 1:1$ i.e. D is midpoint of AB.



$$\Rightarrow \text{ Co-ordinates of } D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Now $AE:EB = 3:1$

$$\Rightarrow \text{ Co-ordinates of } E = \left(\frac{3(x_2)+1(x_1)}{3+1}, \frac{3(y_2)+1(y_1)}{3+1}\right) = \left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$$

Hence $\left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and $\left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$ are the points dividing AB into four equal parts.

Question # 1

The two points P and O' are given in $xy-coordinates$ system. Find the $XY-coordinates$ of P referred to the translated axes $O'X$ and $O'Y$

(i) $P(3,2);O'(1,3)$

(ii) $P(-2,6);O'(-3,2)$

(iii) $P(-6,-8);O'(-4,-6)$

(iv) $P\left(\frac{3}{2},\frac{5}{2}\right);O'\left(-\frac{1}{2},\frac{7}{2}\right)$

Solution

(i) Since $P(x,y) = P(3,2)$

i.e. $x=3$ and $y=2$

$O'(h,k) = O'(1,3)$

i.e. $h=1$ and $k=3$

$\because\ X = x-h$

$= 3-1 = 2$

Also $Y = y-k$

$= 2-3 = -1$

Hence $(2,-1)$ is point P in $XY-$coordinates.

(ii) *Do yourself*

(iii) *Do yourself*

(iv) Since $P(x,y) = P\left(\frac{3}{2},\frac{5}{2}\right)$

i.e. $x = \frac{3}{2}$ and $y = \frac{5}{2}$

$O'(h,k) = O'\left(-\frac{1}{2},\frac{7}{2}\right)$

i.e. $h=-\frac{1}{2}$ and $k=\frac{7}{2}$

$\because\ X = x-h$

$= \frac{3}{2}-\left(-\frac{1}{2}\right) = 2$

And $Y = y-k$

$= \frac{5}{2}-\frac{7}{2} = -1$

Hence $(2,-1)$ are coordinates of P in XY-axes.

Question # 2

The $xy-coordinates$ axes are translated through the point O' whose coordinates are given in $xy-coordinates$. The coordinates of P are given in the $XY-coordinates$ system. Find the coordinates of P in $xy-coordinates$ system.

(i) $P(8,10);O'(3,4)$

(ii) $P(-5,-3);O'(-2,-6)$

(iii) $P\left(-\frac{3}{4},-\frac{7}{6}\right);O'\left(\frac{1}{4},-\frac{1}{6}\right)$

(iv) $P(4,-3);O'(-2,3)$

Solution

(i) $\because\ P(X,Y) = P(8,10)$

$\Rightarrow X=8$ and $Y=10$

$O'(h,k) = O'(3,4)$

$\Rightarrow h=3$ and $k=4$

$\because\ X = x-h$

$\Rightarrow 8 = x-3$

$\Rightarrow x = 8+3 \quad \Rightarrow x = 11$

Also $Y = y-k$

$\Rightarrow 10 = y-4$

$\Rightarrow y = 10+4 \quad \Rightarrow y = 14$

Hence $(11,14)$ are coordinates of P in xy-axes.

(ii) *Do yourself*

(iii) $\because\ P(X,Y) = P\left(-\frac{3}{4},-\frac{7}{6}\right)$

$\Rightarrow X=-\frac{3}{4}$ and $Y=-\frac{7}{6}$

$O'(h,k) = O'\left(\frac{1}{4},-\frac{1}{6}\right)$

$\Rightarrow h=\frac{1}{4}$ and $k=-\frac{1}{6}$

$\because\ X = x-h$

$\Rightarrow -\frac{3}{4} = x-\frac{1}{4}$

$\Rightarrow x = -\frac{3}{4}+\frac{1}{4} \quad \Rightarrow x = -\frac{1}{2}$

Also $Y = y-k$

$$\Rightarrow -\frac{7}{6} = y+\frac{1}{6}$$

$$\Rightarrow y = -\frac{7}{6}-\frac{1}{6} \quad \Rightarrow y = -\frac{4}{3}$$

Hence $\left(-\frac{1}{2},-\frac{4}{3}\right)$ is the required point.

(iv) *Do yourself*

Rotation of Axes

Let (x, y) be the coordinates of point P in xy-coordinate system. If the axes are rotated through at angle of θ and (X,Y) are coordinate of P in new XY-coordinate system then

$$X = x\cos\theta + y\sin\theta$$

$$Y = y\cos\theta - x\sin\theta$$

Question # 3

The $xy-coordinates$ axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the $XY-coordinates$ of the point P with the given $xy-coordinates$.

(i) $P(5,3);\theta=45^\circ$

(ii) $P(3,-7);\theta=30^\circ$

(iii) $P(11,-15);\theta=60^\circ$

(iv) $P(15,10);\theta=\arctan\frac{1}{3}$

Solution

(i) $\because P(x,y) = P(5,3)$

$\Rightarrow x=5$ & $y=3$, $\theta=45^\circ$

Since $X = x\cos\theta + y\sin\theta$

$$= 5\cos 45^\circ + 3\sin 45^\circ$$

$$= 5\left(\frac{1}{\sqrt{2}}\right)+3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(5+3)$$

$$= \frac{8}{\sqrt{2}} = \frac{4\times 2}{\sqrt{2}} = 4\sqrt{2}$$

Now $Y = y\cos\theta - x\sin\theta$

$$= 3\cos 45^\circ - 5\sin 45^\circ$$

$$= 3\left(\frac{1}{\sqrt{2}}\right)-5\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(3-5)$$

$$= -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

Hence the required point is $\left(4\sqrt{2},-\sqrt{2}\right)$.

(ii) $\because P(x,y) = P(3,-7)$

$\Rightarrow x=3$ & $y=-7$, $\theta=30^\circ$

Since $X = x\cos\theta + y\sin\theta$

$$= 3\cos 30^\circ - 7\sin 30^\circ$$

$$\Rightarrow X = 3\left(\frac{\sqrt{3}}{2}\right)-7\left(\frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}-7}{2}$$

Now $Y = y\cos\theta - x\sin\theta$

$$= -7\cos 30^\circ - 3\sin 30^\circ$$

$$= -7\left(\frac{\sqrt{3}}{2}\right)-3\left(\frac{1}{2}\right) = \frac{-7\sqrt{3}-3}{2}$$

Hence the required point is

$$\left(\frac{3\sqrt{3}-7}{2},\frac{-7\sqrt{3}-3}{2}\right).$$

(iv) *Do yourself*

(iv) $\because P(x,y) = P(15,10)$

$\Rightarrow x=15$ & $y=10$

Also $\theta=\tan^{-1}\left(\frac{1}{3}\right)$

$$\Rightarrow \tan\theta=\frac{1}{3}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta}=\frac{1/\sqrt{10}}{3/\sqrt{10}}$$

$$\tan\theta=\frac{y}{x}=\frac{1}{3}$$

$$\Rightarrow x=3\,,\ y=1$$

$$r=\sqrt{x^2+y^2}$$

$$=\sqrt{3^2+1^2}$$

$$=\sqrt{10}$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{10}}\ ,\quad \cos\theta=\frac{3}{\sqrt{10}}$$

Now $X = x\cos\theta + y\sin\theta$

$$= 15\left(\frac{3}{\sqrt{10}}\right)+10\left(\frac{1}{\sqrt{10}}\right)$$

$$= \frac{1}{\sqrt{10}}(45+10) = \frac{55}{\sqrt{10}}$$

$Y = y\cos\theta - x\sin\theta$

$$=10\left(\frac{3}{\sqrt{10}}\right)-15\left(\frac{1}{\sqrt{10}}\right)$$

$$= \frac{1}{\sqrt{10}}(30-15) = \frac{15}{\sqrt{10}}$$

Hence the required point is $\left(\frac{55}{\sqrt{10}},\frac{15}{\sqrt{10}}\right)$.

(iv)

$\because P(x,y) = P(15,10)$

$\Rightarrow x=15 \quad \& \quad y=10$

Also $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\Rightarrow \tan\theta = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \dfrac{1/2}{\sqrt{3}/2}$

$\Rightarrow \sin\theta = \dfrac{1}{2}\ , \qquad \cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{y}{x} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow x = \sqrt{3}\ ,\ y = 1$

$r = \sqrt{x^2+y^2}$

$= \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$

$= \sqrt{4} = 2$

Now $X = x\cos\theta + y\sin\theta$

$= 15\left(\dfrac{\sqrt{3}}{2}\right) + 10\left(\dfrac{1}{2}\right)$

$= \dfrac{15\sqrt{3}+10}{2}$

$Y = y\cos\theta - x\sin\theta$

$= 10\left(\dfrac{\sqrt{3}}{2}\right) - 15\left(\dfrac{1}{2}\right)$

$= \dfrac{10\sqrt{3}-15}{2}$

Hence the required point is

$\left(= \dfrac{15\sqrt{3}+10}{2}, \dfrac{10\sqrt{3}-15}{2}\right).$

Question # 4

The $xy-coordinates$ axes are rotated about the origin through the indicated angle and the new axes are OX and OY , Find the $xy-coordinates$ of P with the given $XY-coordinates$.

(i) $P(-5,3);\theta = 30^\circ$

(ii) $P\left(-7\sqrt{2}, 5\sqrt{2}\right);\theta = 45^\circ$

Solution

(i) $\because P(X,Y)$

$\Rightarrow X = -5 \ \& \ Y$

Also $\theta = 30^\circ$

Therefore $\sin\theta = \dfrac{1}{2} \quad \& \quad \cos\theta = \dfrac{\sqrt{3}}{2}$

Now $X = x\cos\theta + y\sin\theta$

$\Rightarrow -5 = x\left(\dfrac{\sqrt{3}}{2}\right) + y\left(\dfrac{1}{2}\right)$

$\Rightarrow \sqrt{3}x + y = -10$ …….. (i)

Also $Y = y\cos\theta - x\sin\theta$

$\Rightarrow 3 = y\left(\dfrac{\sqrt{3}}{2}\right) - x\left(\dfrac{1}{2}\right)$

$\Rightarrow 6 = \sqrt{3}y - x$

$\Rightarrow x = \sqrt{3}y - 6$ ……… (ii)

Putting value of x in (i)

$\sqrt{3}\left(\sqrt{3}y-6\right) + y = -10$

$\Rightarrow 3y - 6\sqrt{3} + y = -10$

$\Rightarrow 4y = -10 + 6\sqrt{3}$

$\Rightarrow y = \dfrac{-10+6\sqrt{3}}{4}$

$= \dfrac{-5+3\sqrt{3}}{2}$

Putting value of y in (ii)

$x = \sqrt{3}\left(\dfrac{-5+3\sqrt{3}}{2}\right) - 6$

$= \dfrac{-5\sqrt{3}+9}{2} - 6 = \dfrac{-5\sqrt{3}+9-12}{2}$

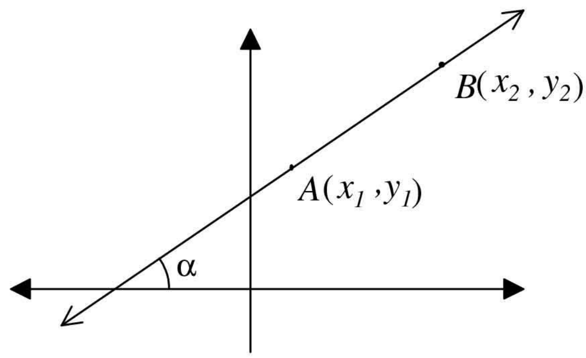
$= \dfrac{-5\sqrt{3}-3}{2}$

Hence $\left(\dfrac{-5\sqrt{3}-3}{2}, \dfrac{-5+3\sqrt{3}}{2}\right)$ is required point.

(ii) *Do yourself*

Inclination of a Line:

The angle α $(0^\circ \le \alpha < 180^\circ)$ measure anti-clockwise from positive $x-$axis to the straight line l is called *inclination* of a line l.



Slope or Gradient of Line

The slope m of the line l is defined by:

$$m = \tan\alpha$$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two distinct points on the line l then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

See proof on book at page: 191

Note: l is horizontal, iff $m = 0$ $(\because \alpha = 0^\circ)$

l is vertical, iff $m = \infty$ i.e. m is not defined. $(\because \alpha = 90^\circ)$

If slope of AB = slope of BC, then the points A, B and C are collinear i.e. lie on the same line.

Theorem

The two lines l_1 and l_2 with respective slopes m_1 and m_2 are

(i) Parallel iff $m_1 = m_2$

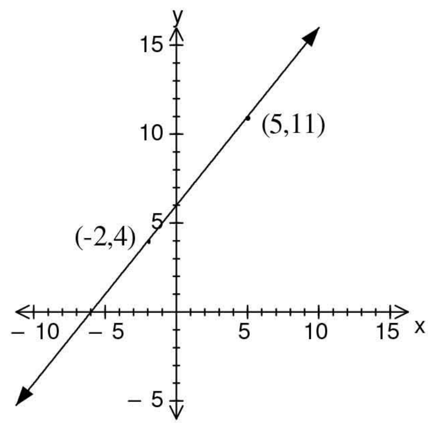
(ii) Perpendicular iff $m_1 m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

Question # 1

Find the slope and inclination of the line joining the points:

(i) $(-2,4)$; $(5,11)$ (ii) $(3,-2)$; $(2,7)$

(iii) $(4,6)$; $(4,8)$



Solution

(i) $(-2,4)$; $(5,11)$

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{11-4}{5+2} = \dfrac{7}{7} = 1$

Since $\tan\alpha = m = 1$

$\Rightarrow \alpha = \tan^{-1}(1) = 45^\circ$

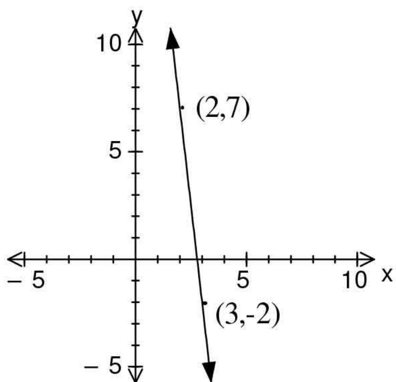
(ii) $(3,-2)$; $(2,7)$

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7+2}{2-3} = \dfrac{9}{-1} = -9$

Since $\tan\alpha = m = -9$

$\Rightarrow -\tan\alpha = 9 \Rightarrow \tan(180-\alpha) = 9$

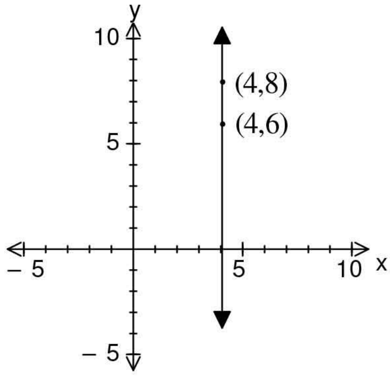
$\Rightarrow 180 - \alpha = \tan^{-1}(9)$



$\Rightarrow\ 180-\alpha = 83^\circ 40'$

$\Rightarrow\ \alpha = 180-83^\circ 40' = 96^\circ 20'$

(ii) $(4,6)$; $(4,8)$



Slope $m=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{8-6}{4-4}=\dfrac{2}{0}=\infty$

Since $\tan\alpha = m = \infty$

$\Rightarrow\ \alpha=\tan^{-1}(\infty)$

$=90^\circ$

Question # 2

In the triangle $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$, find the slope of

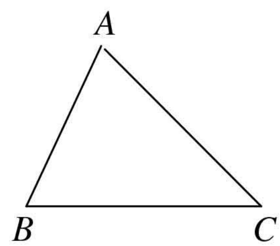
(i) each side of the triangle (ii) each median of the triangle

(iii) each altitude of the triangle

Solution

Since $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$ are vertices of triangle therefore

(i) Slope of side $AB=\dfrac{2-6}{-4-8}=\dfrac{-4}{-12}=\dfrac{1}{3}$

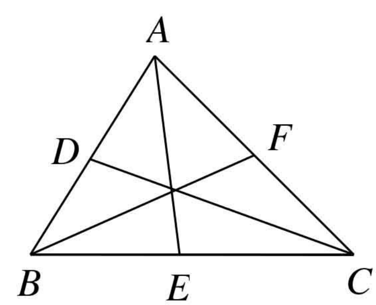


Slope of side $BC=\dfrac{-6-2}{-2+4}=\dfrac{-8}{2}=-4$

Slope of side $CA=\dfrac{6+6}{8+2}=\dfrac{12}{10}=\dfrac{6}{5}$

(ii) Let D,E and F are midpoints of sides AB, BC and CA respectively.

Then



Coordinate of $D=\left(\dfrac{8-4}{2},\dfrac{6+2}{2}\right)=\left(\dfrac{4}{2},\dfrac{8}{2}\right)=(2,4)$

Coordinate of $E=\left(\dfrac{-4-2}{2},\dfrac{2-6}{2}\right)=\left(\dfrac{-6}{2},\dfrac{-4}{2}\right)=(-3,-2)$

Coordinate of $F=\left(\dfrac{-2+8}{2},\dfrac{-6+6}{2}\right)=\left(\dfrac{6}{2},\dfrac{0}{2}\right)=(3,0)$

Hence Slope of median $AE=\dfrac{-2-6}{-3-8}=\dfrac{-8}{-11}=\dfrac{8}{11}$

Slope of median $BF=\dfrac{0-2}{3+4}=\dfrac{-2}{7}$

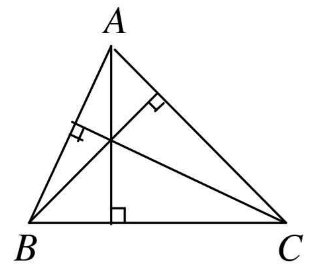
Slope of median $CD=\dfrac{4+6}{2+2}=\dfrac{10}{4}=\dfrac{5}{2}$

(iii) Since altitudes are perpendicular to the sides of a triangle therefore

Slope of altitude from vertex $A = \dfrac{-1}{\text{slope of side } BC} = \dfrac{-1}{-4} = \dfrac{1}{4}$

Slope of altitude from vertex $B = \dfrac{-1}{\text{slope of side } AC} = \dfrac{-1}{6/5} = -\dfrac{5}{6}$

Slope of altitude from vertex $C = \dfrac{-1}{\text{slope of side } AB} = \dfrac{-1}{1/3} = -3$



Question # 3
By means of slopes, show that the following points lie in the same line:
(a) $(-1,-3)$; $(1,5)$; $(2,9)$ (b) $(4,-5);(7,5);(10,15)$
(c) $(-4,6);(3,8);(10,10)$ (d) $(a,2b);(c,a+b);(2c-a,2a)$

Solution
(a) Let $A(-1,-3)$, $B(1,5)$ and $C(2,9)$ be given points

$$\text{Slope of } AB = \frac{5+3}{1+1} = \frac{8}{2} = 4$$
$$\text{Slope of } BC = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

Since slope of AB = slope of BC
Therefore A, B and C lie on the same line.

(b) *Do yourself as above*

(c) *Do yourself as above*

(d) Let $A(a,2b), B(c,a+b)$ and $C(2c-a,2a)$ be given points.

$$\text{Slope of } AB = \frac{(a+b)-2b}{c-a} = \frac{a-b}{c-a}$$
$$\text{Slope of } BC = \frac{2a-(a+b)}{(2c-a)-c} = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$$

Since slope of AB = slope of BC
Therefore A, B and C lie on the same line.

Question # 4
Find k so that the line joining $A(7,3)$; $B(k,-6)$ and the line joining $C(-4,5)$; $D(-6,4)$ are (i) parallel (ii) perpendicular.

Solution
Since $A(7,3)$, $B(k,-6)$, $C(-4,5)$ and $D(-6,4)$

$$\text{Therefore slope of } AB = m_1 = \frac{-6-3}{k-7} = \frac{-9}{k-7}$$
$$\text{Slope of } CD = m_2 = \frac{4-5}{-6+4} = \frac{-1}{-2} = \frac{1}{2}$$

(i) If AB and CD are parallel then $m_1 = m_2$

$$\Rightarrow \frac{-9}{k-7}=\frac{1}{2} \qquad \Rightarrow -18=k-7$$

$$\Rightarrow k=-18+7 \qquad \Rightarrow \boxed{k=-11}$$

(ii) If AB and CD are perpendicular then $m_1m_2=-1$

$$\Rightarrow \left(\frac{-9}{k-7}\right)\left(\frac{1}{2}\right)=-1 \qquad \Rightarrow -9=-2(k-7)$$

$$\Rightarrow 9=2k-14 \qquad \Rightarrow 2k=9+14=23$$

$$\Rightarrow \boxed{k=\frac{23}{2}}$$

Question # 5

Using slopes, show that the triangle with its vertices $A(6,1), B(2,7)$ and $C(-6,-7)$ is a right triangle.

Solution

Since $A(6,1), B(2,7)$ and $C(-6,-7)$ are vertices of triangle therefore

$$\text{Slope of } \overline{AB} = m_1 = \frac{7-1}{2-6}=\frac{6}{-4}=-\frac{3}{2}$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{-7-7}{-6-2}=\frac{-12}{-8}=\frac{7}{4}$$

$$\text{Slope of } \overline{CA} = m_3 = \frac{1+7}{6+6}=\frac{8}{12}=\frac{2}{3}$$

Since $\quad m_1m_3=\left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)=-1$

$\Rightarrow$ The triangle ABC is a right triangle with $m\angle A=90^\circ$

> **REMEMBER**
> The symbols
> (*i*) || stands for 'parallel"
> (*ii*) $\nparallel$ stands for "not parallel"
> (*iii*) $\perp$ stands for "perpendicular"

Question # 6

The three points $A(7,-1), B(-2,2)$ and $C(1,4)$ are consecutive vertices of a parallelogram. Find the fourth vertex.

Solution

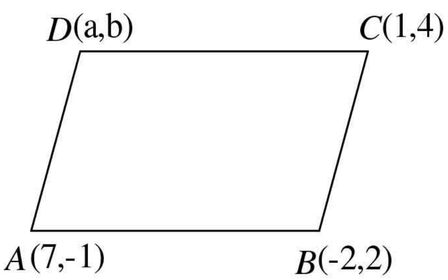
Let $D(a,b)$ be a fourth vertex of the parallelogram.

$$\text{Slope of } \overline{AB}=\frac{2+1}{-2-7}=\frac{3}{-9}=-\frac{1}{3}$$

$$\text{Slope of } \overline{BC}=\frac{4-2}{1+2}=\frac{2}{3}$$

$$\text{Slope of } \overline{CD}=\frac{b-4}{a-1}$$

$$\text{Slope of } \overline{DA}=\frac{-1-b}{7-a}$$



Since $ABCD$ is a parallelogram therefore

Slope of $\overline{AB}=$ Slope of $\overline{CD}$

$$\Rightarrow -\frac{1}{3}=\frac{b-4}{a-1} \qquad \Rightarrow -(a-1)=3(b-4)$$

$$\Rightarrow -a+1-3b+12=0 \qquad \Rightarrow -a-3b+13=0 \ ...(\text{i})$$

Also slope of $\overline{BC}$ = slope of $\overline{DA}$

$$\Rightarrow \frac{2}{3}=\frac{-1-b}{7-a} \qquad \Rightarrow 2(7-a)=3(-1-b) \qquad \Rightarrow 14-2a=-3-3b$$

$$\Rightarrow 14-2a+3+3b=0 \quad \Rightarrow -2a+3b+17=0... \quad (\text{ii})$$

Adding (i) and (ii)

$$\begin{array}{l} -a-3b+13=0 \\ \underline{-2a+3b+17=0} \\ -3a \qquad +30=0 \end{array} \qquad \Rightarrow 3a=30 \quad \Rightarrow \boxed{a=10}$$

Putting value of a in (i)

$$-10-3b+13=0 \quad \Rightarrow -3b+3=0 \quad \Rightarrow 3b=3 \Rightarrow \boxed{b=1}$$

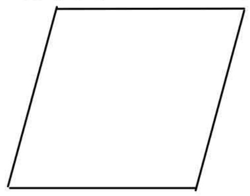
Hence $D(10,1)$ is the fourth vertex of parallelogram.

Question # 7

The points $A(-1,2), B(3,-1)$ and $C(6,3)$ are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.

Solution

Let $D(a,b)$ be a fourth vertex of rhombus.



$$\text{Slope of } \overline{AB}=\frac{-1-2}{3+1}=\frac{-3}{4}$$

$$\text{Slope of } \overline{BC}=\frac{3+1}{6-3}=\frac{4}{3}$$

$$\text{Slope of } \overline{CD}=\frac{b-3}{a-6}$$

$$\text{Slope of } \overline{DA}=\frac{2-b}{-1-a}$$

Since $ABCD$ is a rhombus therefore

Slope of $\overline{AB}$ = Slope of $\overline{CD}$

$$\Rightarrow -\frac{3}{4}=\frac{b-3}{a-6} \qquad \Rightarrow -3(a-6)=4(b-3)$$

$$\Rightarrow -3a+18=4b-12 \qquad \Rightarrow -3a+18-4b+12=0$$

$$\Rightarrow -3a-4b+30=0... \qquad (\text{i})$$

Also slope of $\overline{BC}$ = slope of $\overline{DA}$

$$\Rightarrow \frac{4}{3}=\frac{2-b}{-1-a} \qquad \Rightarrow 4(-1-a)=3(2-b)$$

$$\Rightarrow -4-4a=6-3b \qquad \Rightarrow -4-4a-6+3b=0$$

$$\Rightarrow -4a+3b-10=0 \ ... \quad (\text{ii})$$

$\times$ing eq. (i) by 3 and (ii) by 4 and adding.

$$\begin{array}{l} -9a - 12b + 90 = 0 \\ \underline{-16a + 12b - 40 = 0} \\ -25a \qquad\quad + 50 = 0 \quad \Rightarrow 25a = 50 \quad \Rightarrow \boxed{a = 2} \end{array}$$

Putting value of a in (ii)

$$-4(2) + 3b - 10 = 0 \Rightarrow 3b - 18 = 0 \quad \Rightarrow 3b = 18 \Rightarrow \boxed{b = 6}$$

Hence $D(2,6)$ is the fourth vertex of rhombus.

Now slope of diagonal $\overline{AC} = \dfrac{3-2}{6+1} = \dfrac{1}{7}$

Slope of diagonal $\overline{BD} = \dfrac{b-(-1)}{a-3} = \dfrac{6+1}{2-3} = \dfrac{7}{-1} = -7$

Since

$$(\text{Slope of } \overline{AC})(\text{Slope of } \overline{BD}) = \left(\frac{1}{7}\right)(-7) = -1$$

$\Rightarrow$ Diagonals of a rhombus are $\perp$ to each other.

Question # 8

Two pairs of points are given. Find whether the two lines determined by these points are:

(i) Parallel (ii) perpendicular (iii) none

(a) $(1,-2),(2,4) \text{ and } (4,1),(-8,2)$ (b) $(-3,4),(6,2) \text{ and } (4,5),(-2,-7)$

Solution

(a) Slope of line joining $(1,-2)$ and $(2,4) = m_1 = \dfrac{4+2}{2-1} = \dfrac{6}{1} = 6$

Slope of line joining $(4,1)$ and $(-8,2) = m_2 = \dfrac{2-1}{-8-4} = \dfrac{1}{-12}$

Since $m_1 \neq m_2$

Also $m_1 m_2 = 6 \cdot \dfrac{1}{-12} = -\dfrac{1}{2} \neq -1$

$\Rightarrow$ lines are neither parallel nor perpendicular.

(b) *Do yourself as above.*

Equation of Straight Line:

(i) **Slope-intercept form**

Equation of straight line with slope m and $y-$intercept c is given by:

$$\boxed{y = mx + c}$$

See proof on book at page 194

(ii) **Point-slope form**

Let m be a slope of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{y - y_1 = m(x - x_1)}$$

See proof on book at page 195

(iii) **Symmetric form**

Let α be an inclination of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{\frac{y-y_1}{\cos\alpha} = \frac{x-x_1}{\sin\alpha}}$$

See proof on book at page 195

(iv) **Two-points form**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points lie on a line then it's equation is given by:

$$\boxed{y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)} \quad \text{or} \quad \boxed{y-y_2 = \frac{y_2-y_1}{x_2-x_1}(x-x_2)} \quad \text{or} \quad \boxed{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0}$$

See proof on book at page 196

(v) **Two-intercept form**

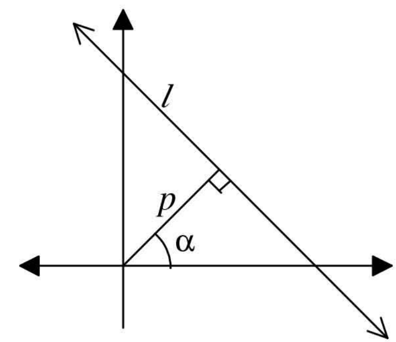
When a line intersect $x-$axis at $x=a$ and $y-$axis at $y=b$ i.e. $x-$intercept $=a$ and $y-$intercept $=b$, then equation of line is given by:

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

See proof on book at page 197

(vi) **Normal form**

Let p denoted length of perpendicular from the origin to the line and α is the angle of the perpendicular from +ive x-axis then equation of line is given by:



$$\boxed{x\cos\alpha + y\sin\alpha = p}$$

See proof on book at page 198

Question # 9

Find an equation of

(a) the horizontal line through $(7,-9)$

(b) the vertical line through $(-5,3)$

(c) the line bisecting the first and third quadrants.

(d) the line bisecting the second and fourth quadrants.

Solution

(a) Since slope of horizontal line $= m = 0$

$$\& \ (x_1, y_1) = (7,-9)$$

therefore equation of line:

$$y-(-9) = 0(x-7)$$

$$\Rightarrow \ y+9=0 \quad \text{Answer}$$

(b) Since slope of vertical line $m = \infty = \dfrac{1}{0}$

$$\& \ (x_1, y_1) = (-5,3)$$

therefore required equation of line

$$y-3=\infty\left(x-(-5)\right)$$

$$\Rightarrow\ y-3=\frac{1}{0}(x+5) \qquad \Rightarrow\ 0(y-3)=1(x+5)$$

$$\Rightarrow\ x+5=0 \qquad \text{Answer}$$

(c) The line bisecting the first and third quadrant makes an angle of 45° with the $x-$axis therefore slope of line $=m=\tan 45^\circ=1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=1(x-0) \quad\Rightarrow\ y=x$$

$$\Rightarrow\ x-y=0 \qquad \text{Answer}$$

(d) The line bisecting the second and fourth quadrant makes an angle of 135° with $x-$axis therefore slope of line $=m=\tan 135^\circ=-1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=-1(x-0) \quad\Rightarrow\ y=-x$$

$$\Rightarrow\ x+y=0 \qquad \text{Answer}$$

Question # 10

Find an equation of the line

(a) through $A(-6,5)$ having slope 7

(b) through $(8,-3)$ having slope 0

(c) through $(-8,5)$ having slope undefined

(d) through $(-5,-3)$ and $(9,-1)$

(e) $y-intercept\ -7$ and $slope-5$

(f) $x-intercept:-3$ and $y-intercept:-4$

(g) $x-intercept:-9$ and $slope:-4$

Solution

(a) $\because\ (x_1,y_1)=(-6,5)$

and slope of line $=m=7$

so required equation

$$y-5=7\left(x-(-6)\right)$$

$$\Rightarrow\ y-5=7(x+6) \qquad \Rightarrow\ y-5=7x+42$$

$$\Rightarrow\ 7x+42-y+5=0 \qquad \Rightarrow\ 7x-y+47=0 \qquad \text{Answer}$$

(b) *Do yourself as above.*

(c) $\because\ (x_1,y_1)=(-8,5)$

and slope of line $=m=\infty$

So required equation

$$y-5=\infty\left(x-(-8)\right)$$

$$\Rightarrow\ y-5=\frac{1}{0}(x+8) \qquad \Rightarrow\ 0(y-5)=1(x+8)$$

$$\Rightarrow\ x+8=0 \qquad \text{Answer}$$

(d) The line through $(-5,-3)$ and $(9,-1)$ is

$$y-(-3)=\frac{-1-(-3)}{9-(-5)}\big(x-(-5)\big) \quad \Rightarrow\ y+3=\frac{2}{14}(x+5)$$

$$\Rightarrow\ y+3=\frac{1}{7}(x+5) \qquad \Rightarrow\ 7y+21=x+5$$

$$\Rightarrow\ x+5-7y-21=0 \qquad \Rightarrow\ x-7y-16=0 \qquad \text{Answer}$$

(e) $\because\ y-$intercept $=-7$

$\Rightarrow\ (0,-7)$ lies on a required line

Also slope $=m=-5$

So required equation

$y-(-7)=-5(x-0)$

$\Rightarrow\ y+7=-5x \qquad \Rightarrow\ 5x+y+7=0 \qquad$ Answer

(f) $\because\ x-$intercept $=-9$

$\Rightarrow\ (-9,0)$ lies on a required line

Also slope $=m=4$

Therefore required line

$$y-0=4(x+9)$$

$$\Rightarrow\ y=4x+9 \qquad \Rightarrow\ 4x-y+9=0 \qquad \text{Answer}$$

(g) $x-$intercept $=a=-3$

$y-$intercept $=b=4$

Using two-intercept form of equation line

$$\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow\ \frac{x}{-3}+\frac{y}{4}=0$$

$$\Rightarrow\ 4x-3y=-12 \qquad \times\text{ing by } -12$$

$$\Rightarrow\ 4x-3y+12=0 \qquad \text{Answer}$$

Question # 11

Find an equation of the perpendicular bisector of the segment joining the points $A(3,5)$ and $B(9,8)$

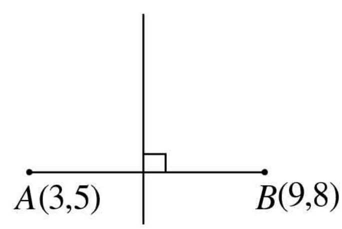
Solution

Given points $A(3,5)$ and $B(9,8)$

$$\text{Midpoint of } \overline{AB}=\left(\frac{3+9}{2},\frac{5+8}{2}\right)=\left(\frac{12}{2},\frac{13}{2}\right)=\left(6,\frac{13}{2}\right)$$

$$\text{Slope of } \overline{AB}=m=\frac{8-5}{9-3}=\frac{3}{6}=\frac{1}{2}$$

$$\text{Slope of line} \perp \text{ to } \overline{AB}=-\frac{1}{m}=-\frac{1}{1/2}=--2$$



Now equation of $\perp$ bisector having slope -2 through $\left(6,\frac{13}{2}\right)$

$$\Rightarrow\ y-\frac{13}{2}=-2(x-6)$$

$$\Rightarrow\ y-\frac{13}{2}=-2x+12 \qquad \Rightarrow\ y-\frac{13}{2}+2x-12=0$$

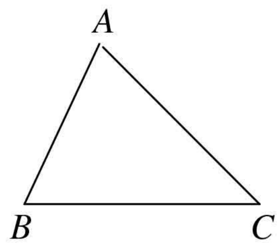
$$\Rightarrow 2x+y-\frac{37}{2}=0 \qquad \Rightarrow 4x+2y-37=0$$

Question # 12
Find equations of the sides, altitudes and medians of the triangle whose vertices are $A(-3,2), B(5,4)$ and $C(3,-8)$.

Solution

Given vertices of triangle are $A(-3,2)$, $B(5,4)$ and $C(3,-8)$.

Equation of sides:



$$\text{Slope of } \overline{AB}=m_1=\frac{4-2}{5-(-3)}=\frac{2}{8}=\frac{1}{4}$$

$$\text{Slope of } \overline{BC}=m_2=\frac{-8-4}{3-5}=\frac{-12}{-2}=6$$

$$\text{Slope of } \overline{CA}=m_3=\frac{2-(-8)}{-3-3}=\frac{10}{-6}=-\frac{5}{3}$$

Now equation of side $\overline{AB}$ having slope $\frac{1}{4}$ passing through $A(-3,2)$

[You may take $B(5,4)$ instead of $A(-3,2)$]

$$y-2=\frac{1}{4}\left(x-(-3)\right) \quad \Rightarrow 4y-8=x+3$$

$$\Rightarrow\ x+3-4y+8=0 \quad \Rightarrow \boxed{x-4y+11=0}$$

Equation of side $\overline{BC}$ having slope 6 passing through $B(5,4)$.

$$y-4=6(x-5) \qquad \Rightarrow\ y-4=6x-30$$

$$\Rightarrow\ 6x-30-y+4=0 \quad \Rightarrow \boxed{6x-y-26=0}$$

Equation of side $\overline{CA}$ having slope $-\frac{5}{3}$ passing through $C(3,-8)$

$$y-(-8)=-\frac{5}{3}(x-3) \qquad \Rightarrow\ 3(y+8)=-5(x-3)$$

$$\Rightarrow\ 3y+24=-5x+15 \quad \Rightarrow\ 5x-15+3y+24=0$$

$$\Rightarrow \boxed{5x+3y+9=0}$$

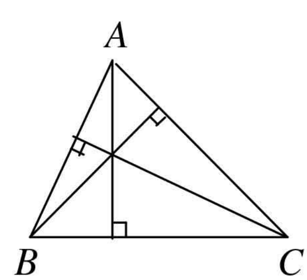
Equation of altitudes:

Since altitudes are perpendicular to the sides of triangle therefore

$$\text{Slope of altitude on } \overline{AB}=-\frac{1}{m_1}=-\frac{1}{1/4}=-4$$

Equation of altitude from $C(3,-8)$ having slope -4

$$y+8=-4(x-3) \qquad \Rightarrow\ y+8=-4x+12$$



$$\Rightarrow\ 4x-12+y+8=0 \quad \Rightarrow \boxed{4x+y-4=0}$$

Slope of altitude on $\overline{BC}=-\dfrac{1}{m_2}=-\dfrac{1}{6}$

Equation of altitude from $A(-3,2)$ having slope $-\dfrac{1}{6}$

$$y-2=-\frac{1}{6}(x+3) \quad \Rightarrow\ 6y-12=-x-3$$

$$\Rightarrow\ x+3+6y-12=0 \quad \Rightarrow \boxed{x+6y-9=0}$$

Slope of altitude on $\overline{CA}=-\dfrac{1}{m_3}=-\dfrac{1}{-5/3}=\dfrac{3}{5}$

Equation of altitude from $B(5,4)$ having slope $\dfrac{3}{5}$

$$y-4=\frac{3}{5}(x-5) \quad \Rightarrow\ 5y-20=3x-15$$

$$\Rightarrow\ 3x-15-5y+20=0 \quad \Rightarrow \boxed{3x-5y+5=0}$$

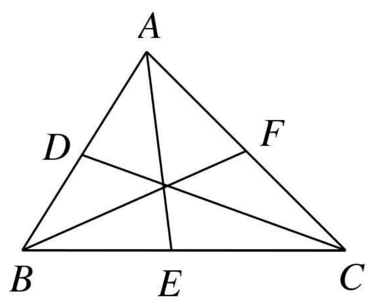
Equation of Medians:

Suppose D,E and F are midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

Then coordinate of $D=\left(\dfrac{-3+5}{2},\dfrac{2+4}{2}\right)=\left(\dfrac{2}{2},\dfrac{6}{2}\right)=(1,3)$

Coordinate of $E=\left(\dfrac{5+3}{2},\dfrac{4-8}{2}\right)=\left(\dfrac{8}{2},\dfrac{-4}{2}\right)=(4,-2)$

Coordinate of $F=\left(\dfrac{3-3}{2},\dfrac{-8+2}{2}\right)=\left(\dfrac{0}{2},\dfrac{-6}{2}\right)=(0,-3)$



Equation of median $\overline{AE}$ by two-point form

$$y-2=\frac{-2-2}{4-(-3)}(x-(-3))$$

$$\Rightarrow\ y-2=\frac{-4}{7}(x+3) \quad \Rightarrow\ 7y-14=-4x-12$$

$$\Rightarrow\ 7y-14+4x+12=0 \quad \Rightarrow \boxed{4x+7y-2=0}$$

Equation of median $\overline{BF}$ by two-point form

$$y-4=\frac{-3-4}{0-5}(x-5)$$

$$\Rightarrow\ y-4=\frac{-7}{-5}(x-5) \quad \Rightarrow\ -5y+20=-7x+35$$

$$\Rightarrow\ -5y+20+7x-35=0 \quad \Rightarrow \boxed{7x-5y-15=0}$$

Equation of median $\overline{CD}$ by two-point form

$$y-(-8)=\frac{3-(-8)}{1-3}(x-3)$$

$$\Rightarrow\ y+8=\frac{11}{-2}(x-3) \qquad \Rightarrow\ -2y-16=11x-33$$
$$\Rightarrow\ 11x-33+2y+16=0 \qquad \Rightarrow\ \boxed{11x+2y-17=0}$$

Question # 13
Find an equation of the line through $(-4,-6)$ and perpendicular to the line having slope $\frac{-3}{2}$.

Solution

Here $(x_1, y_1)=(-4,-6)$

Slope of given line $=m=\frac{-3}{2}$

$\because$ required line is $\perp$ to given line

$\therefore$ slope of required line $=-\frac{1}{m}=-\frac{1}{-3/2}=\frac{2}{3}$

Now equation of line having slope $\frac{2}{3}$ passing through $(-4,-6)$

$$y-(-6)=\frac{2}{3}(x-(-4))$$
$$\Rightarrow\ 3(y+6)=2(x+4) \qquad \Rightarrow\ 3y+18=2x+8$$
$$\Rightarrow\ 2x+8-3y-18=0 \qquad \Rightarrow\ 2x-3y-10=0$$

Question # 14
Find an equation of the line through $(11,-5)$ and parallel to a line with slope -24.

Solution

Here $(x_1, y_1)=(11,-5)$

Slope of given line $=m=-24$

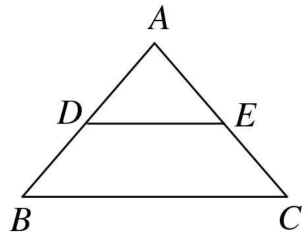
$\because$ required line is $\|$ to given line

$\therefore$ slope of required line $=m=-24$

Now equation of line having slope -24 passing through $(11,-5)$

$$y-(-5)=-24(x-11)$$
$$\Rightarrow\ y+5=-24x+264 \qquad \Rightarrow\ 24x-264+y+5=0$$
$$\Rightarrow\ 24x+y-259=0$$

Question # 15
The points $A(-1,2), B(6,3)$ and $C(2,-4)$ are vertices of a triangle. Show that the line joining the midpoint D of AB and the midpoint E of AC is parallel to BC and $DE=\frac{1}{2}BC$.



Solution Given vertices $A(-1,2)$, $B(6,3)$ and $C(2,-4)$

Since D and E are midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively.

Therefore coordinate of $D=\left(\dfrac{-1+6}{2},\dfrac{2+3}{2}\right)=\left(\dfrac{5}{2},\dfrac{5}{2}\right)$

Coordinate of $E=\left(\dfrac{-1+2}{2},\dfrac{2-4}{2}\right)=\left(\dfrac{1}{2},\dfrac{-2}{2}\right)=\left(\dfrac{1}{2},-1\right)$

Now slope of $\overline{DE}=\dfrac{-1-\frac{5}{2}}{\frac{1}{2}-\frac{5}{2}}=\dfrac{-\frac{7}{2}}{-\frac{4}{2}}=\dfrac{7}{4}$

slope of $\overline{BC}=\dfrac{-4-3}{2-6}=\dfrac{-7}{-4}=\dfrac{7}{4}$

Since slope of $\overline{DE}=$ slope of $\overline{BC}$

Therefore $\overline{DE}$ is parallel to $\overline{BC}$.

Now $\quad\left|\overline{DE}\right|=\sqrt{\left(\dfrac{1}{2}-\dfrac{5}{2}\right)^2+\left(-1-\dfrac{5}{2}\right)^2}=\sqrt{\left(-\dfrac{4}{2}\right)^2+\left(-\dfrac{7}{2}\right)^2}$

$=\sqrt{4+\dfrac{49}{4}}=\sqrt{\dfrac{65}{4}}=\dfrac{\sqrt{65}}{2}$ ……….... (i)

$\left|\overline{BC}\right|=\sqrt{(2-6)^2+(-4-3)^2}=\sqrt{(-4)^2+(-7)^2}$

$=\sqrt{16+49}=\sqrt{65}$ ……….. (ii)

From (i) and (ii)

$$\left|\overline{DE}\right|=\frac{1}{2}\left|\overline{BC}\right|$$

Question # 16

A milkman can sell 560 litres of milk at $Rs12.50$ per litre and 700 litres of milk at $Rs12.00$ per litre. Assuming the graph of the sale price and the milk sold to be a straight line, find the number of litres of milk that the milkman can sell at $Rs12.25$ per litre.

Solution

Let l denotes the number of litres of milk and p denotes the price of milk,

Then $\quad (l_1,p_1)=(560,12.50)\quad \& \quad (l_2,p_2)=(700,12.00)$

Since graph of sale price and milk sold is a straight line

Therefore, from two point form, it's equation

$$p-p_1=\frac{p_2-p_1}{l_2-l_1}(l-l_1)$$

$$\Rightarrow\ p-12.50=\frac{12.00-12.50}{700-560}(l-560)$$

$$\Rightarrow\ p-12.50=\frac{-0.50}{140}(l-560)$$

$$\Rightarrow\ 140p-1750=-0.50l+280$$

$$\Rightarrow\ 140p-1750+0.50l-280=0$$

$$\Rightarrow\ 0.50l+140p-2030=0$$

ALTERNATIVE

You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} l & p & 1\\ l_1 & p_1 & 1\\ l_2 & p_2 & 1\end{vmatrix}=0$$

If $\ p=12.25$

$$\Rightarrow 0.50l + 140(12.25) - 2030 = 0$$

$$\Rightarrow 0.50l + 1715 - 2030 = 0 \qquad \Rightarrow 0.50l - 315 = 0$$

$$\Rightarrow 0.50l = 315 \qquad \Rightarrow l = \frac{315}{0.50} = 630$$

Hence milkman can sell 630 litres milk at Rs. 12.25 per litre.

Question # 17

The population of Pakistan to the nearest million was 60 million in 1961 and 95 million in 1981. Using t as the number of years after 1961, Find an equation of the line that gives the population in terms of t. Use this equation to find the population in

(a) 1947 (b) 1997

Solution

Let p denotes population of Pakistan in million and t denotes year after 1961,

Then $(p_1, t_1) = (60, 1961)$ and $(p_2, t_2) = (95, 1981)$

Equation of line by two point form:

$$t - t_1 = \frac{t_2 - t_1}{p_2 - p_1}(p - p_1)$$

$$\Rightarrow t - 1961 = \frac{1981 - 1961}{95 - 60}(p - 60)$$

$$\Rightarrow t - 1961 = \frac{20}{35}(p - 60) \qquad \Rightarrow t - 1961 = \frac{4}{7}(p - 60)$$

$$\Rightarrow 7t - 13727 = 4p - 240 \qquad \Rightarrow 7t - 13727 + 240 = 4p$$

$$\Rightarrow 4p = 7t - 13487 \quad \Rightarrow p = \frac{7}{4}t - \frac{13487}{4} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

This is the required equation which gives population in term of t.

(a) Put $t = 1947$ in eq. (i)

$$p = \frac{7}{4}(1947) - \frac{13487}{4} = 3407.25 - 3371.75 = 35.5$$

Hence population in 1947 is 35.5 millions.

(b) Put $t = 1997$ in eq. (i)

$$p = \frac{7}{4}(1997) - \frac{13487}{4} = 3494.75 - 3371.75 = 123$$

Hence population in 1997 is 123 millions.

Question # 18

A house was purchased for $Rs1$ million in 1980. It is worth $Rs4$ million in 1996 . Assuming that the value increased by the same amount each year, find an equation that gives the value of the house after t years of the date of purchase. What was the value in 1990?

Solution

Let p denotes purchase price of house in millions and t denotes year then

$$(p_1, t_1) = (1, 1980) \quad \text{and} \quad (p_2, t_2) = (4, 1996)$$

Equation of line by two point form:

$$t - t_1 = \frac{t_2 - t_1}{p_2 - p_1}(p - p_1)$$

$$\Rightarrow\ t - 1980 = \frac{1996 - 1980}{4 - 1}(p - 1)$$

$$\Rightarrow\ t - 1980 = \frac{16}{3}(p - 1)$$

$$\Rightarrow\ 3t - 5940 = 16p - 16$$

$$\Rightarrow\ 3t - 5940 + 16 = 16p \qquad \Rightarrow\ 16p = 3t - 5924$$

$$\Rightarrow\ p = \frac{3}{16}t - \frac{5924}{16} \qquad \Rightarrow\ p = \frac{3}{16}t - \frac{1481}{4} \ \ldots\ldots\ldots\ldots \text{(i)}$$

ALTERNATIVE
You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} p & t & 1 \\ p_1 & t_1 & 1 \\ p_2 & t_2 & 1 \end{vmatrix} = 0$$

This is the required equation which gives value of house in term of t.

Put $t = 1990$ in eq. (i)

$$p = \frac{3}{16}(1990) - \frac{1481}{4} \quad = 373.125 - 370.25 \quad = 2.875$$

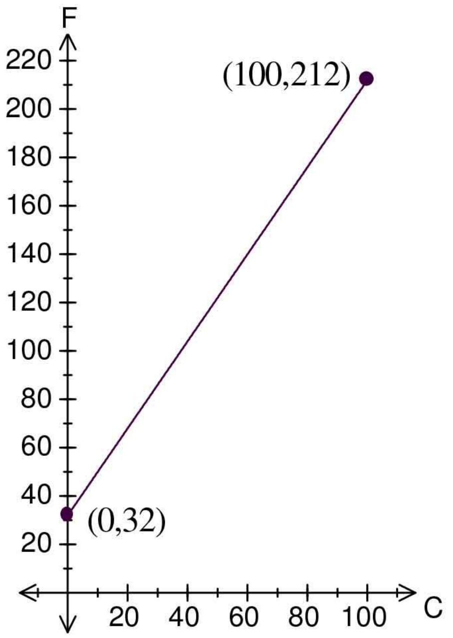
Hence value of house in 1990 is 2.875 millions.

Question # 19

Plot the Celsius (C) and Fahrenheit (F) temperature scales on the horizontal axis and the vertical axis respectively. Draw the line joining the freezing point and the boiling point of water. Find an equation giving F temperature in term of C.

Solution

Since freezing point of water $= 0^\circ C\ =\ 32^\circ F$ and boiling point of water $= 100^\circ C\ =\ 212^\circ F$ therefore we have points $(C_1, F_1) = (0, 32)$ and $(C_2, F_2) = (100, 212)$



Take scale 10ss = 20C and 10ss = 20F on x-axis and y-axis respectively to draw graph.

Equation of line by two point form

$$F - F_1 = \frac{F_2 - F_1}{C_2 - C_1}(C - C_1)$$

$$\Rightarrow\ F - 32 = \frac{212 - 32}{100 - 0}(C - 0)$$

$$\Rightarrow\ F - 32 = \frac{180}{100}C$$

$$\Rightarrow\ F = \frac{9}{5}C + 32$$

Question # 20

The average entry test score of engineering candidates was in the year 1998 while the score was 564 in 2002. Assuming that the relationship between time and score is linear, find the average score for 2006.

Solution

Let s denotes entry test score and y denotes year.

Then we have $(s_1, y_1) = (592, 1998)$ and $(s_2, y_2) = (564, 2002)$

By two point form of equation of line

$$y - y_1 = \frac{y_2 - y_1}{s_2 - s_1}(s - s_1)$$

$$\Rightarrow\ y - 1998 = \frac{2002 - 1998}{564 - 592}(s - 592) \quad \Rightarrow\ y - 1998 = \frac{4}{-28}(s - 592)$$

$$\Rightarrow\ y - 1998 = -\frac{1}{7}(s - 592) \quad \Rightarrow\ 7y - 13986 = -s + 592$$

$$\Rightarrow\ 7y - 13986 + s - 592 = 0 \quad \Rightarrow\ s + 7y - 14578 = 0$$

Put $y = 2006$ in (i)

$$s + 7(2006) - 14578 = 0 \quad \Rightarrow\ s + 14042 - 14578 = 0$$

$$\Rightarrow\ s - 536 = 0 \quad \Rightarrow\ s = 536$$

Hence in 2006 the average score will be 536.

Question # 21

Convert each of the following equation into

(i) Slope intercept form (ii) Two-intercept form (iii) Normal form

(a) $2x - 4y + 11 = 0$ (b) $4x + 7y - 2 = 0$ (c) $15y - 8x + 3 = 0$

Also find the length of the perpendicular from $(0,0)$ to each line.

Solution

(a)

(i) **- Slope-intercept form**

$$\because\ 2x - 4y + 11 = 0$$

$$\Rightarrow\ 4y = 2x + 11 \quad \Rightarrow\ y = \frac{2x + 11}{4}$$

$$\Rightarrow\ y = \frac{1}{2}x + \frac{11}{4}$$

is the intercept form of equation of line with $m = \frac{1}{2}$ and $c = \frac{11}{4}$

(ii) **- Two-intercept form**

$$\because\ 2x - 4y + 11 = 0 \quad \Rightarrow\ 2x - 4y = -11$$

$$\Rightarrow\ \frac{2}{-11}x - \frac{4}{-11}y = 1 \quad \Rightarrow\ \frac{x}{-11/2} + \frac{y}{11/4} = 1$$

is the two-point form of equation of line with $a = -\frac{11}{2}$ and $b = \frac{11}{4}$.

(iii) **- Normal form**

$$\because\ 2x - 4y + 11 = 0 \quad \Rightarrow\ 2x - 4y = -11$$

Dividing above equation by $\sqrt{(2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$$\frac{2x}{2\sqrt{5}} - \frac{4y}{2\sqrt{5}} = \frac{-11}{2\sqrt{5}} \quad \Rightarrow\ \frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}} = \frac{-11}{2\sqrt{5}}$$

Point of intersection of lines

Let $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$ be non-parallel lines.

Let $P(x_1,y_1)$ be the point of intersection of l_1 and l_2. Then

$$a_1x_1+b_1y_1+c_1=0 \text{.............}(i)$$
$$a_2x_1+b_2y_1+c_2=0 \text{.............}(ii)$$

Solving (i) and (ii) simultaneously, we have

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{-y_1}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow \frac{x_1}{b_1c_2-b_2c_1}=\frac{1}{a_1b_2-a_2b_1} \text{ and } \frac{-y_1}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1=-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$$

Hence $\left(\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},-\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}\right)$ is the point of intersection of l_1 and l_2.

Equation of line passing through the point of intersection.

Let $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$

Then equation of line passing through the point of intersection of l_1 and l_2 is

$$l_1+k\,l_2=0 \text{ , where } k \text{ is constant.}$$

i.e. $a_1x+b_1y+c_{11}+k\left(a_2x+b_2y+c_2\right)=0$

Question # 1

Find the point of intersection of the lines

(i) $x-2y+1=0 \quad 2x-y+2=0$ (ii) $3x+y+12=0 \quad x+2y-1=0$

(iii) $x+4y-12=0 \quad x-3y+3=0$

Solution

(i) $l_1:\ x-2y+1=0$

$l_2:\ 2x-y+2=0$

Slope of $l_1=m_1=-\dfrac{1}{-2}=\dfrac{1}{2}$

Slope of $l_2=m_2=-\dfrac{2}{-1}=2$

$\because\ m_1\neq m_2$, therefore lines are intersecting.

Now if (x,y) is the point of intersection of l_1 and l_2 then

$$\frac{x}{(-2)(2)-(-1)(1)}=\frac{-y}{(1)(2)-(2)(1)}=\frac{1}{(1)(-1)-(2)(-2)}$$

$$\Rightarrow \frac{x}{-4+1}=\frac{-y}{2-2}=\frac{1}{-1+4}$$

$$\Rightarrow \frac{x}{-3}=\frac{-y}{0}=\frac{1}{3}$$

$$\Rightarrow \frac{x}{-3}=\frac{1}{3} \quad \text{and} \quad \frac{-y}{0}=\frac{1}{3}$$

$$\Rightarrow x=\frac{-3}{3} \quad \text{and} \quad y=-\frac{0}{3}$$

$$\Rightarrow x=-1 \quad \text{and} \quad y=0$$

Hence $(-1,0)$ is the point of intersection.

(ii) $l_1:\ 3x+y+12=0$

$l_2:\ x+2y-1=0$

Slope of $l_1=m_1=-\frac{3}{1}=-3$

Slope of $l_2=m_2=-\frac{1}{2}$

$\because\ m_1\neq m_2$, therefore lines are intersecting.

Now if (x,y) is the point of intersection of l_1 and l_2 then

$$\frac{x}{-1-24}=\frac{-y}{-3-12}=\frac{1}{6-1}$$

$$\Rightarrow \frac{x}{-25}=\frac{-y}{-15}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{-25}=\frac{1}{5} \quad \text{and} \quad \frac{-y}{-15}=\frac{1}{5}$$

$$\Rightarrow x=\frac{-25}{5}=-5 \quad \text{and} \quad y=\frac{15}{5}=3$$

Hence $(-5,3)$ is the point of intersection.

(iii) *Do yourself as above.*

Question # 2

Find an equation of the line through

(i) the point $(2,-9)$ and the intersection of the lines $2x+5y-8=0$ and $3x-4y-6=0$

(ii) the intersection of the lines
$x-y-4=0$ $\qquad 7x+y+20=0$ $\qquad 6x+y-14=0$

(a) Parallel (ii) Perpendicular
to the line $6x+y-14=0$

(iii) through the intersection of the lines $x+2y+3=0,\ \ 3x+4y+7=0$

And making equal intercepts on the axes.

Solution

(i) Let $l_1:\ 2x+5y-8=0$

$l_2:\ 3x-4y-6=0$

Equation of line passing through point of intersection of l_1 and l_2 is

$$2x+5y-8+k(3x-4y-6)=0 \ldots \text{ (i)}$$

Since $(2,-9)$ lies on (i) therefore put $x=2$ and $y=-9$ in (i)

$$2(2)+5(-9)-8+k\left(3(2)-4(-9)-6\right)=0$$

$$\Rightarrow\ 4-45-8+k(6+36-6)=0$$

$$\Rightarrow\ -49+36k=0$$

$$\Rightarrow\ 36k=49 \qquad \Rightarrow\ k=\frac{49}{36}$$

Putting value of k in (i)

$$2x+5y-8+\frac{49}{36}(3x-4y-6)=0$$

$$\Rightarrow\ 72x+180y-288+49(3x-4y-6)=0 \qquad \times\text{ing by } 36$$

$$\Rightarrow\ 72x+180y-288+147x-196y-294=0$$

$\Rightarrow\ 219x-16y-582=0$ is the required equation.

(ii) Let $l_1:\ x-y-4=0$

$l_2:\ 7x+y+20=0$

$l_3:\ 6x+y-14=0$

Let l_4 be a line passing through point of intersection of l_1 and l_2, then

$$l_4:\ l_1+kl_2=0$$

$$\Rightarrow\ x-y-4+k(7x+y+20)=0 \ldots \text{ (i)}$$

$$\Rightarrow\ (1+7k)x+(-1+k)y+(-4+20k)=0$$

$$\text{Slope of } l_4=m_1=-\frac{1+7k}{-1+k}$$

$$\text{Slope of } l_3=m_2=-\frac{6}{1}=-6$$

(a) If l_3 and l_4 are parallel then

$$m_1=m_2$$

$$\Rightarrow\ -\frac{1+7k}{-1+k}=-6$$

$$\Rightarrow\ 1+7k=6(-1+k) \qquad \Rightarrow\ 1+7k=-6+6k$$

$$\Rightarrow\ 7k-6k=-6-1 \qquad \Rightarrow\ k=-7$$

Putting value of k in (i)

$$x-y-4-7(7x+y+20)=0$$

$$\Rightarrow\ x-y-4-49x-7y-140=0$$

$$\Rightarrow\ -48x-8y-144=0$$
$$\Rightarrow\ 6x+y+18=0$$

is the required equation

(b) If l_3 and l_4 are $\perp$ then

$$m_1m_2=-1$$
$$\Rightarrow\ \left(-\frac{1+7k}{-1+k}\right)(-6)=-1$$
$$\Rightarrow\ 6(1+7k)=-(-1+k) \qquad \Rightarrow\ 6+42k=1-k$$
$$\Rightarrow\ 42k+k=1-6 \qquad \Rightarrow\ 43k=-5 \qquad \Rightarrow\ k=-\frac{5}{43}$$

Putting in (i) we have

$$x-y-4-\frac{5}{43}(7x+y+20)=0$$
$$\Rightarrow\ 43x-43y-172-5(7x+y+20)=0$$
$$\Rightarrow\ 43x-43y-172-35x-5y-100=0$$
$$\Rightarrow\ 8x-48y-272=0$$
$$\Rightarrow\ x-6y-34=0 \text{ is the required equation.}$$

(iii) Suppose $l_1:\ x+2y+3=0$

$l_2:\ 3x+4y+7=0$

Equation of line passing through the intersection of l_1 and l_2 is given by:

$$x+2y+3+k(3x+4y+7)=0 \ \ldots\ldots\ldots\ldots \text{(i)}$$
$$\Rightarrow\ (1+3k)x+(2+4k)y+(3+7k)=0$$
$$\Rightarrow\ (1+3k)x+(2+4k)y=-(3+7k)$$
$$\Rightarrow\ \frac{(1+3k)x}{-(3+7k)}+\frac{(2+4k)y}{-(3+7k)}=1$$
$$\Rightarrow\ \frac{x}{-(3+7k)\big/(1+3k)}+\frac{y}{-(3+7k)\big/(2+4k)}=1$$

Which is two-intercept form of equation of line with

$$x-\text{intercept}=\frac{-(3+7k)}{(1+3k)} \quad \text{and} \quad y-\text{intercept}=\frac{-(3+7k)}{(2+4k)}$$

We have given

$$x-\text{intercept} = y-\text{intercept}$$
$$\Rightarrow\ \frac{-(3+7k)}{(1+3k)}=\frac{-(3+7k)}{(2+4k)}$$
$$\Rightarrow\ \frac{1}{(1+3k)}=\frac{1}{(2+4k)} \qquad \Rightarrow\ (2+4k) = (1+3k)$$
$$\Rightarrow\ 4k-3k = 1-2 \quad \Rightarrow\ k=-1$$

Putting value of k in (i)

$$x+2y+3-1(3x+4y+7) = 0$$
$$\Rightarrow x+2y+3-3x-4y-7 = 0 \quad \Rightarrow -2x-2y-4 = 0$$
$$\Rightarrow x+y+2 = 0$$

is the required equation.

Question # 3

Find an equation of the line through the intersection of $16x-10y-33=0$; $12x+14y+29=0$ and the intersection of $x-y+4=0$; $x-7y+2=0$

Solution

Let $l_1:\ 16x-10y-33=0$
$l_2:\ 12x+14y+29=0$
$l_3:\ x-y+4=0$
$l_4:\ x-7y+2=0$

For point of intersection of l_1 and l_2

$$\frac{x}{-290+462}=\frac{-y}{464+396}=\frac{1}{224+120}$$
$$\Rightarrow \frac{x}{172}=\frac{-y}{860}=\frac{1}{334}$$
$$\Rightarrow \frac{x}{172}=\frac{1}{334} \quad \text{and} \quad \frac{-y}{860}=\frac{1}{334}$$
$$\Rightarrow x=\frac{172}{334}=\frac{1}{2} \quad \text{and} \quad y=-\frac{860}{334}=-\frac{5}{2}$$
$$\Rightarrow \left(\frac{1}{2},-\frac{5}{2}\right) \text{ is a point of intersection of } l_1 \text{ and } l_2.$$

For point of intersection of l_3 and l_4.

$$\frac{x}{-2+28}=\frac{-y}{2-4}=\frac{1}{-7+1}$$
$$\Rightarrow \frac{x}{26}=\frac{-y}{-2}=\frac{1}{-6}$$
$$\Rightarrow \frac{x}{26}=\frac{1}{-6} \quad \text{and} \quad \frac{-y}{-2}=\frac{1}{-6}$$
$$\Rightarrow x=\frac{26}{-6}=-\frac{13}{3} \quad \text{and} \quad y=\frac{2}{-6}=-\frac{1}{3}$$
$$\Rightarrow \left(-\frac{13}{3},-\frac{1}{3}\right) \text{ is a point of intersection of } l_3 \text{ and } l_4.$$

Now equation of line passing through $\left(\frac{1}{2},-\frac{5}{2}\right)$ and $\left(-\frac{13}{3},-\frac{1}{3}\right)$

$$y+\frac{5}{2}=\frac{-\frac{1}{3}+\frac{5}{2}}{-\frac{13}{3}-\frac{1}{2}}\left(x-\frac{1}{2}\right)$$

$$\Rightarrow\ y+\frac{5}{2}=\frac{13/6}{-29/6}\left(x-\frac{1}{2}\right)\quad\Rightarrow\ y+\frac{5}{2}=-\frac{13}{29}\left(x-\frac{1}{2}\right)$$

$$\Rightarrow\ 29y+\frac{145}{2}=-13x+\frac{13}{2}\qquad\Rightarrow\ 13x-\frac{13}{2}+29y+\frac{145}{2}=0$$

$$\Rightarrow\ 13x+29y+66=0$$

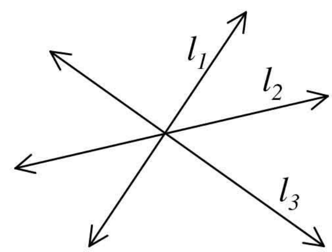
is the required equation.

Three Concurrent Lines

Suppose $l_1:\ a_1x+b_1y+c_1=0$

$l_2:\ a_2x+b_2y+c_2=0$

$l_3:\ a_3x+b_3y+c_3=0$



If $l_1,\ l_2$ and l_3 are concurrent (intersect at one point) then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

See proof on book at page 208

Question # 4

Find the condition that the lines $y=m_1x+c_1$; $y=m_2x+c_2$ and $y=m_3x+c_3$ are concurrent.

Solution

Assume that $l_1:\ y=m_1x+c_1$

$\Rightarrow\ m_1x-y+c_1=0$

$l_2:\ y=m_2x+c_2$

$\Rightarrow\ m_2x-y+c_2=0$

$l_3:\ y=m_3x+c_3$

$\Rightarrow\ m_3x-y+c_3=0$

If $l_1,\ l_2$ and l_3 are concurrent then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix}=0$$

$$\Rightarrow\ \begin{vmatrix} m_1 & -1 & c_1 \\ m_2-m_1 & 0 & c_2-c_1 \\ m_3-m_1 & 0 & c_3-c_1 \end{vmatrix}=0 \quad \text{by}\ \begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

Expanding by C_2

$$-(-1)\left[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)\right]+0-0=0$$

$$\Rightarrow\ \left[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)\right]=0$$

$$\Rightarrow\ (m_2-m_1)(c_3-c_1)=(m_3-m_1)(c_2-c_1)$$

is the required condition.

Question # 5

Determine the value of P such that the lines $2x-3y-1=0$, $3x-y-5=0$ and $3x+py+8=0$ meet at a point .

Solution

Let $l_1:\ 2x-3y-1=0$

$l_2:\ 3x-y-5=0$

$l_3:\ 3x+py+8=0$

Since l_1, l_2 and l_3 meets at a point i.e. concurrent therefore

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & p & 8 \end{vmatrix} = 0$$

$\Rightarrow\ 2(-8+5p)+3(24+15)-1(3p+3)=0$

$\Rightarrow\ -16+10p+72+45-3p-3=0$

$\Rightarrow\ 7p+98=0 \quad \Rightarrow\ 7p=-98$

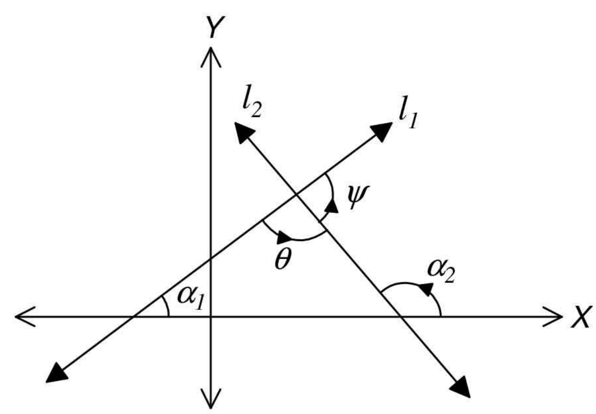
$\Rightarrow\ p=-\dfrac{98}{7} \quad \Rightarrow\ \boxed{p=-14}$

Angle between lines

Let l_1 and l_2 be two lines. If α_1 and α_2 be inclinations and m_1 and m_2 be slopes of lines l_1 and l_2 respectively, Let θ be a angle from line l_1 to l_2 then θ is given by

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$$

See proof on book at page 219



Question # 6

Show that the lines $4x-3y-8=0,$ $3x-4y-6=0$ and $x-y-2=0$ are concurrent and the third-line bisects the angle formed by the first two lines.

Solution

Let $l_1:\ 4x-3y-8=0$

$l_2:\ 3x-4y-6=0$

$l_3:\ x-y-2=0$

To check l_1, l_2 and l_3 are concurrent, let

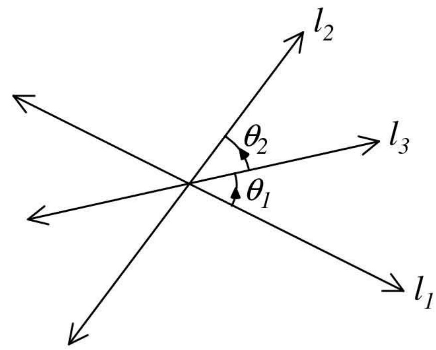
$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$=4(8-6)+3(-6+6)-8(-3+4)$

$=4(2)+3(0)-8(1)$

$=8+0-8=0$

Hence l_1, l_2 and l_3 are concurrent.



Slope of $l_1 = m_1 = -\dfrac{4}{-3} = \dfrac{4}{3}$

Slope of $l_2 = m_2 = -\dfrac{3}{-4} = \dfrac{3}{4}$

Slope of $l_3 = m_3 = -\dfrac{1}{-1} = 1$

Now let θ_1 be angle from l_1 to l_3 and θ_2 be a angle from l_3 to l_2. Then

$$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{1 - 4/3}{1 + (1)(4/3)} = \frac{-1/3}{7/3} = -\frac{1}{7} \quad \text{...........} \; (i)$$

And $$\tan\theta_2 = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{3/4 - 1}{1 + (3/4)(1)} = \frac{-1/4}{7/4} = -\frac{1}{7} \quad \text{...........} \; (ii)$$

From (i) and (ii)

$$\tan\theta_1 = \tan\theta_2 \quad \Rightarrow \theta_1 = \theta_2$$

$\Rightarrow$ l_3 bisect the angle formed by the first two lines.

Question # 7

The vertices of a triangle are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$.

Find coordinates of the

(i) centroid (ii) orthocentre (iii) circumcentre of the triangle

Are these three points are collinear?

Solution

Given vertices of triangles are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$.

(i) Centroid of triangle is the intersection of medians and is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

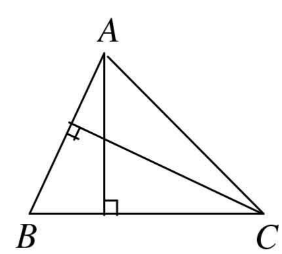
$$= \left(\frac{-2-4+3}{3}, \frac{3+1+5}{3}\right) = \left(\frac{-3}{3}, \frac{9}{3}\right) = (-1,3)$$

Hence $(-1,3)$ is the centroid of the triangle.

(ii) Orthocentre is the point of intersection of altitudes.

Slope of $\overline{AB} = m_1 = \dfrac{1-3}{-4+2} = \dfrac{-2}{-2} = 1$

Slope of $\overline{BC} = m_2 = \dfrac{5-1}{3+4} = \dfrac{4}{7}$



Since altitudes are $\perp$ to sides therefore

Slope of altitude on $\overline{AB} = -\dfrac{1}{m_1} = -\dfrac{1}{1} = -1$

Slope of altitude on $\overline{BC} = -\dfrac{1}{m_2} = -\dfrac{1}{4/7} = -\dfrac{7}{4}$

Equation of altitude on $\overline{AB}$ with slope -1 from $C(3,5)$

$$y-5=-1(x-3)$$
$$\Rightarrow y-5=-x+3 \quad \Rightarrow x-3+y-5=0$$
$$\Rightarrow x+y-8=0 \text{ ………. (i)}$$

Now equation of altitude on $\overline{BC}$ with slope $-\dfrac{7}{4}$ from $A(-2,3)$

$$y-3=-\frac{7}{4}(x+2)$$
$$\Rightarrow 4y-12=-7x-14 \quad \Rightarrow 7x+14+4y-12=0$$
$$\Rightarrow 7x+4y+2=0 \text{ ………….. (ii)}$$

For point of intersection of (i) and (ii)

$$\frac{x}{2+32}=\frac{-y}{2+56}=\frac{1}{4-7}$$
$$\Rightarrow \frac{x}{34}=\frac{-y}{58}=\frac{1}{-3}$$
$$\Rightarrow \frac{x}{34}=\frac{1}{-3} \quad \text{and} \quad \frac{-y}{58}=\frac{1}{-3}$$
$$\Rightarrow x=-\frac{34}{3} \quad \text{and} \quad y=-\frac{58}{-3}=\frac{58}{3}$$

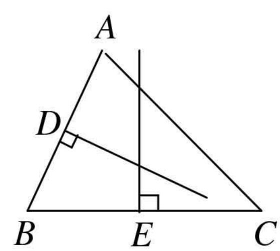
Hence $\left(-\dfrac{34}{3},\dfrac{58}{3}\right)$ is orthocentre of triangle ABC.

(iii) Circumcentre of the triangle is the point of intersection of perpendicular bisector.

Let D and E are midpoints of side $\overline{AB}$ and $\overline{BC}$ respectively.

Then coordinate of $D=\left(\dfrac{-4-2}{2},\dfrac{1+3}{2}\right)=\left(\dfrac{-6}{2},\dfrac{4}{2}\right)=(-3,2)$

Coordinate of $E=\left(\dfrac{-4+3}{2},\dfrac{1+5}{2}\right)=\left(\dfrac{-1}{2},\dfrac{6}{2}\right)=\left(-\dfrac{1}{2},3\right)$



Slope of $\overline{AB}=m_1=\dfrac{1-3}{-4+2}=\dfrac{-2}{-2}=1$

Slope of $\overline{BC}=m_2=\dfrac{5-1}{3+4}=\dfrac{4}{7}$

Slope of $\perp$ bisector on $\overline{AB}=-\dfrac{1}{m_1}=-\dfrac{1}{1}=-1$

Slope of $\perp$ bisector on $\overline{BC}=-\dfrac{1}{m_2}=-\dfrac{1}{4/7}=-\dfrac{7}{4}$

Now equation of $\perp$ bisector having slope -1 through $D(-3,2)$

$$y-2=-1(x+3)$$
$$\Rightarrow\ y-2=-x-3 \quad \Rightarrow\ x+3+y-2=0$$
$$\Rightarrow\ x+y+1=0 \ \ldots\ldots\ldots\ldots \text{(iii)}$$

Now equation of $\perp$ bisector having slope $-\dfrac{7}{4}$ through $E\left(-\dfrac{1}{2},3\right)$

$$y-3=-\frac{7}{4}\left(x+\frac{1}{2}\right) \quad \Rightarrow 4y-12=-7x-\frac{7}{2}$$
$$\Rightarrow\ 7x+\frac{7}{2}+4y-12=0 \quad \Rightarrow 7x+4y-\frac{17}{2}=0$$
$$\Rightarrow 14x+8y-17=0 \ \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

For point of intersection of (iii) and (iv)

$$\frac{x}{-17-8}=\frac{-y}{-17-14}=\frac{1}{8-14}$$
$$\Rightarrow\ \frac{x}{-25}=\frac{-y}{-31}=\frac{1}{-6}$$
$$\Rightarrow\ \frac{x}{-25}=\frac{1}{-6} \quad \text{and} \quad \frac{-y}{-31}=\frac{1}{-6}$$
$$\Rightarrow\ x=\frac{-25}{-6}=\frac{25}{6} \quad \text{and} \quad y=-\frac{31}{6}$$

Hence $\left(\dfrac{25}{6},-\dfrac{31}{6}\right)$ is the circumcentre of the triangle.

Now to check $(-1,3)$, $\left(-\dfrac{34}{3},\dfrac{58}{3}\right)$ and $\left(\dfrac{25}{6},-\dfrac{31}{6}\right)$ are collinear, let

$$\begin{vmatrix} -1 & 3 & 1 \\ -34/3 & 58/3 & 1 \\ 25/6 & -31/6 & 1 \end{vmatrix}$$
$$=-1\left(\frac{58}{3}+\frac{31}{6}\right)-3\left(-\frac{34}{3}-\frac{25}{6}\right)+1\left(\frac{1054}{18}-\frac{1450}{18}\right)$$
$$=-1\left(\frac{49}{2}\right)-3\left(-\frac{31}{2}\right)+1(-22)$$
$$=-\frac{49}{2}+\frac{93}{2}-22=0$$

Hence centroid, orthocentre and circumcentre of triangle are collinear.

Question # 8

Check whether the lines $4x-3y-8=0$; $3x-4y-6=0$; $x-y-2=0$ are concurrent. If so, find the point where they meet.

Solution

Let $l_1:\ 4x-3y-8=0$

$l_2:\ 3x-4y-6=0$

$l_3:\ x-y-2=0$

To check lines are concurrent, let

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$$=4(8-6)+3(-6+6)-8(-3+4)$$

$$=4(2)+3(0)-8(1)\quad =8+0-8\ =0$$

Hence l_1 , l_2 and l_3 are concurrent.

For point of concurrency, we find intersection of l_1 and l_2 (You may choose any two lines)

$$\frac{x}{18-32}=\frac{-y}{-24+24}=\frac{1}{-16+9}$$

$$\Rightarrow \frac{x}{-14}=\frac{-y}{0}=\frac{1}{-7}$$

$$\Rightarrow \frac{x}{-14}=\frac{1}{-7}\quad \text{and}\quad \frac{-y}{0}=\frac{1}{-7}$$

$$\Rightarrow x=\frac{-14}{-7}=2\quad \text{and}\quad y=-\frac{0}{-7}=0$$

Hence $(2,0)$ is the point of concurrency.

Question # 9

Find the coordinates of the vertices of the triangle formed by the lines $x-2y-6=0;\quad 3x-y+3=0;\quad 2x+y-4=0$. Also find measures of the angles of the triangle.

Solution

Let $l_1:\ x-2y-6=0$

$l_2:\ 3x-y+3=0$

$l_3:\ 2x+y-4=0$

For point of intersection of l_1 and l_2

$$\frac{x}{-6-6}=\frac{-y}{3+18}=\frac{1}{-1+6}$$

$$\Rightarrow \frac{x}{-12}=\frac{-y}{21}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{-12}=\frac{1}{5}\quad \text{and}\quad \frac{-y}{21}=\frac{1}{5}$$

$$\Rightarrow x=-\frac{12}{5}\quad \text{and}\quad y=-\frac{21}{5}$$

$\Rightarrow \left(-\frac{12}{5},-\frac{21}{5}\right)$ is the point of intersection of l_1 and l_2.

For point of intersection of l_2 and l_3.

$$\frac{x}{4-3}=\frac{-y}{-12-6}=\frac{1}{3+2}$$

$$\Rightarrow \frac{x}{1}=\frac{-y}{-18}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{1}=\frac{1}{5} \quad \text{and} \quad \frac{-y}{-18}=\frac{1}{5}$$

$$\Rightarrow x=\frac{1}{5} \quad \text{and} \quad y=\frac{18}{5}$$

$\Rightarrow \left(\frac{1}{5},\frac{18}{5}\right)$ is the point of intersection of l_2 and l_3.

Now for point of intersection of l_1 and l_3

$$\frac{x}{8+6}=\frac{-y}{-4+12}=\frac{1}{1+4}$$

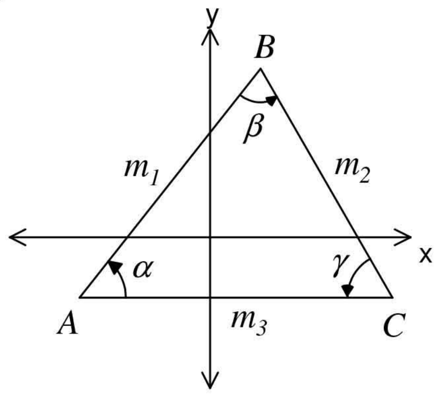
$$\Rightarrow \frac{x}{14}=\frac{-y}{8}=\frac{1}{5}$$

$$\Rightarrow \frac{x}{14}=\frac{1}{5} \quad \text{and} \quad \frac{-y}{8}=\frac{1}{5}$$

$$\Rightarrow x=\frac{14}{5} \quad \text{and} \quad y=-\frac{8}{5}$$

$\Rightarrow \left(\frac{14}{5},-\frac{8}{5}\right)$ is the point of intersection of l_1 and l_3.

Hence $\left(-\frac{12}{5},-\frac{21}{5}\right)$, $\left(\frac{1}{5},\frac{18}{5}\right)$ and $\left(\frac{14}{5},-\frac{8}{5}\right)$ are vertices of triangle made by l_1, l_2 and l_3. We say these vertices as A,B and C respectively.



Slope of side $AB = m_1 = \dfrac{18/5+21/5}{1/5+12/5} = \dfrac{39/5}{13/5} = \dfrac{39}{13} = 3$

Slope of side $BC = m_2 = \dfrac{-8/5-18/5}{14/5-1/5} = \dfrac{-26/5}{13/5} = -\dfrac{26}{13} = -2$

Slope of side $CA = m_3 = \dfrac{-21/5+8/5}{-12/5-14/5} = \dfrac{-13/5}{-26/5} = \dfrac{13}{26} = \dfrac{1}{2}$

Let α,β and γ denotes angles of triangle at vertices A,B and C respectively. Then

$$\tan\alpha = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{3 - \frac{1}{2}}{1 + (3)\left(\frac{1}{2}\right)} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1$$

$$\Rightarrow \alpha = \tan^{-1}(1) \quad \Rightarrow \boxed{\alpha = 45^\circ}$$

Now $$\tan\beta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-2-3}{1 + (-3)(3)} = \frac{-5}{-5} = 1$$

$$\Rightarrow \beta = \tan^{-1}(1) \quad \Rightarrow \boxed{\beta = 45^\circ}$$

Now $$\tan\gamma = \frac{m_3 - m_2}{1 + m_3 m_2} = \frac{\frac{1}{2} + 2}{1 + \left(\frac{1}{2}\right)(-2)} = \frac{\frac{5}{2}}{0} = \infty$$

$$\Rightarrow \gamma = \tan^{-1}(\infty) \quad \Rightarrow \boxed{\gamma = 90^\circ}.$$

Question # 10

Find the angle measured from the line l_1 to the line l_2 where

(a) l_1: joining $(2,7)$ and $(7,10)$
l_2: joining $(1,1)$ and $(-5,3)$

(b) l_1: joining $(3,-1)$ and $(5,7)$
l_2: joining $(2,4)$ and $(-8,2)$

(c) l_1: joining $(1,-7)$ and $(6,-4)$
l_2: joining $(-1,2)$ and $(-6,-1)$

(d) l_1: joining $(-9,-1)$ and $(3,-5)$
l_2: joining $(2,7)$ and $(-6,-7)$

Solution

(a) Since l_1: joining $(2,7)$ and $(7,10)$

Therefore slope of $l_1 = m_1 = \dfrac{10-7}{7-2} = \dfrac{3}{5}$

Also l_2: joining $(1,1)$ and $(-5,3)$

Therefore slope of $l_2 = m_2 = \dfrac{3-1}{-5-1} = \dfrac{2}{-6} = -\dfrac{1}{3}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-\frac{1}{3} - \frac{3}{5}}{1 + \left(\frac{3}{5}\right)\left(-\frac{1}{3}\right)}$$

$$= \frac{-\frac{14}{15}}{1 - \frac{1}{5}} = \frac{-\frac{14}{15}}{\frac{4}{5}} = -\frac{14}{15} \times \frac{5}{4} = -\frac{7}{6}$$

$$\Rightarrow -\tan\theta = \frac{7}{6} \quad \Rightarrow \tan(180 - \theta) = \frac{7}{6}$$

$$\because \tan(180 - \theta) = -\tan\theta$$

$$\Rightarrow\ 180-\theta=\tan^{-1}\left(\frac{7}{6}\right)=49.4$$

$\Rightarrow\ \theta=180-49.4 \quad \Rightarrow \boxed{\theta=130.6^\circ}$

Now acute angle between lines $=180-130.6=49.4^\circ$

(b) *Do yourself as above.*

(c) Since l_1 : joining $(1,-7)$ and $(6,-4)$

Therefore slope of $l_1=m_1=\dfrac{-4+7}{6-1}=\dfrac{3}{5}$

Also l_2 : joining $(-1,2)$ and $(-6,-1)$

Therefore slope of $l_2=m_2=\dfrac{-1-2}{-6+1}=\dfrac{-3}{-5}=\dfrac{3}{5}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{3/5-3/5}{1+\left(3/5\right)\left(3/5\right)}=\frac{0}{1+9/25}=0$$

$$\Rightarrow\ \theta=\tan^{-1}(0) \quad \Rightarrow \boxed{\theta=0^\circ}$$

Also acute angle between lines $=0^\circ$

(d) Since l_1 : joining $(-9,-1)$ and $(3,-5)$

Therefore slope of $l_1=m_1=\dfrac{-5+1}{3+9}=\dfrac{-4}{12}=-\dfrac{1}{3}$

Also l_2 : joining $(2,7)$ and $(-6,-7)$

Therefore slope of $l_2=m_2=\dfrac{-7-7}{-6-2}=\dfrac{-14}{-8}=\dfrac{7}{4}$

Let θ be a angle from l_1 to l_2 then

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{7/4-\left(-1/3\right)}{1+\left(7/4\right)\left(-1/3\right)}$$

$$=\frac{7/4+1/3}{1-7/12}=\frac{25/12}{5/12}=\frac{25}{12}\times\frac{12}{5}=5$$

$$\Rightarrow\ \theta=\tan^{-1}(5) \quad \Rightarrow \boxed{\theta=78.69^\circ}$$

Also acute angle between lines $=78.69^\circ$

Question # 11

Find the interior angle of the triangle whose vertices are

(a) $A(-2,11)$, $B(-6,-3)$, $C(4,-9)$

(b) $A(6,1)$, $B(2,7)$, $C(-6,-7)$
(c) $A(2,-5)$, $B(-4,-3)$, $C(-1,5)$
(d) $A(2,8)$, $B(-5,4)$ and $C(4,-9)$

Solution

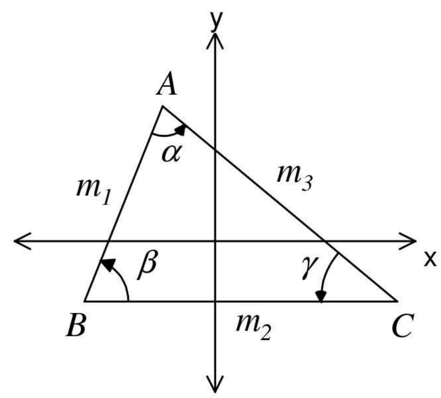
(a) Given vertices $A(-2,11)$, $B(-6,-3)$ and $C(4,-9)$

Let m_1, m_2 and m_3 denotes the slopes of side AB, BC and CA respectively. Then

$$m_1=\frac{-3-11}{-6+2}=\frac{-14}{-4}=\frac{7}{2}$$

$$m_2=\frac{-9+3}{4+6}=\frac{-6}{10}=-\frac{3}{5}$$

$$m_3=\frac{11+9}{-2-4}=\frac{20}{-6}=-\frac{10}{3}$$



Let α,β and γ denotes angles of triangle at vertex A, B and C respectively. Then

$$\tan\alpha=\frac{m_3-m_1}{1+m_3m_1}=\frac{-10/3-7/2}{1+\left(-10/3\right)\left(7/2\right)}$$

$$=\frac{-41/6}{1-35/3}=\frac{-41/6}{-32/3}=\frac{41}{6}\times\frac{3}{32}=\frac{41}{64}$$

$$\Rightarrow\ \alpha=\tan^{-1}\left(\frac{41}{64}\right)\qquad\Rightarrow\ \boxed{\alpha=32.64^\circ}$$

$$\tan\beta=\frac{m_1-m_2}{1+m_1m_2}=\frac{7/2-\left(-3/5\right)}{1+\left(7/2\right)\left(-3/5\right)}$$

$$=\frac{7/2+3/5}{1-21/10}=\frac{41/10}{-11/10}=-\frac{41}{10}\times\frac{10}{11}=-\frac{41}{11}$$

$$\Rightarrow\ -\tan\beta=\frac{41}{11}\qquad\Rightarrow\ \tan(180-\beta)=\frac{41}{11}\qquad\because\ \tan(180-\theta)=-\tan\theta$$

$$\Rightarrow\ 180-\beta=\tan^{-1}\left(\frac{41}{11}\right)=74.98$$

$$\Rightarrow\ \beta=180-74.98\qquad\Rightarrow\ \boxed{\beta=105.02}$$

$$\tan\gamma=\frac{m_2-m_3}{1+m_2m_3}=\frac{-3/5-\left(-10/3\right)}{1+\left(-3/5\right)\left(-10/3\right)}$$

$$=\frac{-3/5+10/3}{1+2}=\frac{41/15}{3}=\frac{41}{15\times3}=\frac{41}{45}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{41}{45}\right) \quad \Rightarrow \boxed{\gamma = 42.34^\circ}$$

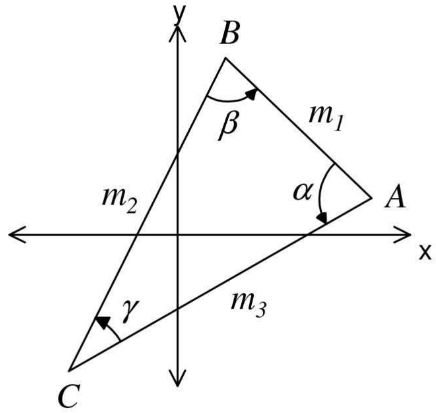
(b) Given vertices $A(6,1)$, $B(2,7)$ and $C(-6,-7)$

Let m_1, m_2 and m_3 denotes the slopes of side AB, BC and CA respectively. Then

$$m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

$$m_3 = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$$



Let α, β and γ denotes angles of triangle at vertex A, B and C respectively. Then

$$\tan\alpha = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{2/3 - \left(-3/2\right)}{1 + \left(2/3\right)\left(-3/2\right)}$$

$$= \frac{2/3 + 3/2}{1-1} = \frac{13/6}{0} = \infty$$

$$\Rightarrow \alpha = \tan^{-1}(\infty) \quad \Rightarrow \boxed{\alpha = 90^\circ}$$

$$\tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-3/2 - 7/4}{1 + \left(-3/2\right)\left(7/4\right)}$$

$$= \frac{-13/4}{1 - 21/8} = \frac{-13/4}{-13/8} = \frac{13}{4} \times \frac{8}{13} = 2$$

$$\Rightarrow \beta = \tan^{-1}(2) \quad \Rightarrow \boxed{\beta = 63.43^\circ}$$

$$\tan\gamma = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{7/4 - 2/3}{1 + \left(7/4\right)\left(2/3\right)}$$

$$= \frac{13/12}{1 + 7/6} = \frac{13/12}{13/6} = \frac{13}{12} \times \frac{6}{13} = \frac{1}{2}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{1}{2}\right) \quad \Rightarrow \boxed{\gamma = 26.57^\circ}$$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 12
Find the interior angles of the quadrilateral whose vertices are $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$

Solution Given vertices are $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$

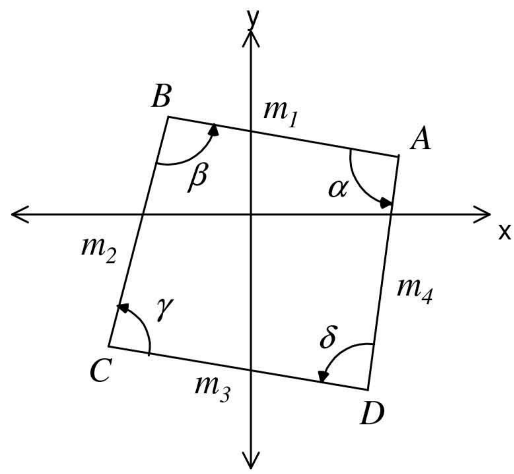
Let m_1, m_2, m_3 and m_4 be slopes of side AB, BC, CD and DA. Then

$$m_1=\frac{3-2}{-2-5}=\frac{1}{-7}$$

$$m_2=\frac{-4-3}{-3+2}=\frac{-7}{-1}=7$$

$$m_3=\frac{-5+4}{4+3}=\frac{-1}{7}$$

$$m_4=\frac{2+5}{5-4}=\frac{7}{1}=7$$



Now suppose α,β,γ and δ are angels of quadrilateral at vertices A, B, C and D respectively. Then

$$\tan\alpha=\frac{m_4-m_1}{1+m_4m_1}=\frac{7-\left(-\frac{1}{7}\right)}{1+(7)\left(-\frac{1}{7}\right)}=\frac{7+\frac{1}{7}}{1-1}=\frac{\frac{50}{7}}{0}=\infty$$

$$\Rightarrow\ \alpha=\tan^{-1}(\infty)\qquad\Rightarrow\ \boxed{\alpha=90^\circ}$$

$$\tan\beta=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\frac{1}{7}-7}{1+\left(-\frac{1}{7}\right)(7)}=\frac{-\frac{50}{7}}{0}=\infty$$

$$\Rightarrow\ \beta=\tan^{-1}(\infty)\qquad\Rightarrow\ \boxed{\beta=90^\circ}$$

$$\tan\gamma=\frac{m_2-m_3}{1+m_2m_3}=\frac{7-\left(-\frac{1}{7}\right)}{1+(7)\left(-\frac{1}{7}\right)}=\frac{7+\frac{1}{7}}{1-1}=\frac{\frac{50}{7}}{0}=\infty$$

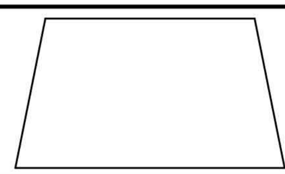
$$\Rightarrow\ \gamma=\tan^{-1}(\infty)\qquad\Rightarrow\ \boxed{\gamma=90^\circ}$$

$$\tan\delta=\frac{m_3-m_4}{1+m_3m_4}=\frac{-\frac{1}{7}-7}{1+\left(-\frac{1}{7}\right)(7)}=\frac{-\frac{50}{7}}{0}=\infty$$

$$\Rightarrow\ \delta=\tan^{-1}(\infty)\qquad\Rightarrow\ \boxed{\delta=90^\circ}$$

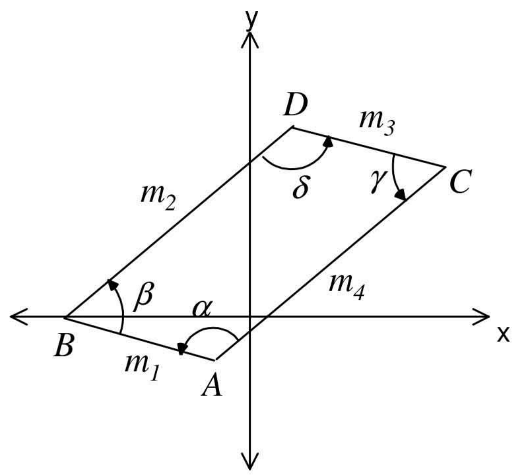
Trapezium

If any two opposite sides of the quadrilateral are parallel then it is called *trapezium.*



Question # 13
Show that the points $A(-1,-1)$, $B(-3,0)$, $C(3,7)$ and $D(1,8)$ are the vertices of the rhombus and find its interior angle.
Solution Given vertices are $A(-1,-1)$, $B(-3,0)$, $C(3,7)$ and $D(1,8)$
Let m_1, m_2, m_3 and m_4 be slopes of side $\overline{AB}, \overline{BD}, \overline{DC}$ and $\overline{CA}$. Then

$$m_1 = \frac{0+1}{-3+1} = \frac{1}{-2}$$
$$m_2 = \frac{8-0}{1+3} = \frac{8}{4} = 2$$
$$m_3 = \frac{7-8}{3-1} = \frac{-1}{2}$$
$$m_4 = \frac{-1-7}{-1-3} = \frac{-8}{-4} = 2$$



Since $m_2 = m_4$ or $m_1 = m_3$
Hence A, B, C and D are vertices of trapezium.
Now suppose α, β, γ and δ are angels of quadrilateral at vertices A, B, C and D respectively. Then
Now do yourself as above in Question # 12

Question # 14
Find the area of the region bounded by the triangle, whose sides are
$$7x - y - 10 = 0; \quad 10x + y - 41 = 0; \quad 3x + 2y + 3 = 0$$
Solution
Let $l_1 : 7x - y - 10 = 0$
$l_2 : 10x + y - 41 = 0$
$l_3 : 3x + 2y + 3 = 0$
For intersection of l_1 and l_2
$$\frac{x}{41+10} = \frac{-y}{-287+100} = \frac{1}{7+10}$$
$$\Rightarrow \frac{x}{51} = \frac{-y}{-187} = \frac{1}{17}$$
$$\Rightarrow \frac{x}{51} = \frac{1}{17} \quad \text{and} \quad \frac{y}{187} = \frac{1}{17}$$
$$\Rightarrow x = \frac{51}{17} = 3 \quad \text{and} \quad y = \frac{187}{17} = 11$$
$\Rightarrow (3,11)$ is the point of intersection of l_1 and l_2.
Now for point of intersection of l_2 and l_3
$$\frac{x}{3+82} = \frac{-y}{30+123} = \frac{1}{20-3}$$
$$\Rightarrow \frac{x}{85} = \frac{-y}{153} = \frac{1}{17}$$

$$\Rightarrow \frac{x}{85}=\frac{-y}{153}=\frac{1}{17}$$

$$\Rightarrow \frac{x}{85}=\frac{1}{17} \quad \text{and} \quad \frac{-y}{153}=\frac{1}{17}$$

$$\Rightarrow x=\frac{85}{17}=5 \quad \text{and} \quad y=-\frac{153}{17}=-9$$

$\Rightarrow (5,-9)$ is the point of intersection of l_2 and l_3.

For point of intersection of l_1 and l_3

$$\frac{x}{-3+20}=\frac{-y}{21+30}=\frac{1}{14+3}$$

$$\Rightarrow \frac{x}{17}=\frac{-y}{51}=\frac{1}{17}$$

$$\Rightarrow \frac{x}{17}=\frac{1}{17} \quad \text{and} \quad \frac{-y}{51}=\frac{1}{17}$$

$$\Rightarrow x=\frac{17}{17}=1 \quad \text{and} \quad y=-\frac{51}{17}=3$$

$\Rightarrow (1,-3)$ is the point of intersection of l_1 and l_3.

Now area of triangle having vertices $(3,11)$, $(5,-9)$ and $(1,-3)$ is given by:

$$\frac{1}{2}\begin{vmatrix} 3 & 11 & 1 \\ 5 & -9 & 1 \\ 1 & -3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left|3(-9+3)-11(5-1)+1(-15+9)\right|$$

$$=\frac{1}{2}\left|3(-6)-11(4)+1(-6)\right| =\frac{1}{2}\left|-18-44-6\right|$$

$$=\frac{1}{2}\left|-68\right| =\frac{1}{2}(68) \quad =34 \text{ sq. unit}$$

Question # 15

The vertices of a triangle are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$. Find the centre of the circum centre of the triangle?

Solution *Same Question # 7(c)*

Question # 16

Express the given system of equations in matrix form. Find in each case whether in lines are concurrent.

(a) $x+3y-2=0$; $2x-y+4=0$; $x-11y+14=0$

(b) $2x+3y+4=0$; $x-2y-3=0$; $3x+y-8=0$

(c) $3x-4y-2=0$; $x+2y-4=0$; $3x-2y+5=0$

Solution

(a)

$$x+3y-2=0$$
$$2x-y+4=0$$
$$x-11y+14=0$$

In matrix form

$$\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Coefficient matrix of the system is

$$A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}$$

$$\Rightarrow |A| = 1(-14+44)-3(28-4)-2(-22+1)$$
$$=1(30)-3(24)-2(-21)$$
$$=30-72+42=0$$

Hence given lines are concurrent.

(b)

$$2x+3y+4=0$$
$$x-2y-3=0$$
$$3x+y-8=0$$

In matrix form

$$\begin{bmatrix}2 & 3 & 4\\ 1 & -2 & -3\\ 3 & 1 & -8\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Coefficient matrix of the system is

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & -2 & -3\\ 3 & 1 & -8\end{bmatrix}$$

$$\Rightarrow |A| = 2(16+3)-3(-8+9)+4(1+6)$$
$$=2(19)-3(1)+4(7) \quad =38-3+28 \quad =63\neq 0$$

Hence given lines are not concurrent.

(c) *Do yourself as above*

Question # 17

Find a system of linear equations corresponding to the given matrix form .Check whether the lines responded by the system are concurrent.

$$(a)\begin{bmatrix}1 & 0 & -1\\ 2 & 0 & 1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (b)\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Solution

(a)

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+0-1 \\ 2x+0+1 \\ 0-y+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x-1 \\ 2x+1 \\ -y+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equating the elements

$$x-1=0$$
$$2x+1=0$$
$$-y+2=0$$

are the required equation of lines.

Coefficients matrix of the system

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \det A = 1(0+1) - 0 - 1(-2-0)$$
$$= 1+2 = 3 \neq 0$$

Hence system is not concurrent.

(b) *Do yourself as above.*

Homogenous 2nd Degree Equation
Every homogenous second degree equation
$$ax^2 + 2hxy + by^2 = 0$$
represents straight lines through the origin.

Consider the equations are $y = m_1x$ and $y = m_2x$
$$\Rightarrow \quad m_1x - y = 0 \quad \text{and} \quad m_2x - y = 0$$
Taking product
$$(m_1x - y)(m_2x - y) = 0$$
$$\Rightarrow \quad m_1m_2x^2 - m_1xy - m_2xy + y^2 = 0$$
$$\Rightarrow \quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0 \quad \ldots\ldots\ldots \text{(i)}$$
Also we have
$$ax^2 + 2hxy + by^2 = 0$$
$$\Rightarrow \quad \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \qquad \div\text{ing by } b$$
$$\Rightarrow \quad \frac{a}{b}x^2 - \left(-\frac{2h}{b}\right)xy + y^2 = 0$$
Comparing it with (i), we have
$$\boxed{m_1m_2 = \frac{a}{b}} \quad \text{and} \quad \boxed{m_1 + m_2 = -\frac{2h}{b}}$$
Let θ be the angles between the lines then
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$
$$= \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1m_2} = \frac{\sqrt{m_1^2 + m_2^2 - 2m_1m_2}}{1 + m_1m_1}$$
$$= \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 - 4m_1m_2}}{1 + m_1m_1} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_1}$$
$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{b + a}{b}}$$
$$\Rightarrow \quad \tan\theta = \frac{\sqrt{4(h^2 - ab)}}{b + a} \quad \Rightarrow \quad \boxed{\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}}$$

Find the lines represented by each of the following and also find measure of the angle between them (problem 1-6)

Question # 1
$10x^2 - 23xy - 5y^2 = 0$
Solution

$$10x^2 - 23xy - 5y^2 = 0 \;\ldots\ldots\ldots\ldots\; (i)$$
$$\Rightarrow\; 10x^2 - 25xy + 2xy - 5y^2 = 0$$
$$\Rightarrow\; 5x(2x-5y) + y(2x-5y) = 0 \;\Rightarrow\; (2x-5y)(5x+y) = 0$$
$$\Rightarrow\; 2x-5y = 0 \quad \text{and} \quad 5x+y = 0$$

are the required lines.

Comparing eq. (i) with
$$ax^2 + 2hxy + by^2 = 0$$
So $a = 10$, $2h = -23 \Rightarrow h = -\dfrac{23}{2}$, $b = -5$

Let θ be angle between lines then
$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$
$$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10-5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$
$$= \frac{2\sqrt{729/4}}{5} = \frac{2\left(27/2\right)}{5} = \frac{27}{5}$$
$$\Rightarrow\; \theta = \tan^{-1}\left(\frac{27}{5}\right) = 79^\circ 31'$$

Hence acute angle between the lines $= 79^\circ 31'$

Question # 2
$3x^2 + 7xy + 2y^2 = 0$
Solution *Do yourself as above*

Question # 3
$9x^2 + 24xy + 16y^2 = 0$
Solution *Do yourself as above*

Question # 4
$2x^2 + 3xy - 5y^2 = 0$
Solution $2x^2 + 3xy - 5y^2 = 0 \;\ldots\ldots\ldots\ldots\; (i)$
$$\Rightarrow\; 2x^2 + 5xy - 2xy - 5y^2 = 0$$
$$\Rightarrow\; x(2x+5y) - y(2x+5y) = 0$$
$$\Rightarrow\; (2x+5y)(x-y) = 0$$
$$\Rightarrow\; 2x+5y = 0 \;\text{ and }\; x-y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \ a = 2 \quad , \quad 2h = 3 \ \Rightarrow \ h = \frac{3}{2} \quad , \quad b = -5$$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5} = \frac{2\sqrt{\frac{9}{4}+10}}{-3}$$

$$= -\frac{2\sqrt{49/4}}{3} = -\frac{2\left(7/2\right)}{3} = -\frac{7}{3}$$

$$\Rightarrow \ -\tan\theta = \frac{7}{3}$$

$$\Rightarrow \ \tan(180-\theta) = \frac{7}{3} \qquad \because \ \tan(180-\theta) = -\tan\theta$$

$$\Rightarrow \ 180-\theta = \tan^{-1}\left(\frac{7}{3}\right) \qquad \Rightarrow \ 180-\theta = 66^\circ 48'$$

$$\Rightarrow \ \theta = 180 - 66^\circ 48' = 113^\circ 12'$$

Hence acute angle between the lines $= 180 - 113^\circ 12' = 66^\circ 48'$

Question # 5

$$6x^2 - 19xy + 15y^2 = 0$$

Solution

Do yourself as above

Question # 6

$$x^2 - 2xy\sec\alpha + y^2 = 0$$

Solution

$$x^2 - 2xy\sec\alpha + y^2 = 0 \ \ldots\ldots\ldots \text{(i)}$$

$\div$ing by y^2

$$\frac{x^2}{y^2} - \frac{2xy\sec\alpha}{y^2} + \frac{y^2}{y^2} = 0$$

$$\Rightarrow \ \left(\frac{x}{y}\right)^2 - 2\sec\alpha\left(\frac{x}{y}\right) + 1 = 0$$

This is quadric equation in $\frac{x}{y}$ with $a = 1$, $b = -2\sec\alpha$, $c = 1$

$$\frac{x}{y} = \frac{2\sec\alpha \pm \sqrt{(-2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2(1)} \quad = \frac{2\sec\alpha \pm \sqrt{4\left(\sec^2\alpha - 1\right)}}{2}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\tan^2\alpha}}{2} \qquad \because\ 1+\tan^2\alpha = \sec^2\alpha$$

$$= \frac{2\sec\alpha \pm 2\tan\alpha}{2}$$

$$\Rightarrow \frac{x}{y} = \sec\alpha \pm \tan\alpha$$

$$= \frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha} = \frac{1 \pm \sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{1+\sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{x}{y} = \frac{1-\sin\alpha}{\cos\alpha}$$

$$\Rightarrow x\cos\alpha = (1+\sin\alpha)y \quad \text{and} \quad x\cos\alpha = (1-\sin\alpha)y$$

$$\Rightarrow x\cos\alpha - (1+\sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (1-\sin\alpha)y = 0$$

These are required equations of lines.
Now comparing (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$a = 1 \quad , \quad 2h = -2\sec\alpha \ \Rightarrow\ h = -\sec\alpha \quad , \quad b = 1\text{b}$$

If θ is angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1+1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha}$$

$$\Rightarrow \tan\theta = \tan\alpha \qquad \Rightarrow \theta = \alpha.$$

Note:
If one wish to get the answer similar to given at the end of textbook, then follow the solution as follows after getting:

$$x\cos\alpha - (1+\sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (1-\sin\alpha)y = 0$$

Multiplying equation at left with $\dfrac{1-\sin\alpha}{\cos\alpha}$ and equation at right with $\dfrac{1+\sin\alpha}{\cos\alpha}$ to get

$$x(1-\sin\alpha) - \frac{\left(1-\sin^2\alpha\right)}{\cos\alpha}y = 0 \quad \text{and} \quad x(1+\sin\alpha) - \frac{\left(1-\sin^2\alpha\right)}{\cos\alpha}y = 0$$

$$\Rightarrow x(1-\sin\alpha) - y\cos\alpha = 0 \quad \text{and} \quad x(1+\sin\alpha) - y\cos\alpha = 0$$

Question # 7
Find a joint equation of the lines through the origin and perpendicular to the lines:
$x^2 - 2xy\tan\alpha - y^2 = 0$

Solution Given: $x^2 - 2xy\tan\alpha - y^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b}$$
$$= -\frac{-2\tan\alpha}{-1}$$
$$\Rightarrow m_1 + m_2 = -2\tan\alpha$$

$a = 1$,
$2h = -2\tan\alpha$
$\Rightarrow h = -\tan\alpha$
$b = -1$

$$\& \quad m_1 m_2 = \frac{a}{b} = \frac{1}{-1} \quad \Rightarrow m_1 m_2 = -1$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$
$$\Rightarrow m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow x^2 + (-2\tan\alpha)xy + (-1)y^2 = 0$$
$$\Rightarrow x^2 - 2xy\tan\alpha - y^2 = 0$$

Question # 8

Find a joint equation of the lines through the origin and perpendicular to the lines: $ax^2 + 2hxy + by^2 = 0$

Solution

Given: $ax^2 + 2hxy + by^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1 m_2 = \frac{a}{b}$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$
$$\Rightarrow m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow x^2 + \left(-\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)y^2 = 0$$
$$\Rightarrow bx^2 - 2hxy + ay^2 = 0$$

Question # 9

Find the area of the region bounded by:

$$10x^2 - xy - 21y^2 = 0 \quad \text{and} \quad x + y + 1 = 0$$

Solution $\quad 10x^2 - xy - 21y^2 = 0 \quad , \quad x + y + 1 = 0$

$$\Rightarrow 10x^2 - 15xy + 14xy - 21y^2 = 0$$

$$\Rightarrow 5x(2x - 3y) + 7y(2x - 3y) = 0$$

$$\Rightarrow (2x - 3y)(5x + 7y) = 0$$

$$\Rightarrow 2x - 3y = 0 \quad \text{or} \quad 5x + 7y = 0$$

So we have equation of lines

$l_1 :\ 2x - 3y = 0$ ………… (i)

$l_2 :\ 5x + 7y = 0$ ………. (ii)

$l_3 :\ x + y + 1 = 0$ ……… (iii)

Now do yourself as Q # 14 (Ex. 4.4)

Chapter 5

LINEAR INEQUALITIES AND LINEAR PROGRAMMING

EXERCISE 5.1

Q.1: Graph the solution set of each of the following linear inequality in xy-plane.

(i)	$2x + y \leq 6$	**(ii)**	$3x + 7y \geq 21$	**(iii)**	$3x - 2y \geq 6$
(iv)	$5x - 4y \leq 20$	**(v)**	$2x + 1 \geq 0$	**(vi)**	$3y - 4 \leq 0$

Solution:

(i) $2x + y \leq 6$

The associated equation is

$2x + y = 6$ ...... (1)

x-intercept

Put $y = 0$ in eq. (1)

$2x + 0 = 6$

$x = \frac{6}{2} = 3$

$\therefore$ Point is (3, 0)

y-intercept

Put $x = 0$ in eq. (1)

$2(0) + y = 6$

$y = 6$

$\therefore$ Point is (0, 6)

Test Point

Put (0, 0) in

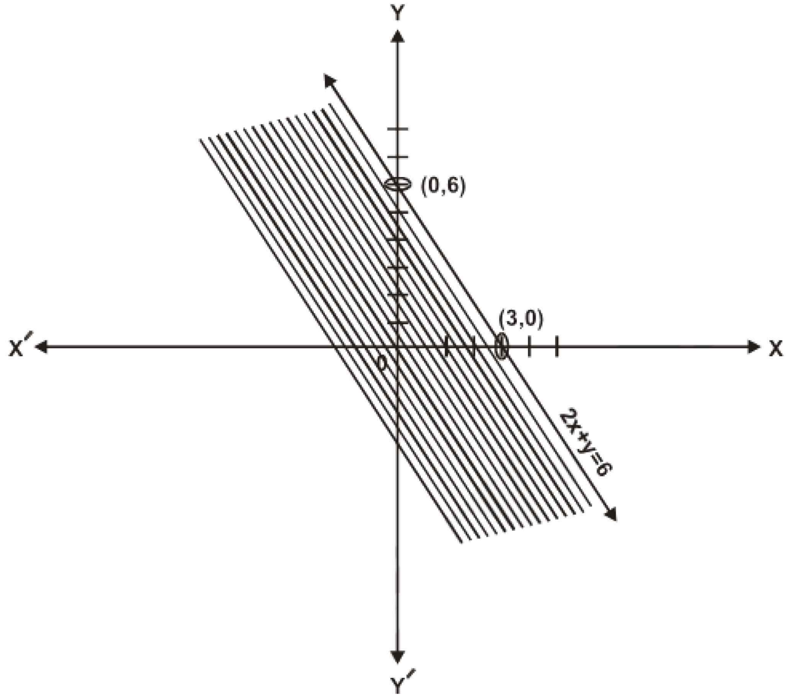
$2x + y < 6$

$2(0) + 0 < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 6$ will be towards the origin side.



(ii) $3x + 7y \geq 21$

The associated equation is

$3x + 7y = 21$ .... (1)

x-intercept

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \frac{21}{3} = 7$

$\therefore$ Point is (7, 0)

y-intercept

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$$y = \frac{21}{7} = 3$$

$\therefore$ Point is (0, 3)

<u>Test Point</u>

Put (0, 0) in

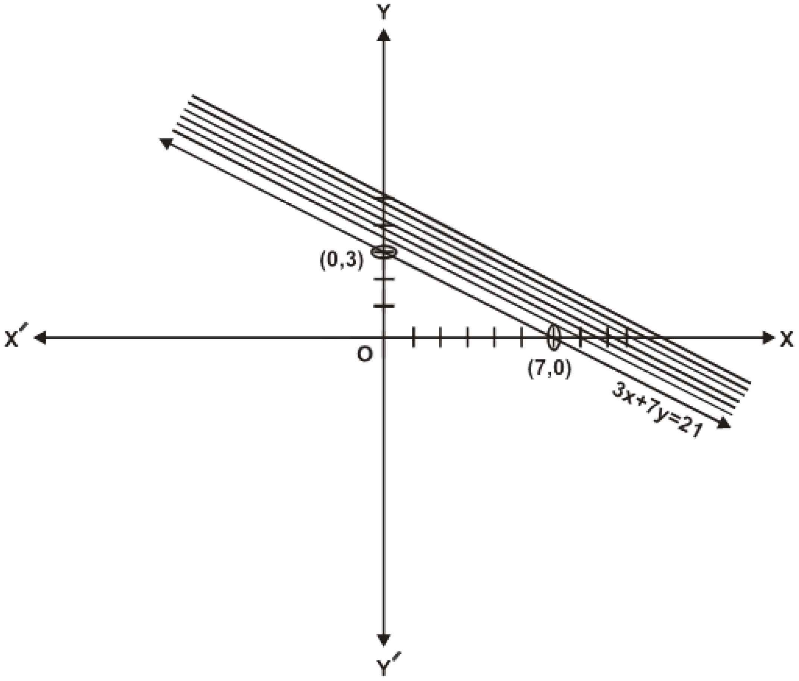
$3x + 7y > 21$

$3(0) + 7(0) > 21$

$0 > 21$

Which is false.

$\therefore$ Graph of an inequality $3x + 7y \geq 21$ will not be towards the origin side.



(iii) $3x - 2y \geq 6$

The associated equation is

$3x - 2y = 6$ .... (1)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x - 2(0) = 6$

$3x = 6$

$$x = \frac{6}{3} = 2$$

$\therefore$ Point is (2, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$3(0) - 2y = 6$

$-2y = 6$

$y = \dfrac{6}{-2} = -3$

$\therefore$ Point is $(0, -3)$

<u>Test Point</u>

Put $(0, 0)$ in

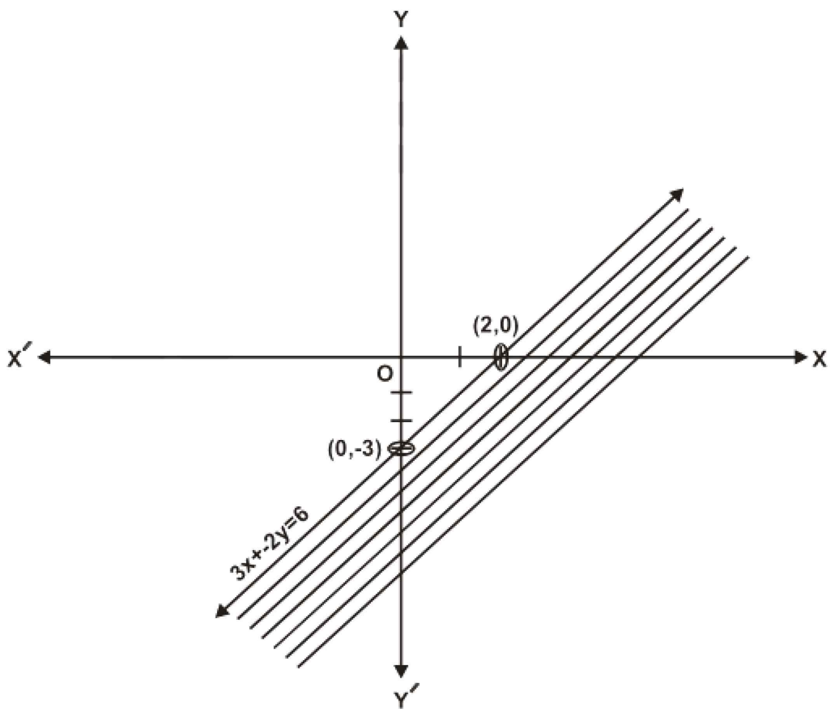
$3x - 2y > 6$

$3(0) + 2(0) > 6$

$0 > 6$

Which is false.

$\therefore$ Graph of an inequality $3x - 2y \geq 6$ will not be towards the origin side.



(iv) $\mathbf{5x - 4y \leq 20}$

The associated equation is

$5x - 4y = 20$ ....(1)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$5x - 4(0) = 20$$
$$5x = 20$$
$$x = \frac{20}{5} = 4$$

$\therefore$ Point is (4, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$5(0) - 4y = 20$$
$$-4y = 20$$
$$y = \frac{20}{-4} = -5$$

$\therefore$ Point is (0, – 5)

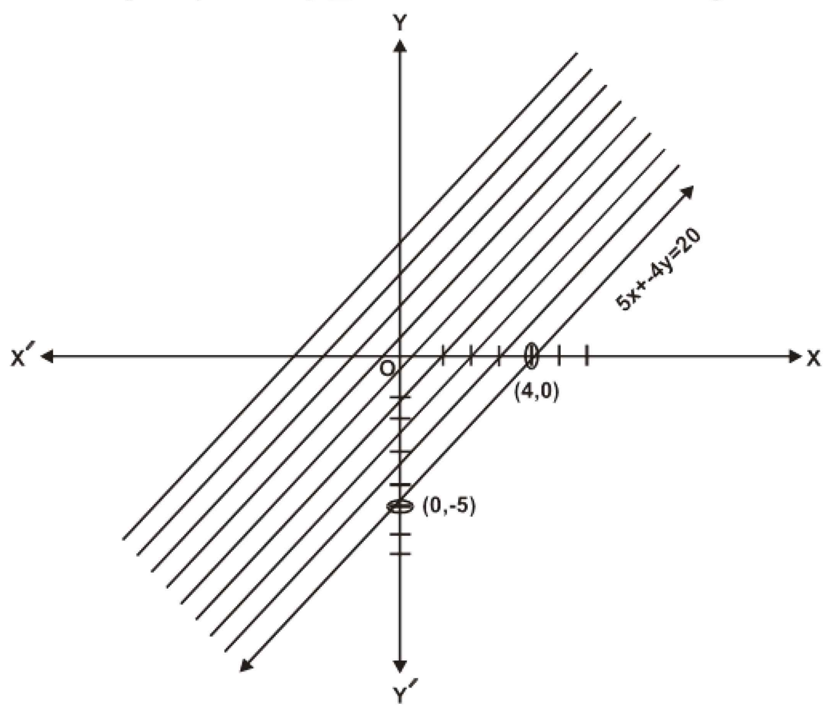
<u>Test Point</u>

Put (0, 0) in

$$5x - 4y < 20$$
$$5(0) - 4(0) < 20$$
$$0 < 20$$

Which is true.

$\therefore$ Graph of an inequality $5x - 4y \leq 20$ will be towards the origin side.



(v) $2x + 1 \geq 0$

The associated equation is

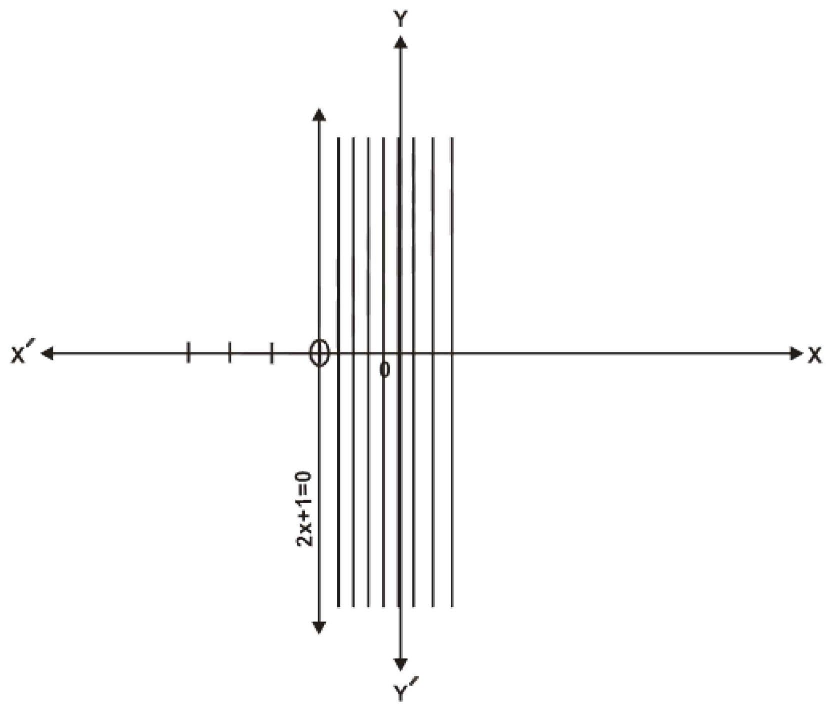
$$2x + 1 = 0$$
$$2x = -1$$
$$x = \frac{-1}{2}$$

Put $x = 0$ in

$$2x + 1 > 0$$
$$2(0) + 1 > 0$$
$$1 > 0$$

Which is true.

$\therefore$ Graph of an inequality $2x + 1 \geq 0$ will be towards the origin side.



(vi) $3y - 4 \leq 0$

The associated equation is

$$3y - 4 = 0$$
$$3y = 4$$
$$y = \frac{4}{3}$$

Put $y = 0$ in

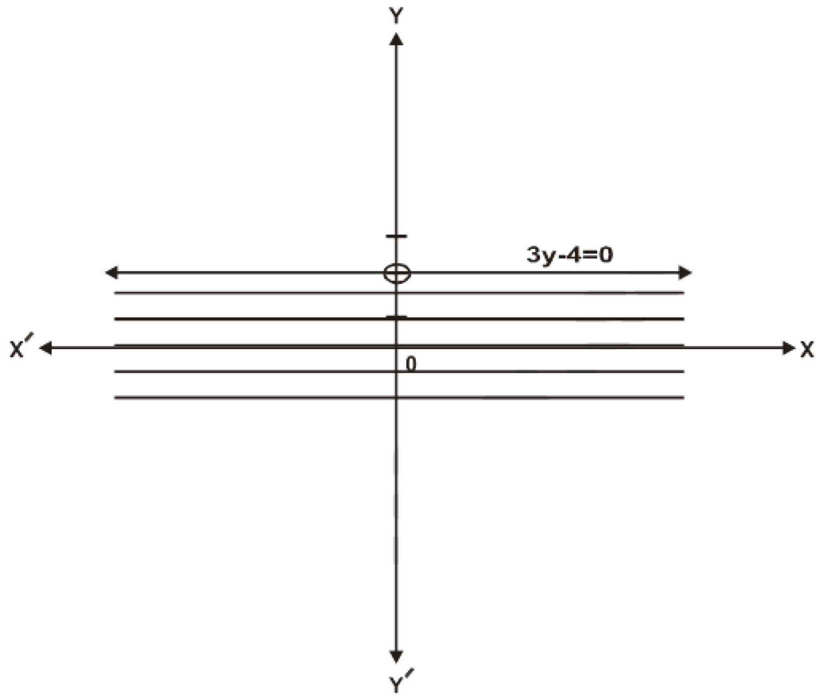
$3y - 4 < 0$

$3(0) - 4 < 0$

$-4 < 0$

Which is true.

$\therefore$ Graph of an inequality $3y - 4 \leq 0$ will be towards the origin side.



Q.2: Indicate the solution set of the following systems of linear inequalities by shading.

(i) $2x - 3y \leq 6$, $2x + 3y \leq 12$ **(ii)** $x + y \geq 5$, $x - y \leq 1$ **(iii)** $3x + 7y \geq 21$, $x - y \leq 2$

(iv) $4x - 3y \leq 12$

$x \geq \dfrac{-3}{2}$ **(Lhr. Board 2011) (Guj. Board 2008)**

(v) $3x + 7y \geq 21$ **(Lhr. Board 2011)**

$y \leq 4$

Solution:

(i) $2x - 3y \leq 6$

$2x + 3y \leq 12$

The associated equations are

$$2x - 3y = 6 \quad \text{...... (1)}$$
$$2x + 3y = 12 \quad \text{...... (2)}$$

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$2x - 3(0) = 6$$
$$2x = 6$$
$$x = \frac{6}{2} = 3$$

$\therefore$ Point is (3, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$2(0) - 3y = 6$$
$$-3y = 6$$
$$y = \frac{6}{-3} = -2$$

$\therefore$ Point is (0, – 2)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$2x + 3(0) = 12$$
$$2x = 12$$
$$x = \frac{12}{2} = 6$$

$\therefore$ Point is (6, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$2(0) + 3y = 12$$
$$3y = 12$$
$$y = \frac{12}{3} = 4$$

$\therefore$ Point is (0, 4)

<u>Test Point</u>

Put (0, 0) in

$$2x - 3y < 6$$

$2(0) - 3(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $2x - 3y \leq 6$ will be towards the origin side.

Put (0, 0) in

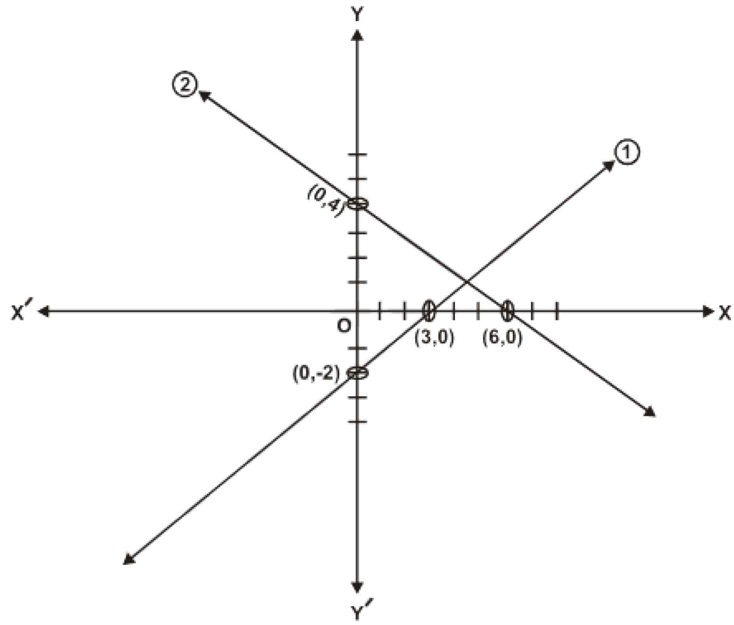
$2x + 3y < 12$

$2(0) - 3(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 12$ will be towards the origin side.



(ii) $\mathbf{x + y \geq 5}$

$\mathbf{x - y \leq 1}$

The associated equations are

$x + y = 5$ ...... (1)

$x - y = 1$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$x + 0 = 5$

$x = 5$

$\therefore$ Point is (5, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$0 + y = 5$

$y = 5$

$\therefore$ Point is (0, 5)

x-intercept

Put $y = 0$ in eq. (2)

$x - 0 = 1$

$x = 1$

$\therefore$ Point is (1, 0)

y-intercept

Put $x = 0$ in eq. (2)

$0 - y = 1$

$y = -1$

$\therefore$ Point is (0, – 1)

Test Point

Put (0, 0) in

$x + y > 5$

$0 + 0 > 5$

$0 > 5$

Which is false.

$\therefore$ Graph of an inequality $x + y \geq 5$ will not be towards the origin side.

Put (0, 0) in

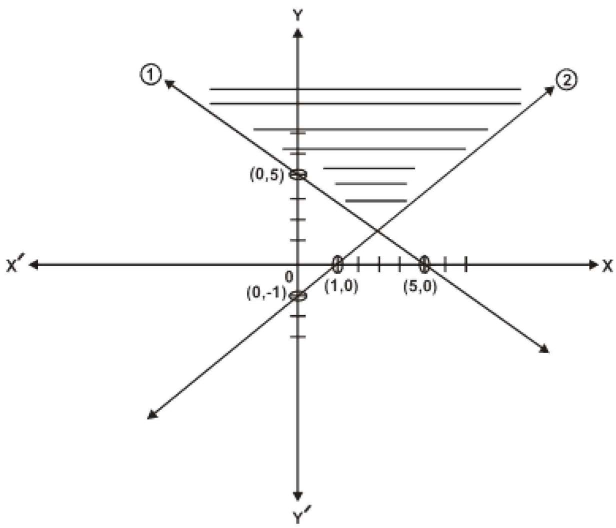
$x - y < 1$

$0 - 0 < 1$

$0 < 1$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 1$ will be towards the origin side.



(iii) $\mathbf{3x + 7y \geq 21}$

$\mathbf{x - y \leq 2}$

The associated equations are

$3x + 7y = 21$ ..... (1)

$x - y = 2$ ..... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \frac{21}{3} = 7$

$\therefore$ Point is (7, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$y = \frac{21}{7} = 3$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x - 0 = 2$

$x = 2$

$\therefore$ Point is (2, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 - y = 2$

$y = -2$

$\therefore$ Point is (0, – 2)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y > 21$

$3(0) + 7(0) > 21$

$0 > 21$

Which is false.

$\therefore$ Graph of an inequality $3x + 7y \geq 21$ will not be towards the origin side.

Put (0, 0) in

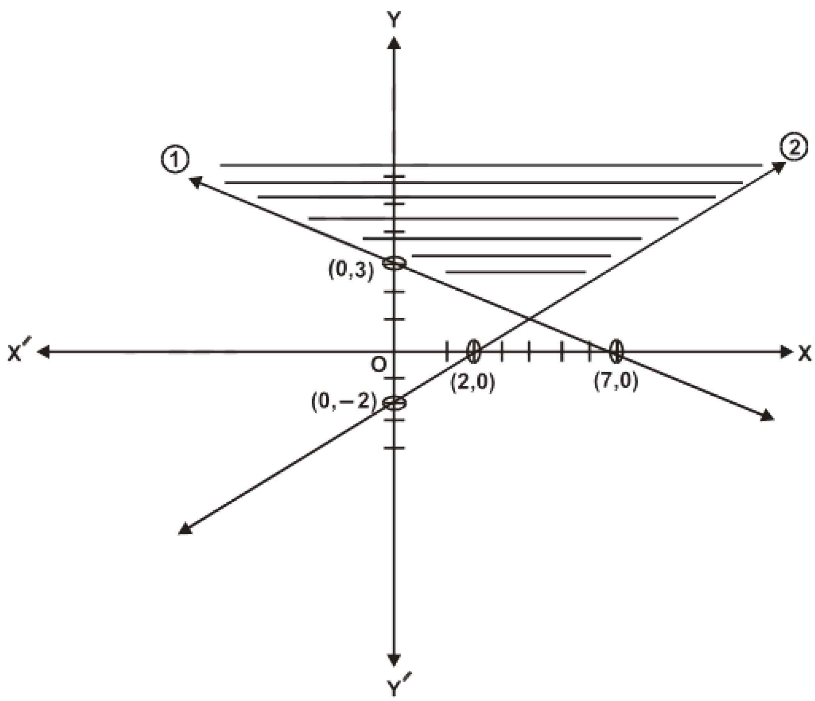
$x - y < 2$

$0 - 0 < 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 2$ will be towards the origin side.



(iv) $\mathbf{4x - 3y \leq 12}$

$\mathbf{x \geq \dfrac{-3}{2}}$

The associated equations are

$4x - 3y = 12$ ..... (1)

$$x = \frac{-3}{2} \quad ..... (2)$$

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$4x - 3(0) = 12$$

$$4x = 12$$

$$x = \frac{21}{4} = 3$$

$\therefore$ Point is (3, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$4(0) - 3y = 12$$

$$-3y = 12$$

$$y = \frac{12}{-3} = -4$$

$\therefore$ Point is (0, – 4)

<u>Test Point</u>

Put (0, 0) in

$$4x - 3y < 12$$

$$4(0) - 3(0) < 12$$

$$0 < 12$$

Which is true.

$\therefore$ Graph of an inequality $4x - 3y \leq 12$ will be towards the origin side.

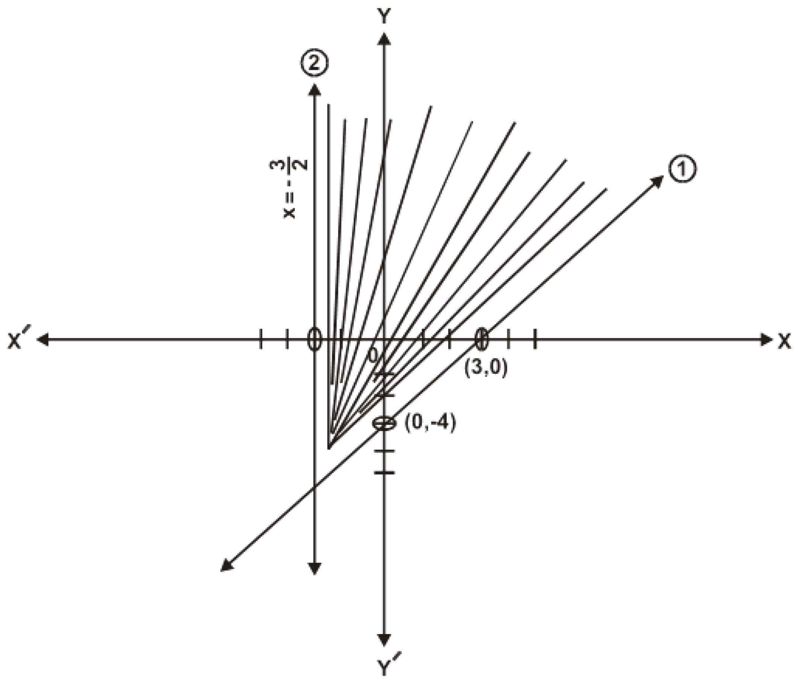
Put $x = 0$ in

$$x > \frac{-3}{2}$$

$$0 > \frac{-3}{2}$$

Which is true.

$\therefore$ Graph of an inequality $x \geq \frac{-3}{2}$ will be towards the origin side.



(v) $\mathbf{3x + 7y \geq 21}$

$\mathbf{y \leq 4}$

The associated equations are

$3x + 7y = 21$ ...... (1)

$y = 4$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \frac{21}{3} = 7$

$\therefore$ Point is (7, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$y = \frac{21}{7} = 3$

$\therefore$ Point is (0, 3)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y > 21$

$3(0) + 7(0) > 21$

$0 > 21$

Which is false.

$\therefore$ Graph of an inequality $3x + 7y \geq 21$ will not be towards the origin side.

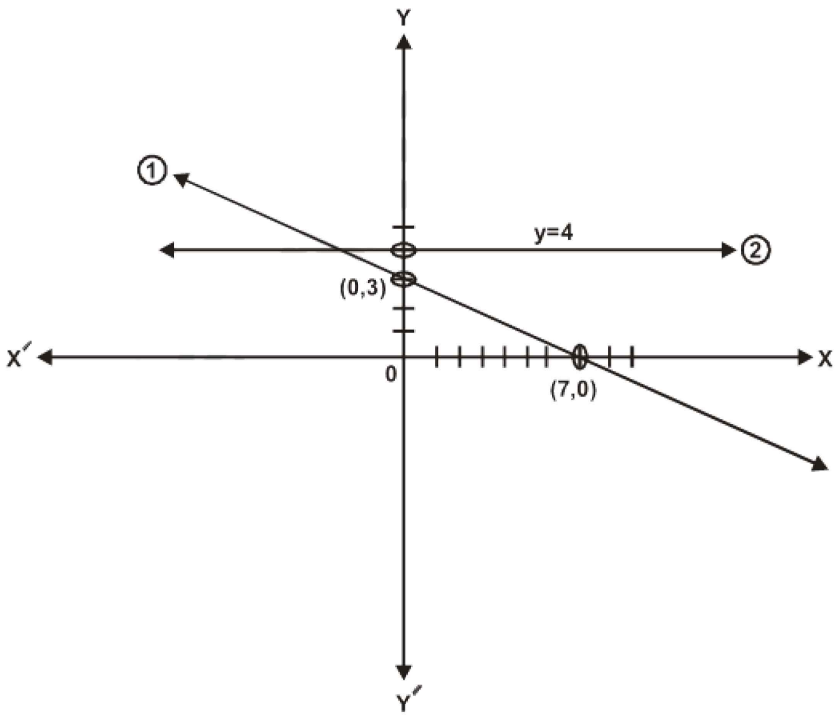
Put $y = 0$ in

$y < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $y \leq 4$ will be towards the origin side.



Q.3: Indicate the solution region of the following systems of linear inequalities by shading.

(i)	$2x - 3y \leq 6$	**(ii)**	$x + y \leq 5$	**(iii)**	$x + y \geq 5$
	$2x + 3y \leq 12$		$y - 2x \leq 2$		$x - y \geq 1$
	$y \geq 0$		$x \geq 0$		$y \geq 0$

(iv) $3x + 7y \le 21$, $x - y \le 2$, $x \ge 0$ **(v)** $3x + 7y \le 21$, $x - y \le 2$, $y \ge 0$ **(vi)** $3x + 7y \le 21$, $2x - y \ge -3$, $x \ge 0$

Solution:

(i) $2x - 3y \le 6$ **(Lhr. Board 2007)**

$2x + 3y \le 12$

$y \ge 0$

The associated equations are

$2x - 3y = 6$ ...... (1)

$2x + 3y = 12$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$2x - 3(0) = 6$

$2x = 6$

$x = \frac{6}{2} = 3$

$\therefore$ Point is (3, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$2(0) - 3y = 6$

$-3y = 6$

$y = \frac{6}{-3} = -2$

$\therefore$ Point is (0, – 2)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$2x + 3(0) = 12$

$2x = 12$

$x = \frac{12}{2} = 6$

$\therefore$ Point is (6, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$2(0) + 3y = 12$$
$$3y = 12$$
$$y = \frac{12}{3} = 4$$

$\therefore$ Point is (0, 4)

<u>Test Point</u>

Put (0, 0) in

$2x - 3y < 6$

$2(0) - 3(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $2x - 3y \leq 6$ will be towards the origin side.

Put (0, 0) in

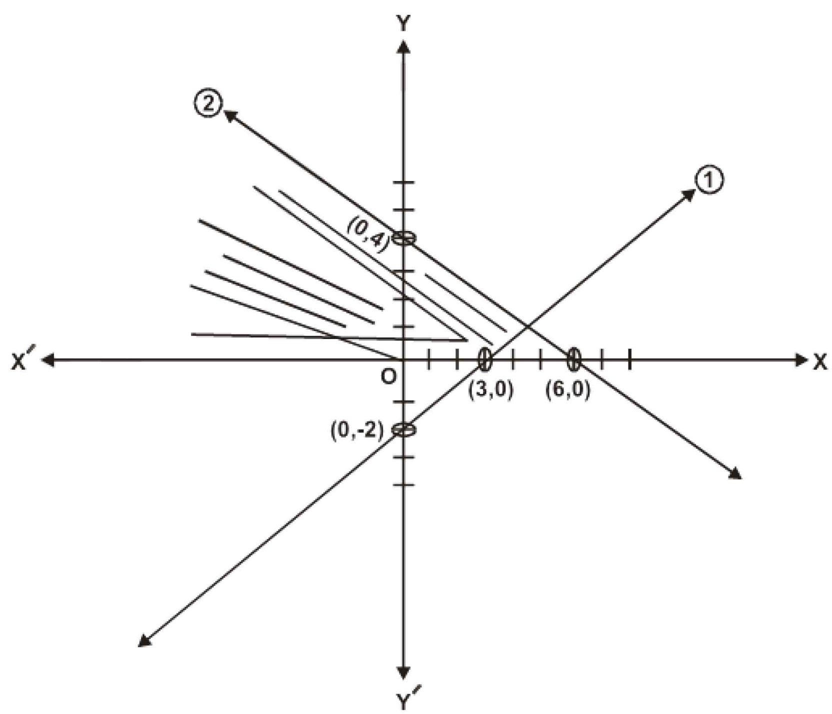
$2x + 3y < 12$

$2(0) + 3(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 12$ will be towards the origin side.



(ii) $\mathbf{x + y \leq 5}$

$\mathbf{y - 2x \leq 2}$

$\mathbf{x \geq 0}$

The associated equations are

$x + y = 5$ ...... (1)

$y - 2x = 2$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$x + 0 = 5$

$x = 5$

$\therefore$ Point is (5, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$0 + y = 5$

$y = 5$

$\therefore$ Point is (0, 5)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$0 - 2x = 2$

$$x = \frac{2}{-2} = -1$$

$\therefore$ Point is (– 1, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$y - 2(0) = 2$

$y = 2$

$\therefore$ Point is (0, 2)

<u>Test Point</u>

Put (0, 0) in

$x + y < 5$

$0 + 0 < 5$

$0 < 5$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 5$ will towards the origin side.

Put (0, 0) in

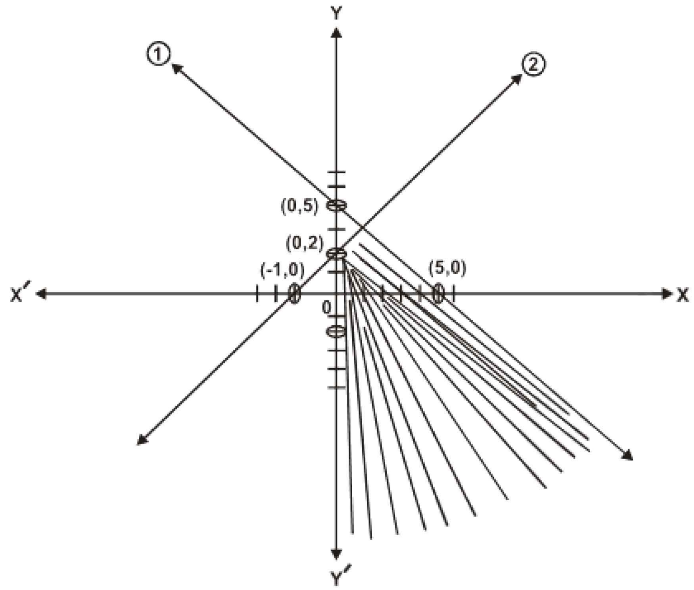
$y - 2x \quad < \quad 2$

$0 - 2(0) \quad < \quad 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $y - 2x \leq 2$ will towards the origin side.



(iii) $\mathbf{x + y \geq 5}$

$\mathbf{x - y \geq 1}$

$\mathbf{y \geq 0}$

The associated equations are

$x + y = 5 \quad .... (1)$

$x - y = 1 \quad .... (2)$

x–intercept

Put $y = 0$ in eq. (1)

$x + 0 = 5$

$x = 5$

$\therefore$ Point is (5, 0)

y–intercept

Put $x = 0$ in eq. (1)

$0 + y = 5$

$y = 5$

$\therefore$ Point is (0, 5)

x–intercept

Put $y = 0$ in eq. (2)

$x - 0 = 1$

$x = 1$

$\therefore$ Point is (1, 0)

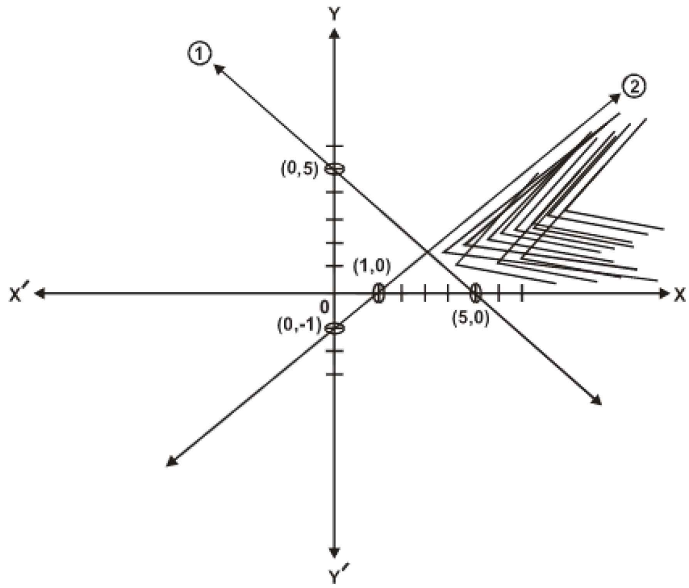
y–intercept

Put $x = 0$ in eq. (2)

$0 - y = 1$

$y = -1$

$\therefore$ Point is (0, – 1)



Test Point

Put (0, 0) in

$x + y > 5$

$0 + 0 > 5$

$0 > 5$

Which is false.

$\therefore$ Graph of an inequality $x + y \geq 5$ will not be towards the origin side.

Put (0, 0) in

$x - y > 1$

$0 - 0 > 1$

$0 > 1$

Which is false.

$\therefore$ Graph of an inequality $x - y \geq 1$ will not be towards the origin side.

(iv) $\mathbf{3x + 7y \leq 21}$

$\mathbf{x - y \leq 2}$

$\mathbf{x \geq 0}$

The associated equations are

$3x + 7y = 21$ ...... (1)

$x - y = 2$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$3x + 7(0) = 21$$
$$3x = 21$$
$$x = \frac{21}{3} = 7$$

$\therefore$ Point is (7, 0)

<u>y–intercept</u>

Put $x = 0$ in eq. (1)

$$3(0) + 7y = 21$$
$$7y = 21$$
$$y = \frac{21}{7} = 3$$

$\therefore$ Point is (0, 3)

<u>x–intercept</u>

Put $y = 0$ in eq. (2)

$$x - 0 = 2$$
$$x = 2$$

$\therefore$ Point is (2, 0)

<u>y–intercept</u>

Put $x = 0$ in eq. (2)

$$0 - y = 2$$
$$y = -2$$

$\therefore$ Point is (0, – 2)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y < 21$

$3(0) + 7(0) < 21$

$0 < 21$

Which is true.

$\therefore$ Graph of an inequality $3x + 7y \leq 21$ will be towards the origin side.

Put (0, 0) in

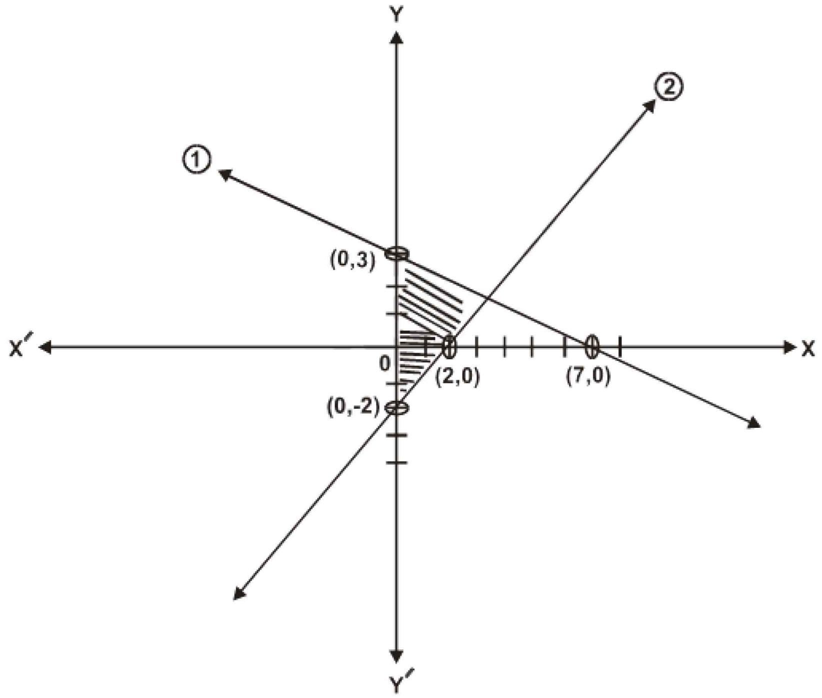
$x - y < 2$

$0 - 0 < 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 2$ will be towards the origin side.



(v) **$3x + 7y \leq 21$** **(Gujranwala Board 2007)**

$x - y \leq 2$

$y \geq 0$

The associated equations are

$3x + 7y = 21$ ...... (1)

$x - y = 2$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \frac{21}{3} = 7$

$\therefore$ Point is (7, 0)

<u>y–intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$y = \frac{21}{7} = 3$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x - 0 = 2$

$x = 2$

$\therefore$ Point is (2, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 - y = 2$

$y = -2$

$\therefore$ Point is (0, – 2)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y < 21$

$3(0) + 7(0) < 21$

$0 < 21$

Which is true.

$\therefore$ Graph of an inequality $3x + 7y \leq 21$ will be towards the origin side.

Put (0, 0) in

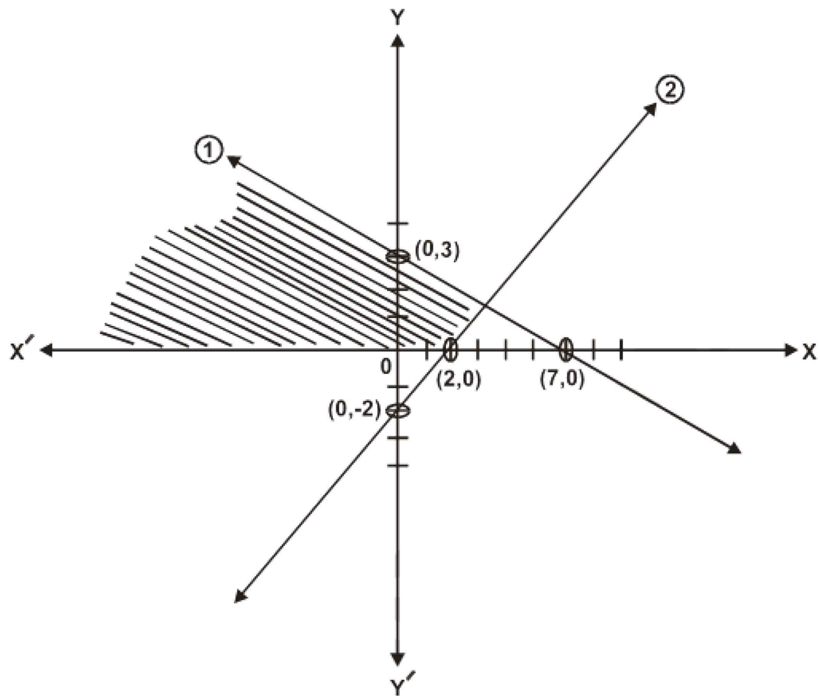
$x - y < 2$

$0 - 0 < 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 2$ will be towards the origin side.



(vi) $\mathbf{3x + 7y \leq 21}$ **(Gujranwala Board 2006)**

$\mathbf{2x - y \geq -3}$

$\mathbf{x \geq 0}$

The associated equations are

$3x + 7y = 21$ ...... (1)

$2x - y = -3$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \frac{21}{3} = 7$

$\therefore$ Point is (7, 0)

<u>y–intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$y = \frac{21}{7} = 3$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$2x - 0 = -3$

$x = \frac{-3}{2}$

$\therefore$ Point is $\left(\frac{-3}{2}, 0\right)$

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$2(0) - y = -3$

$-y = -3$

$y = 3$

$\therefore$ Point is (0, 3)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y < 21$

$3(0) + 7(0) < 21$

$0 < 21$

Which is true.

$\therefore$ Graph of an inequality $3x + 7y \leq 21$ will be towards the origin side.

Put (0, 0) in

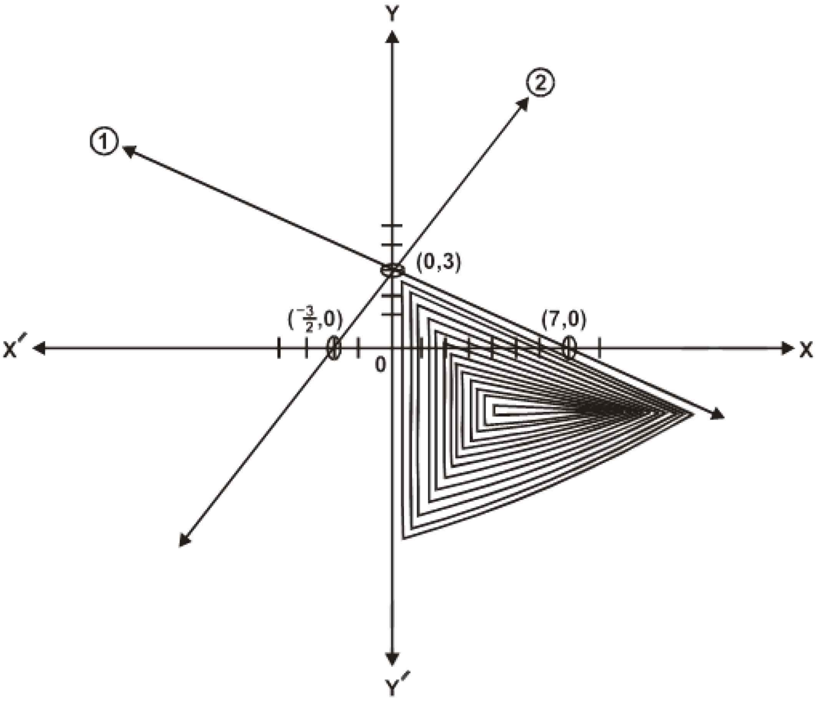
$2x - y > -3$

$2(0) - 0 > -3$

$0 > -3$

Which is true.

$\therefore$ Graph of an inequality $2x - y \geq -3$ will be towards the origin side.



Q.4: Graph the solution region of the following system of linear inequalities and find the corner points in each case.

(i)	$2x - 3y \leq 6$	**(ii)**	$x + y \leq 5$	**(iii)**	$3x + 7y \leq 21$
	$2x + 3y \leq 12$		$-2x + y \leq 2$		$2x - y \leq -3$
	$x \geq 0$		$y \geq 0$		$y \geq 0$
(iv)	$3x + 2y \geq 6$	**(v)**	$5x + 7y \leq 35$	**(vi)**	$5x + 7y \leq 35$
	$x + 3y \leq 6$		$-x + 3y \leq 3$		$x - 2y \leq 2$
	$y \geq 0$		$x \geq 0$		$x \geq 0$

Solution:

(i) $2x - 3y \leq 6$

$2x + 3y \leq 12$

$x \geq 0$

The associated equations are

$2x - 3y = 6$ ...... (1)

$2x + 3y = 12 \quad ...... (2)$

x-intercept

Put $y = 0$ in eq. (1)

$2x - 3(0) = 6$

$2x = 6$

$x = \frac{6}{2} = 3$

$\therefore$ Point is (3, 0)

y-intercept

Put $x = 0$ in eq. (1)

$2(0) - 3y = 6$

$-3y = 6$

$y = \frac{6}{-3} = -2$

$\therefore$ Point is (0, – 2)

x-intercept

Put $y = 0$ in eq. (2)

$2x + 3(0) = 12$

$x = 12$

$x = \frac{12}{2} = 6$

$\therefore$ Point is (6, 0)

y-intercept

Put $x = 0$ in eq. (2)

$2(0) + 3y = 12$

$3y = 12$

$y = \frac{12}{3} = 4$

$\therefore$ Point is (0, 4)

Test Point

Put (0, 0) in

$2x - 3y < 6$

$2(0) - 3(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $2x - 3y \leq 6$ will be towards the origin side.

Put (0, 0) in

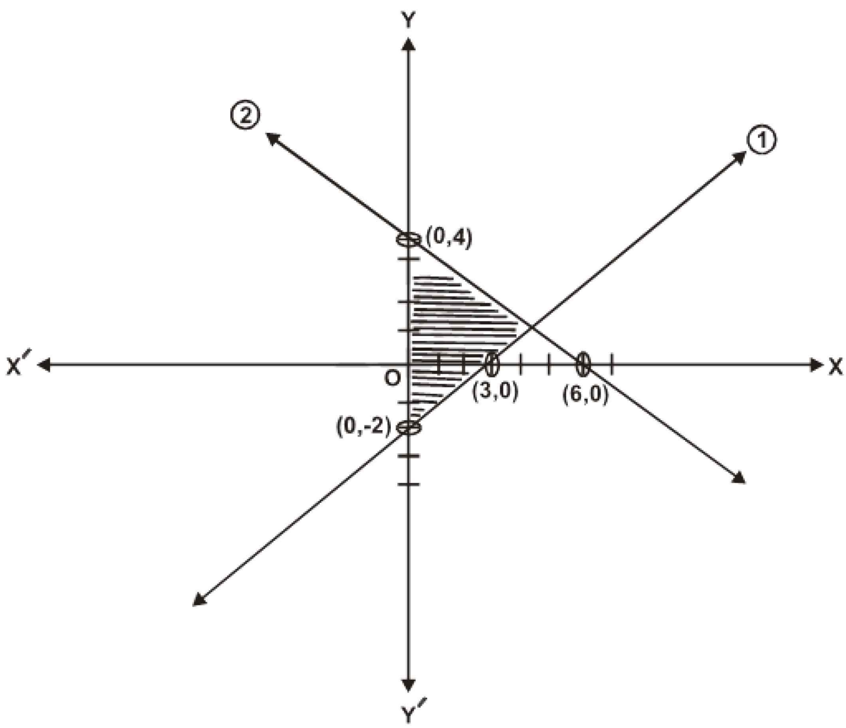
$2x + 3y < 12$

$2(0) + 3(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 12$ will be towards the origin side.



To find the intersection of both the lines solving eq. (1) & eq. (2)

Adding eq. (1) and eq. (2)

$$2x - 3y = 6$$
$$\underline{2x + 3y = 12}$$
$$4x = 18$$
$$x = \frac{18}{4} = \frac{9}{2}$$

Put $x = \frac{9}{2}$ in eq. (1)

$$2\left(\frac{9}{2}\right) - 3y = 6$$

$$9 - 3y = 6$$

$$y = \frac{8}{3} = 1$$

$\therefore$ Point $\left(\frac{9}{2}, 1\right)$

So the corner points are $(0, -2), \left(\frac{9}{2}, 1\right)(0, 4)$

(ii) $\mathbf{x + y \leq 5}$

$\mathbf{-2x + y \leq 2}$

$\mathbf{y \geq 0}$

The associated equations are

$x + y = 5$ ...... (1)

$y - 2x = 2$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$x + 0 = 5$$

$$x = 5$$

$\therefore$ Point is $(5, 0)$

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$0 + y = 5$$

$$y = 5$$

$\therefore$ Point is $(0, 5)$

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$0 - 2x = 2$$

$$x = \frac{2}{-2} = -1$$

$\therefore$ Point is $(-1, 0)$

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$y - 2(0) = 2$

$y = 2$

$\therefore$ Point is (0, 2)

Test Point

Put (0, 0) in

$x + y < 5$

$0 + 0 < 5$

$0 < 5$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 5$ will towards the origin side.

Put (0, 0) in

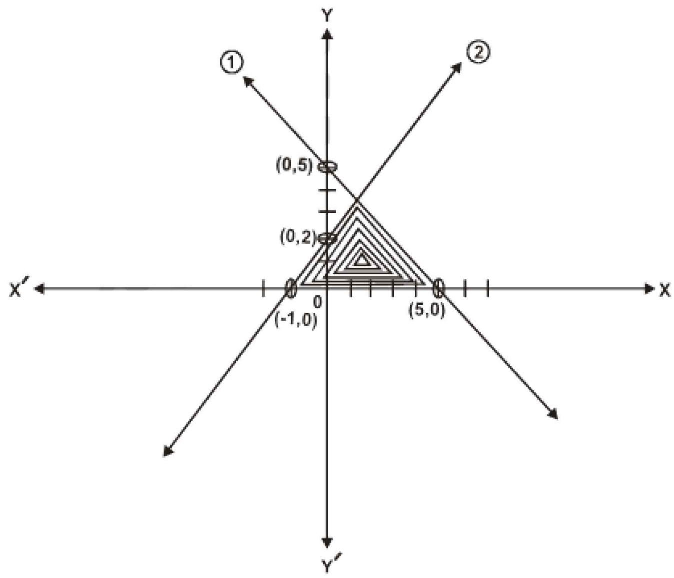
$y - 2x < 2$

$0 - 2(0) < 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $y - 2x \leq 2$ will towards the origin side.



To find the intersection of both the lines solving eq. (1) & eq. (2).

Equation (1) – Eq. (2), we get

$$\begin{aligned} x + y &= 5 \\ \underline{-2x \pm y} &= \underline{-2} \\ 3x &= 3 \end{aligned}$$

$$x = \frac{3}{3} = 1$$

Put $x = 1$ in eq. (1)

$1 + y = 5$

$y = 5 - 1 = 4$

$\therefore$ Point (1, 4)

So the corner points are (– 1, 0), (5, 0), (1, 4)

(iii) $\mathbf{3x + 7y \leq 21}$

$\mathbf{2x - y \leq -3}$

$\mathbf{y \geq 0}$

The associated equations are

$3x + 7y = 21$ ...... (1)

$2x - y = -3$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x + 7(0) = 21$

$3x = 21$

$x = \dfrac{21}{3} = 7$

$\therefore$ Point is (7, 0)

<u>y–intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 7y = 21$

$7y = 21$

$y = \dfrac{21}{7} = 3$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$2x - 0 = -3$

$x = \dfrac{-3}{2}$

$\therefore$ Point is $\left(\dfrac{-3}{2}, 0\right)$

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$2(0) - y = -3$

$-y = -3$

$y = 3$

$\therefore$ Point is (0, 3)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y < 21$

$3(0) + 7(0) < 21$

$0 < 21$

Which is true.

$\therefore$ Graph of an inequality $3x + 7y \leq 21$ will not be towards the origin side.

Put (0, 0) in

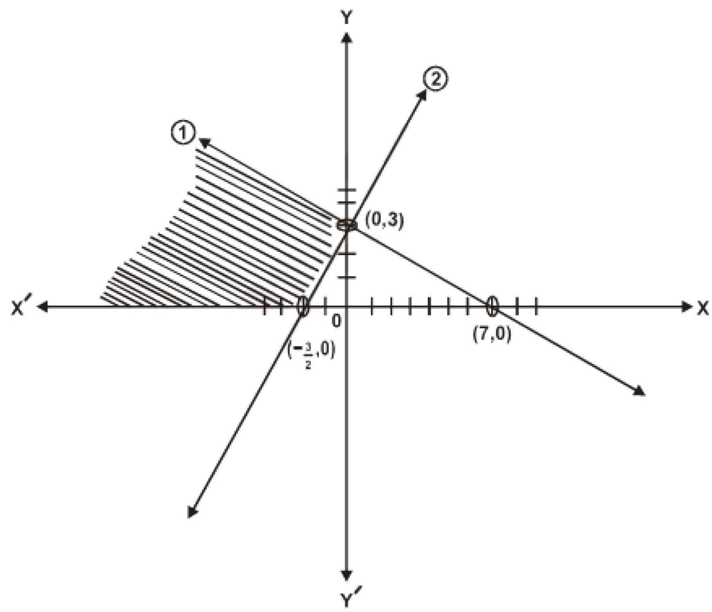
$2x - y < -3$

$2(0) - 0 < -3$

$0 < -3$

Which is false.

$\therefore$ Graph of an inequality $2x - y \leq -3$ will not be towards the origin side.



So the corner points are $\left(\dfrac{-3}{2}\ ,\ 0\right)(0, 3)$

(iv) $\mathbf{3x + 2y \geq 6}$

$\mathbf{x + 3y \leq 6}$

$\mathbf{y \geq 0}$

The associated equations are

$3x + 2y = 6$ ...... (1)

$x + 3y = 6$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$$3x + 2(0) = 6$$
$$3x = 6$$
$$x = \frac{6}{3} = 2$$

$\therefore$ Point is (2, 0)

y–intercept

Put $x = 0$ in eq. (1)

$$3(0) + 2y = 6$$
$$y = \frac{6}{2} = 3$$

$\therefore$ Point is (0, 3)

x-intercept

Put $y = 0$ in eq. (2)

$$x + 3(0) = 6$$
$$x = 6$$

$\therefore$ Point is (6, 0)

y-intercept

Put $x = 0$ in eq. (2)

$$0 + 3y = 6$$
$$y = \frac{6}{3}$$
$$y = 2$$

$\therefore$ Point is (0, 2)

Test Point

Put (0, 0) in

$$3x + 2y > 6$$
$$3(0) + 2(0) > 6$$
$$0 < 6$$

Which is false.

$\therefore$ Graph of an inequality $3x + 2y \geq 6$ will not be towards the origin side.

Put (0, 0) in

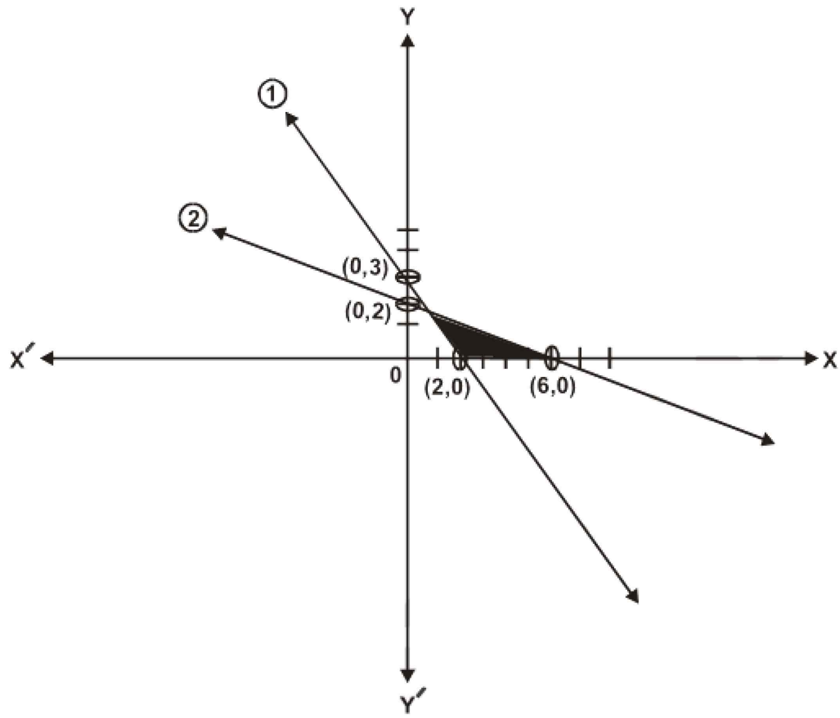
$$x + 3y < 6$$

$0 - 3(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $x + 3y \leq 6$ will be towards the origin side.



To find the intersection of both the equations solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 3, we get

$$\begin{array}{r} 3x + 2y = 6 \\ \underline{-3x \pm 9y = -18} \\ -7y = -12 \end{array}$$

$$y = \frac{12}{7}$$

Put $y = \frac{12}{7}$ in eq. (2)

$$x + 3\left(\frac{12}{7}\right) = 6$$

$$x + \frac{36}{7} = 6$$

$$x = 6 - \frac{36}{7}$$

$$x = \frac{42-36}{7}$$

$$= \frac{6}{7}$$

$\therefore$ Point $\left(\frac{6}{7}, \frac{12}{7}\right)$

So the corner points are (2, 0), (6, 0), $\left(\frac{6}{7}, \frac{12}{7}\right)$

(v) $\mathbf{5x + 7y \leq 35}$

$\mathbf{-x + 3y \leq 3}$

$\mathbf{x \geq 0}$

The associated equations are

$5x + 7y = 35$ ...... (1)

$-x + 3y = 3$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$5x + 7(0) = 35$

$5x = 35$

$x = \frac{35}{5} = 7$

$\therefore$ Point is (7, 0)

y–intercept

Put $x = 0$ in eq. (1)

$5(0) + 7y = 35$

$y = \frac{35}{7} = 5$

$\therefore$ Point is (0, 5)

x-intercept

Put $y = 0$ in eq. (2)

$-x + 3(0) = 3$

$-x = 3$

$x = -3$

$\therefore$ Point is (– 3, 0)

y-intercept

Put $x = 0$ in eq. (2)

$-0 + 3y = 3$

$$y = \frac{3}{3} = 1$$

$\therefore$ Point is (0, 1)

<u>Test Point</u>

Put (0, 0) in

$5x + 7y < 35$

$5(0) + 7(0) < 35$

$0 < 35$

Which is true.

$\therefore$ Graph of an inequality $5x + 7y \leq 35$ will be towards the origin side.

Put (0, 0) in

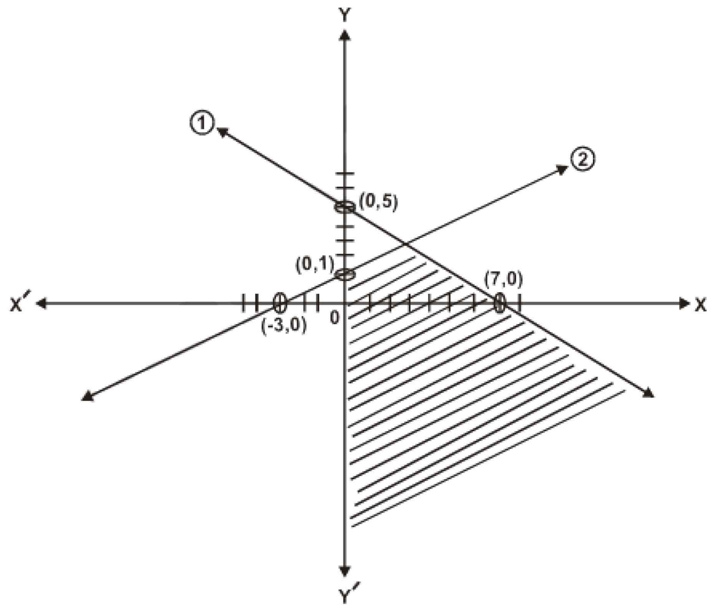
$-x + 3y < 3$

$-0 + 3(0) < 3$

$0 < 3$

Which is true.

$\therefore$ Graph of an inequality $-x + 3y \leq 6$ will be towards the origin side.



To find the intersection of both the equations solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 5, we get

$$\begin{array}{rcl} 5x + 7y &=& 35 \\ -5x + 15y &=& 15 \\ \hline 22y &=& 50 \end{array}$$

$$y = \frac{50}{22} = \frac{25}{11}$$

Put $y = \frac{25}{11}$ in eq. (2)

$$-x + 3\left(\frac{25}{11}\right) = 3$$

$$\frac{75}{11} - 3 = x$$

$$x = \frac{42}{11}$$

$\therefore$ Point $\left(\frac{42}{11}, \frac{25}{11}\right)$

So the corner points are (0,1), $\left(\frac{42}{11}, \frac{25}{11}\right)$

(vi) $\mathbf{5x + 7y \leq 35}$

$\mathbf{x - 2y \leq 2}$

$\mathbf{x \geq 0}$

The associated equations are

$5x + 7y = 35$ ...... (1)

$x - 2y = 2$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$5x + 7(0) = 35$$

$$x = \frac{35}{5} = 7$$

$\therefore$ Point is (7, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$5(0) + 7y = 35$$

$$x = \frac{35}{7} = 5$$

$\therefore$ Point is (0, 5)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$x - 2(0) = 2$$

$$x = 2$$

$\therefore$ Point is (2, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 - 2y = 2$

$$y = \frac{2}{-2} = -1$$

$\therefore$ Point is $(0, -1)$

<u>Test Point</u>

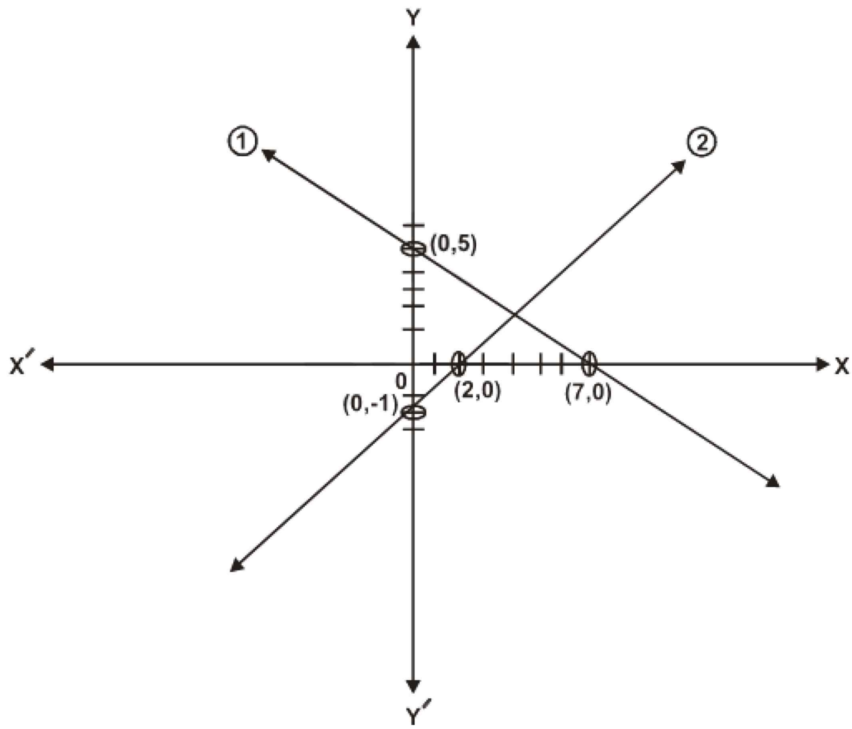
Put (0, 0) in

$5x + 7y < 35$

$5(0) + 7(0) < 35$

$0 < 35$

Which is true.



$\therefore$ Graph of an inequality $5x + 7y \leq 35$ will be towards the origin side.

Put (0, 0) in

$x - 2y < 2$

$0 - 2(0) < 2$

$0 < 2$

Which is true.

$\therefore$ Graph of an inequality $x - 2y \leq 4$ will be towards the origin side.

To find the intersection of both is the equations solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 5, we get

$$\begin{aligned} 5x + 7y &= 35 \\ \underline{-5x \mp 10y} &\underline{= -10} \\ 17y &= 25 \\ y &= \frac{25}{17} \end{aligned}$$

Put $y = \frac{25}{17}$ in eq. (2), we get

$$x - 2\left(\frac{25}{17}\right) = 2$$

$$x - \frac{50}{17} = 2$$

$$x = 2 + \frac{50}{17}$$

$$x = \frac{34 + 50}{17}$$

$$x = \frac{84}{17}$$

$\therefore$ Point $\left(\frac{84}{17}, \frac{25}{17}\right)$

So the corner points are $\left(\frac{84}{17}, \frac{25}{17}\right)$, (0, 5), (0, – 2)

Q.5: Graph the solution region of the following system of linear inequalities by shading.

(i) $3x - 4y \le 12$, $3x + 2y \ge 3$, $x + 2y \le 9$

(ii) $3x - 4y \le 12$, $x + 2y \le 6$, $x + y \ge 1$

(iii) $2x + y \le 4$, $2x - 3y \ge 12$, $x + 2y \le 6$

(iv) $2x + y \le 10$, $x + y \le 7$, $-2x + y \le 4$

(v) $2x + 3y \le 18$, $2x + y \le 10$, $-2x + y \le 2$

(vi) $3x - 2y \ge 3$, $x + 4y \le 12$, $3x + y \le 12$

Solution:

(i) $\mathbf{3x - 4y \leq 12}$

$\mathbf{3x + 2y \geq 3}$

$\mathbf{x + 2y \leq 9}$

The associated equations are

$3x - 4y = 12$ .... (1)

$3x + 2y = 3$ .... (2)

$x + 2y = 9$ .... (3)

<u>x-intercept</u>

Put $y = 0$ in eqs. (1), (2) and (3)

$3x - 4(0) = 12$

$3x = 12$

$x = \frac{12}{3} = 4$

$\therefore$ Point is $(4, 0)$

$3x + 2(0) = 3$

$3x = 3$

$x = \frac{3}{3} = 1$

$\therefore$ Point is $(1, 0)$

$x + 2(0) = 9$

$x = 9$

$\therefore$ Point is $(9, 0)$

<u>y-intercept</u>

Put $x = 0$ in eqs. (1), (2) and (3)

$3(0) - 4y = 12$

$y = \frac{12}{-4} = -3$

$\therefore$ Point is $(0, -3)$

$3(0) + 2y = 3$

$y = \frac{3}{2}$

$\therefore$ Point is $\left(0, \frac{3}{2}\right)$

$0 + 2y = 9$

$y = \frac{9}{2}$

$\therefore$ Point is $\left(0, \frac{9}{2}\right)$

<u>**Test Point**</u>

Put (0, 0) in

$3x - 4y < 12$

$3(0) - 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $3x - 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

$3x + 2y > 3$

$3(0) + 2(0) > 3$

$0 > 3$

which is false.

$\therefore$ Graph of an inequality $3x + 2y \geq 3$ will not be towards the origin side.

Put (0, 0) in

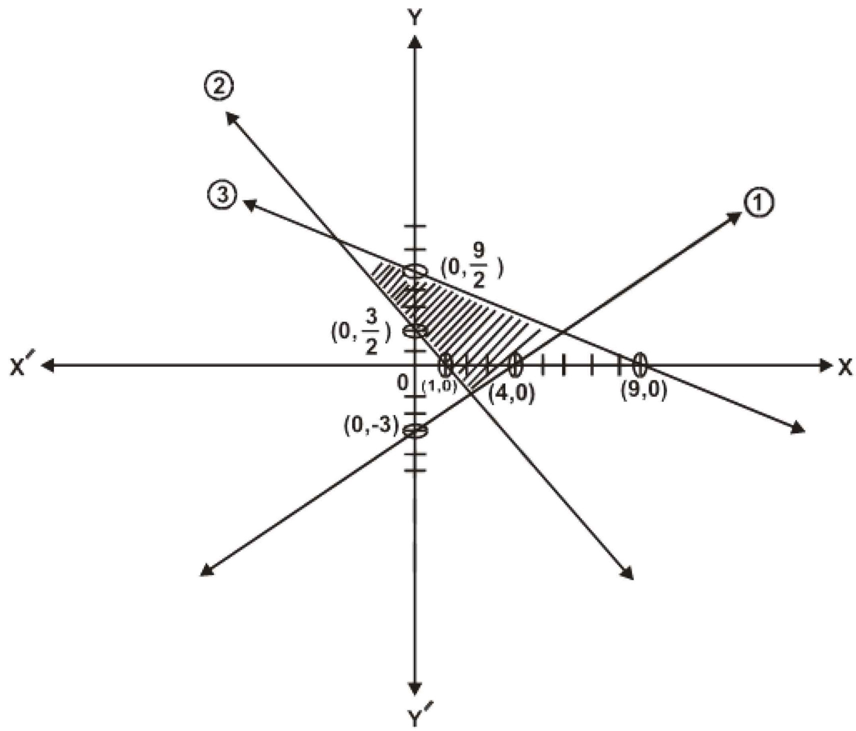
$x + 2y < 9$

$0 + 2(0) < 9$

$0 < 9$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \leq 9$ will be towards the origin side.



(ii) $\mathbf{3x - 4y \leq 12}$

$\mathbf{x + 2y \leq 6}$

$\mathbf{x + y \geq 1}$

The associated equations are

$3x - 4y = 12$ .... (1)

$x + 2y = 6$ .... (2)

$x + y = 1$ .... (3)

<u>x-intercept</u>

Put $y = 0$ in equations (1), (2) and (3)

$3x - 4(0) = 12$	$x + 2(0) = 6$	$x + 0 = 1$
$3x = 12$	$x = 6$	$x = 1$

$x = \frac{12}{3} = 4$

$\therefore$ Point is (4, 0)

$\therefore$ Point is (6, 0)

$\therefore$ Point is (1, 0)

<u>y-intercept</u>

Put x = 0 in equations (1), (2) and (3)

$3(0) - 4y = 12$

$y = \frac{12}{-4} = -3$

$\therefore$ Point is (0, – 3)

$0 + 2y = 6$

$y = \frac{6}{2} = 3$

$\therefore$ Point is (0, 3)

$0 + y = 1$

$y = 1$

$\therefore$ Point is (0, 1)

<u>Test Point</u>

Put (0, 0) in

$3x - 4y < 12$

$3(0) - 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $3x - 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

$x + 2y < 6$

$0 + 2(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \leq 6$ will not be towards the origin side.

Put (0, 0) in

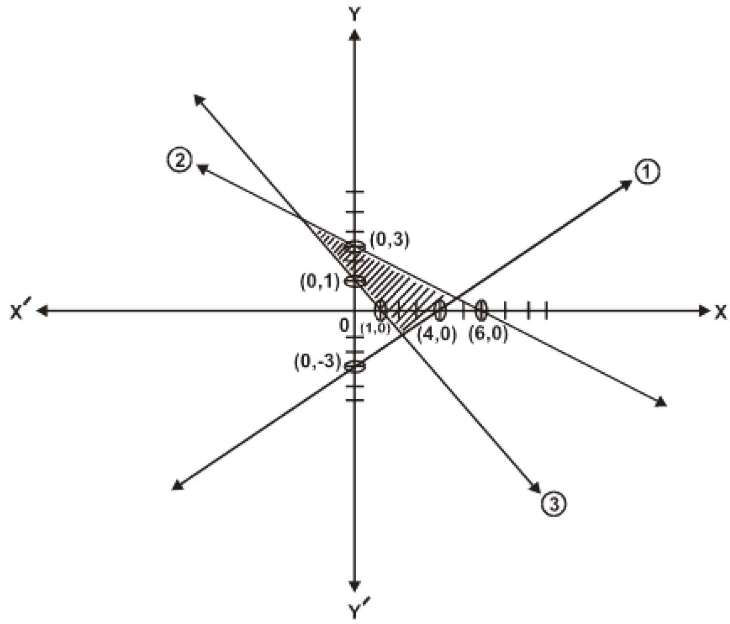
$x + y > 1$

$0 + 0 > 1$

$0 > 1$

Which is false.

$\therefore$ Graph of an inequality $x + y \geq 1$ will not be towards the origin side.



(iii) $\mathbf{2x + y \leq 4}$

$\mathbf{2x - 3y \geq 12}$

$\mathbf{x + 2y \leq 6}$

The associated equations are

$2x + y = 4$ .... (1)

$2x - 3y = 12$ .... (2)

$x + 2y = 6$ .... (3)

<u>x-intercept</u>

Put y = 0 in equations (1), (2) and (3)

$2x + 0 = 4$

$x = \frac{4}{2} = 2$

$\therefore$ Point is (2, 0)

$2x - 3(0) = 12$

$2x = 12$

$x = \frac{12}{2} = 6$

$\therefore$ Point is (6, 0)

$x + 2(0) = 6$

$x = 6$

$\therefore$ Point is (6, 0)

<u>y-intercept</u>

Put x = 0 in equations (1), (2) and (3)

$2(0) + y = 4$

$y = 4$

$\therefore$ Point is (0, 4)

$2(0) - 3y = 12$

$y = \frac{12}{-3} = -4$

$\therefore$ Point is (0, – 4)

$0 + 2y = 6$

$y = \frac{6}{2} = 3$

$\therefore$ Point is (0, 3)

Test Point

Put (0, 0) in

$2x + y < 4$

$2(0) + 0 < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 4$ will be towards the origin side.

Put (0, 0) in

$2x - 3y > 12$

$2(0) - 3(0) > 12$

$0 > 12$

Which is false.

$\therefore$ Graph of an inequality $2x - 3y \geq 12$ will not be towards the origin side.

Put (0, 0) in

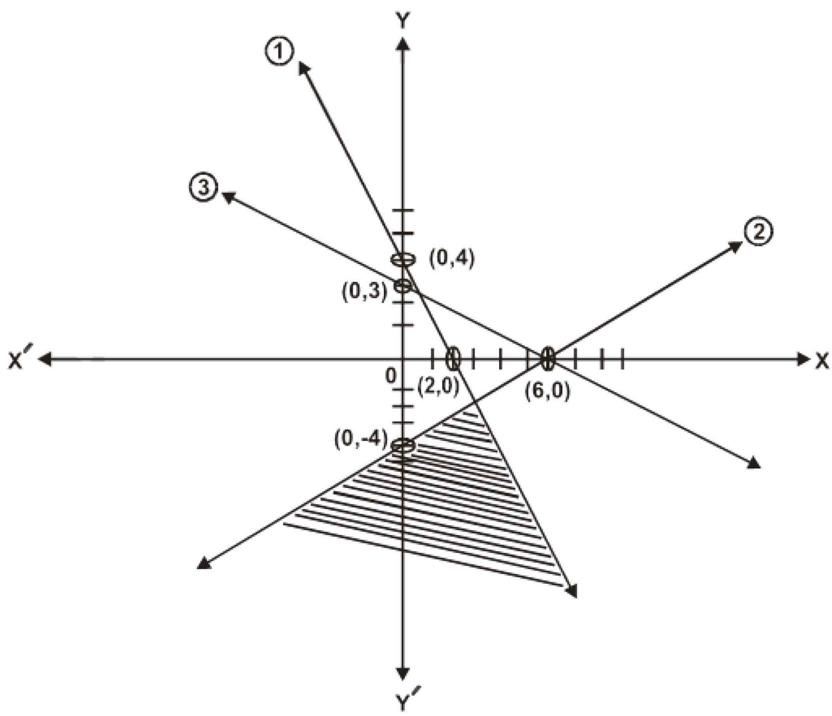
$x + 2y < 6$

$0 + 2(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \leq 6$ will be towards the origin side.



(iv) $2x + y \leq 10$

$x + y \leq 7$

$-2x + y \leq 4$

The associated equations are

$2x + y = 10$ .... (1)

$x + y = 7$ .... (2)

$-2x + y = 4$ .... (3)

<u>x-intercept</u>

Put y = 0 in equations (1), (2) and (3)

$2x + 0 = 10$	$x + 0 = 7$	$-2x + 0 = 4$
$x = \dfrac{10}{2} = 5$	$x = 7$	$x = \dfrac{4}{-2} = -2$
$\therefore$ Point is (5, 0)	$\therefore$ Point is (7, 0)	$\therefore$ Point is (– 2, 0)

<u>y-intercept</u>

Put x = 0 in equations (1), (2) and (3)

$2(0) + y = 10$	$0 + y = 7$	$-2(0) + y = 4$
$y = 10$	$y = 7$	$y = 4$
$\therefore$ Point is (0, 10)	$\therefore$ Point is (0, 7)	$\therefore$ Point is (0, 4)

<u>Test Point</u>

Put (0, 0) in

$2x + y < 10$

$2(0) + 0 < 10$

$0 < 10$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 10$ will be towards the origin side.

Put (0, 0) in

$x + y < 7$

$0 + 0 < 7$

$0 < 7$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 7$ will be towards the origin side.

Put (0, 0) in

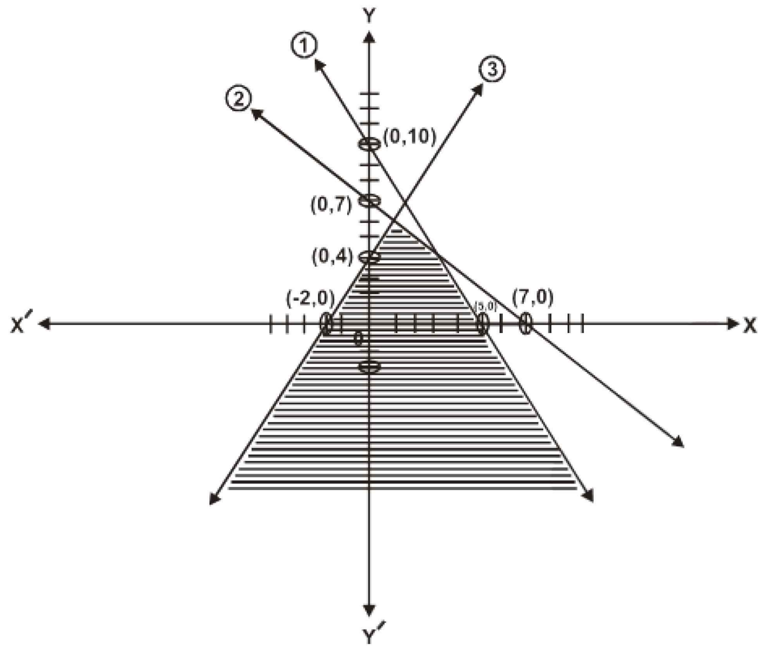
$-2x + y < 4$

$-2(0) + 0 < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $-2x + y \leq 4$ will be towards the origin side.



(v) $\mathbf{2x + 3y \leq 18}$

$\mathbf{2x + y \leq 10}$

$\mathbf{-2x + y \leq 2}$

The associated equations are

$2x + 3y = 18$ .... (1)

$2x + y = 10$ .... (2)

$-2x + y = 2$ .... (3)

x-intercept

Put $y = 0$ in equations (1), (2) and (3)

$2x + 3(0) = 18$

$2x = 18$

$x = \frac{18}{2} = 9$

$\therefore$ Point is (9, 0)

$2x + 0 = 10$

$2x = 10$

$x = \frac{10}{2} = 5$

$\therefore$ Point is (5, 0)

$-2x + 0 = 2$

$x = \frac{2}{-2} = -1$

$\therefore$ Point is (– 1, 0)

y-intercept

Put x = 0 in equations (1), (2) and (3)

$2(0) + 3y = 18$
$3y = 18$
$y = \frac{18}{3} = 6$
$\therefore$ Point is (0, 6)

$2(0) + y = 10$
$y = 10$
$\therefore$ Point is (0, 10)

$-2(0) + y = 2$
$y = 2$
$\therefore$ Point is (0, 2)

Test Point

Put (0, 0) in
$2x + 3y < 18$
$2(0) + 3(0) < 18$
$0 < 18$
Which is true.

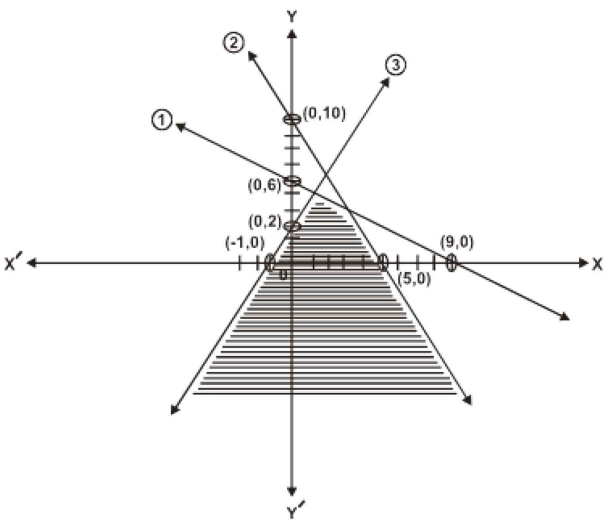
$\therefore$ Graph of an inequality $2x + 3y \leq 18$ will be towards the origin side.

Put (0, 0) in
$2x + y < 10$
$2(0) + 0 < 10$
$0 < 10$
Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 10$ will be towards the origin side.

Put (0, 0) in
$-2x + y < 2$
$-2(0) + 0 < 2$
$0 < 2$
Which is true.

$\therefore$ Graph of an inequality $-2x + y \leq 2$ will be towards the origin side.



(vi) $\mathbf{3x - 2y \geq 3}$

$\mathbf{x + 4y \leq 12}$

$\mathbf{3x + y \leq 12}$

The associated equations are

$3x - 2y = 3$ .... (1)

$x + 4y = 12$ .... (2)

$3x + y = 12$ .... (3)

<u>x-intercept</u>

Put y = 0 in equations (1), (2) and (3)

$3x - 2(0) = 3$

$3x = 3$

$x = \frac{3}{3} = 1$

$\therefore$ Point is (1, 0)

$x + 4(0) = 12$

$x = 12$

$\therefore$ Point is (12, 0)

$3x + 0 = 12$

$x = \frac{12}{3} = 4$

$\therefore$ Point is (4, 0)

<u>y-intercept</u>

Put x = 0 in equations (1), (2) and (3)

$3(0) - 2y = 3$

$y = \frac{3}{-2}$

$\therefore$ Point is $\left(0, \frac{-3}{2}\right)$

$0 + 4y = 12$

$y = \frac{12}{4} = 3$

$\therefore$ Point is (0, 3)

$3(0) + y = 12$

$y = 12$

$\therefore$ Point is (0, 12)

<u>Test Point</u>

Put (0, 0) in

$3x - 2y > 3$

$3(0) - 2(0) > 3$

$0 > 3$

Which is false.

$\therefore$ Graph of an inequality $3x - 2y \geq 3$ will not be towards the origin side.

Put (0, 0) in

$x + 4y < 12$

$0 + 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $x + 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

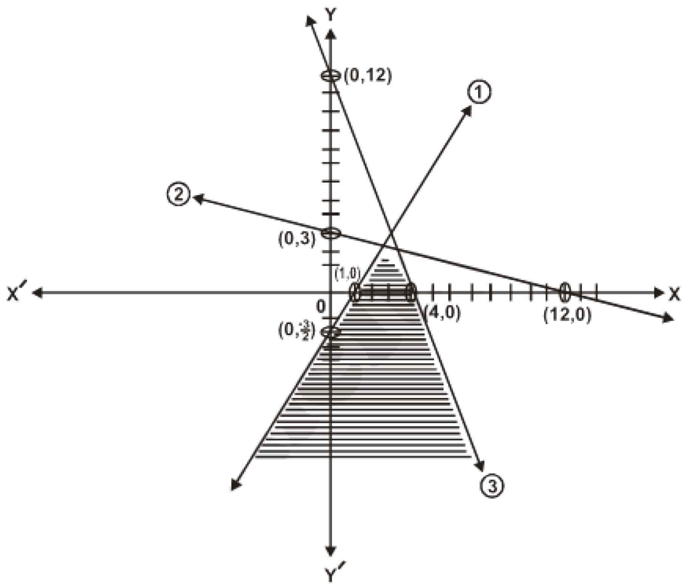
$3x + y < 12$

$3(0) + 0 < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $3x + y \leq 12$ will be towards the origin side.



EXERCISE 5.2

Q.4: Graph the feasible region of the following system of linear inequalities and find the corner points in each case.

(i)	$2x - 3y \leq 6$	**(ii)**	$x + y \leq 5$	**(iii)**	$x + y \leq 5$
	$2x + 3y \leq 12$		$-2x + y \leq 2$		$-2x + y \geq 2$
	$x \geq 0$, $y \geq 0$		$x \geq 0$, $y \geq o$		$x \geq 0$, $y \geq 0$
(iv)	$3x + 7y \leq 21$	**(v)**	$3x + 2y \geq 6$	**(vi)**	$5x + 7y \leq 35$
	$x - y \leq 3$		$x + y \leq 4$		$x - 2y \leq 4$
	$x \geq 0$, $y \geq 0$		$x \geq 0$, $y \geq 0$		$x \geq 0$, $y \geq 0$

Solution:

(i) $\mathbf{2x - 3y \leq 6}$ **(Lhr. Board 2005)**

$\mathbf{2x + 3y \leq 12}$

$\mathbf{x \geq 0 \quad , \quad y \geq 0}$

The associated equations are

$2x - 3y = 6$ ...... (1)

$2x + 3y = 12$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$2x - 3(0) = 6$

$2x = 6$

$x = \frac{6}{2} = 3$

$\therefore$ Point is (3, 0)

y-intercept

Put $x = 0$ in eq. (1)

$2(0) - 3y = 6$

$-3y = 6$

$y = \frac{6}{-3} = -2$

$\therefore$ Point is (0, – 2)

x-intercept

Put $y = 0$ in eq. (2)

$2x + 3(0) = 12$

$x = 12$

$x = \frac{12}{2} = 6$

$\therefore$ Point is (6, 0)

y-intercept

Put $x = 0$ in eq. (2)

$2(0) + 3y = 12$

$3y = 12$

$$y = \frac{12}{3} = 4$$

$\therefore$ Point is (0, 4)

<u>Test Point</u>

Put (0, 0) in

$2x - 3y < 6$

$2(0) - 3(0) < 6$

$0 < 6$

Which is true.

$\therefore$ Graph of an inequality $2x - 3y \leq 6$ will be towards the origin side.

Put (0, 0) in

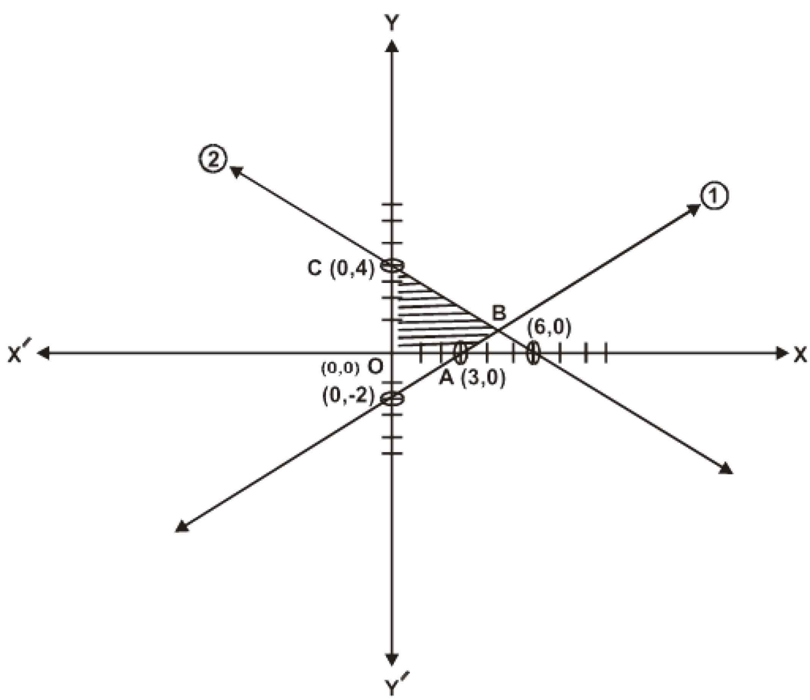
$2x + 3y < 12$

$2(0) + 3(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 12$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so corner points are

O(0, 0), A(3, 0), C(0, 4)

To find B solving eq. (1) & eq. (2)

Adding eq. (1) & eq. (2)

$$\begin{aligned} 2x - 3y &= 6 \\ \underline{2x + 3y} &\underline{= 12} \\ 4x &= 18 \\ x &= \frac{18}{4} = \frac{9}{2} \end{aligned}$$

Put

$x = \frac{9}{2}$ in eq. (1)

$$2\left(\frac{9}{2}\right) - 3y = 6$$

$$9 - 6 = 3y$$

$$y = \frac{3}{3} = 1$$

$\therefore$ $B\left(\frac{9}{2}, 1\right)$

(ii) $\mathbf{x + y \leq 5}$

$\mathbf{-2x + y \leq 2}$

$\mathbf{x \geq 0}$, $\mathbf{y \geq 0}$

The associated equations are

$x + y = 5$ ...... (1)

$y - 2x = 2$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$$x + 0 = 5$$

$$x = 5$$

$\therefore$ Point is (5, 0)

y-intercept

Put $x = 0$ in eq. (1)

$$0 + y = 5$$

$$y = 5$$

$\therefore$ Point is (0, 5)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$0 - 2x = 2$$

$$x = \frac{2}{-2} = -1$$

$\therefore$ Point is (– 1, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$y - 2(0) = 2$$

$$y = 2$$

$\therefore$ Point is (0, 2)

<u>Test Point</u>

Put (0, 0) in

$$x + y < 5$$

$$0 + 0 < 5$$

$$0 < 5$$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 5$ will towards the origin side.

Put (0, 0) in

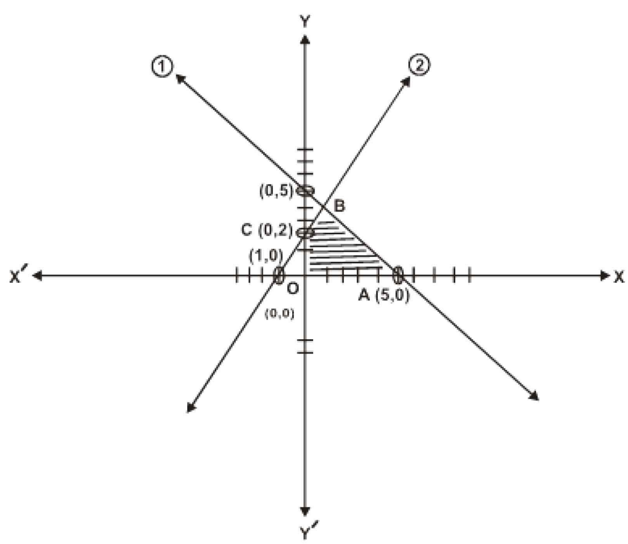
$$y - 2x < 2$$

$$0 - 2(0) < 2$$

$$0 < 2$$

Which is true.

$\therefore$ Graph of an inequality $y - 2x \leq 2$ will towards the origin side.



$\therefore$ OABC is the feasible solution region so corner points are

O(0, 0), A(5, 0), C(0, 2)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) we get

$$\begin{array}{rcrcr} x & + & y & = & 5 \\ \mp 2x & \pm & y & = & _{-}2 \\ \hline & & 3x & = & 3 \end{array}$$

$$x = \frac{3}{3} = 1$$

Put

$x = 1$ in eq. (1)

$1 + y = 5$

$y = 5 - 1 = 4$

$\therefore$ B (1, 4)

(iii) $\mathbf{x + y \leq 5}$

$\mathbf{-2x + y \geq 2}$

$\mathbf{x \geq 0 \quad , \quad y \geq 0}$

The associated equations are

$x + y = 5$ ..... (1)

$-2x + y = 2$ ..... (2)

x-intercept

Put $y = 0$ in eq. (1)

$x + 0 = 5$

$x = 5$

$\therefore$ Point is (5, 0)

y-intercept

Put $x = 0$ in eq. (1)

$0 + y = 5$

$y = 5$

$\therefore$ Point is (0, 5)

x-intercept

Put $y = 0$ in eq. (2)

$-2x + 0 = 2$

$$x = \frac{2}{-2} = -1$$

$\therefore$ Point is $(-1, 0)$

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$-2(0) + y = 2$

$y = 2$

$\therefore$ Point is $(0, 2)$

<u>Test Point</u>

Put (0, 0) in

$x + y < 5$

$0 + 0 < 5$

$0 < 5$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 5$ will be towards the origin side.

Put (0, 0) in

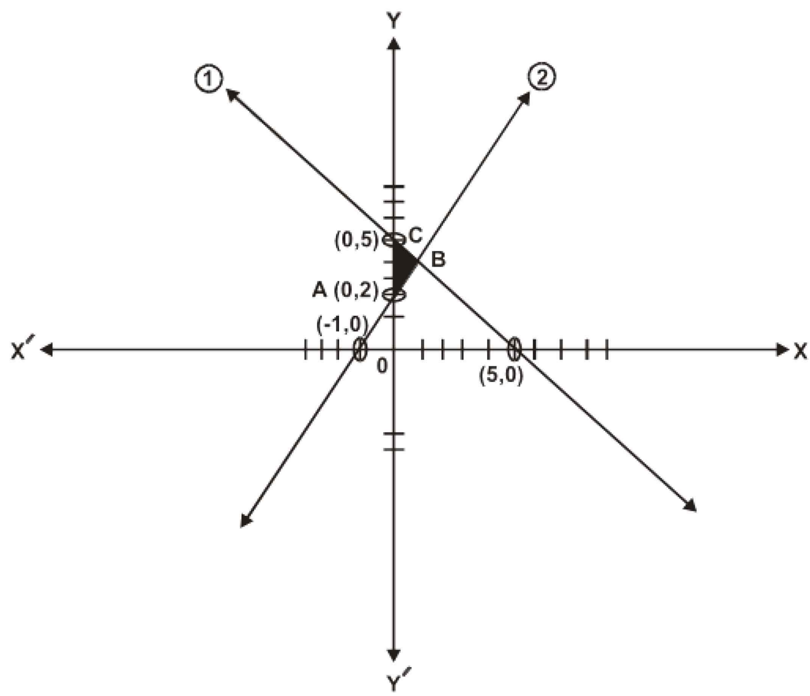
$-2x + y > 2$

$-2(0) + 0 > 2$

$0 > 2$

Which is false.

$\therefore$ Graph of an inequality $-2x + y \geq 2$ will not be towards the origin side.



$\therefore$ ABC is the feasible solution region. So corner points are A (0, 2), C (0, 5). To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2), we get

$$\begin{aligned} x + y &= 5 \\ \underline{\mp 2x \pm y} &\underline{= 2} \\ 3x &= 3 \\ x = \frac{3}{3} &= 1 \end{aligned}$$

Put $x = 1$ in eq. (1)

$$1 + y = 5$$

$$y = 5 - 1 = 4$$

$\therefore$ B (1, 4)

(iv) $\mathbf{3x + 7y \leq 21}$

$\mathbf{x - y \leq 3}$

$\mathbf{x \geq 0 \;,\; y \geq 0}$

The associated equations are

$$3x + 7y = 21 \quad \text{...... (1)}$$

$$x - y = 3 \quad \text{...... (2)}$$

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$3x + 7(0) = 21$$

$$3x = 21$$

$$x = \frac{21}{3} = 7$$

$\therefore$ Point is (7, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$3(0) + 7y = 21$$

$$y = \frac{21}{7} = 3$$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x - 0 = 3$

$x = 3$

$\therefore$ Point is (3, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 - y = 3$

$y = -3$

$\therefore$ Point is (0, – 3)

<u>Test Point</u>

Put (0, 0) in

$3x + 7y < 21$

$3(0) + 7(0) < 21$

$0 < 21$

Which is true.

$\therefore$ Graph of an inequality $3x + 7y \leq 21$ will be towards the origin side.

Put (0, 0) in

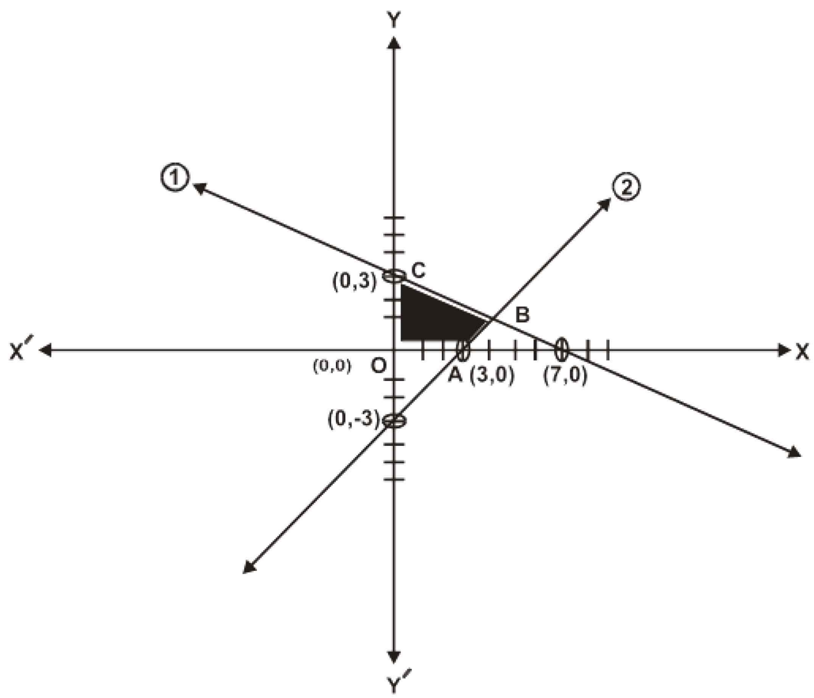
$x - y < 3$

$0 - 0 < 3$

$0 < 3$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 3$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so corner points are

O(0, 0), A(3, 0), C(0, 3)

To find B solving eq. (1) & eq. (2)

Eq. (1) + Eq. (2) × 7, we get

$$\begin{aligned} 3x + 7y &= 21 \\ \underline{7x - 7y} &\underline{= 21} \\ 10x &= 42 \\ x &= \frac{42}{10} = \frac{21}{5} \end{aligned}$$

Put $x = \frac{21}{5}$ in eq. (2)

$$\frac{21}{5} - y = 3$$

$$\frac{21}{5} - 3 = y$$

$$y = \frac{21 - 15}{5}$$

$$y = \frac{6}{5}$$

$\therefore$ $B\left(\frac{21}{5}, \frac{6}{5}\right)$

(v) $\mathbf{3x + 2y \geq 6}$

$\mathbf{x + y \leq 4}$

$\mathbf{x \geq 0 \;,\; y \geq 0}$

The associated equations are

$3x + 2y = 6$ .... (1)

$x + y = 4$ .... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$3x + 2(0) = 6$$

$$x = \frac{6}{3} = 2$$

$\therefore$ Point is (2, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$3(0) + 2y = 6$$

$$y = \frac{6}{2} = 3$$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$x + 0 = 4$$

$$x = 4$$

$\therefore$ Point is (4, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$0 + y = 4$$

$$y = 4$$

$\therefore$ Point is (0, 4)

<u>Test Point</u>

Put (0, 0) in

$$3x + 2y > 6$$

$$3(0) + 2(0) > 6$$

$$0 > 6$$

Which is false.

$\therefore$ Graph of an inequality $3x + 2y \geq 6$ will not be towards the origin side.

Put (0, 0) in

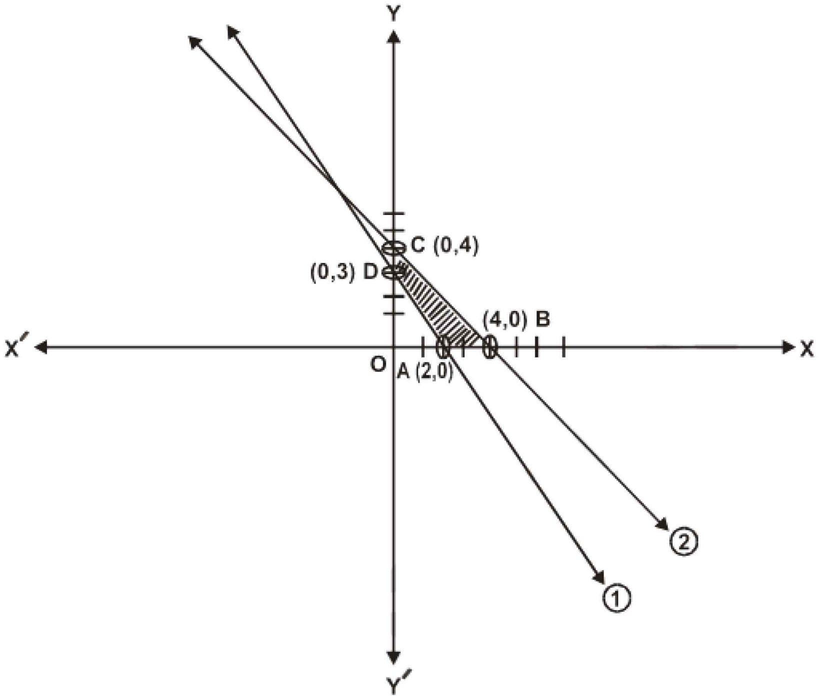
$$x + y < 4$$

$$0 + 0 < 4$$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $x + y \le 4$ will be towards the origin side.



$\therefore$ ABCD is the feasible solution region so corner points are

A (2, 0), B (4, 0), C(0, 4), D (0, 3)

(vi) $\mathbf{5x + 7y \le 35}$

$\mathbf{x - 2y \le 4}$

$\mathbf{x \ge 0 \ , \ y \ge 0}$

The associated equations are

$5x + 7y = 35$ .... (1)

$x - 2y = 4$ .... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$5x + 7(0) = 35$$

$$x = \frac{35}{5} = 7$$

$\therefore$ Point is (7, 0)

y-intercept

Put $x = 0$ in eq. (1)

$5(0) + 7y = 35$

$y = \frac{35}{7} = 5$

$\therefore$ Point is (0, 5)

x-intercept

Put $y = 0$ in eq. (2)

$x - 2(0) = 4$

$x = 4$

$\therefore$ Point is (4, 0)

y-intercept

Put $x = 0$ in eq. (2)

$0 - 2y = 4$

$y = \frac{4}{-2} = -2$

$\therefore$ Point is (0, – 2)

Test Point

Put (0, 0) in

$5x + 7y < 35$

$5(0) + 7(0) < 35$

$0 < 35$

Which is true.

$\therefore$ Graph of an inequality $5x + 7y \leq 35$ will be towards the origin side.

Put (0, 0) in

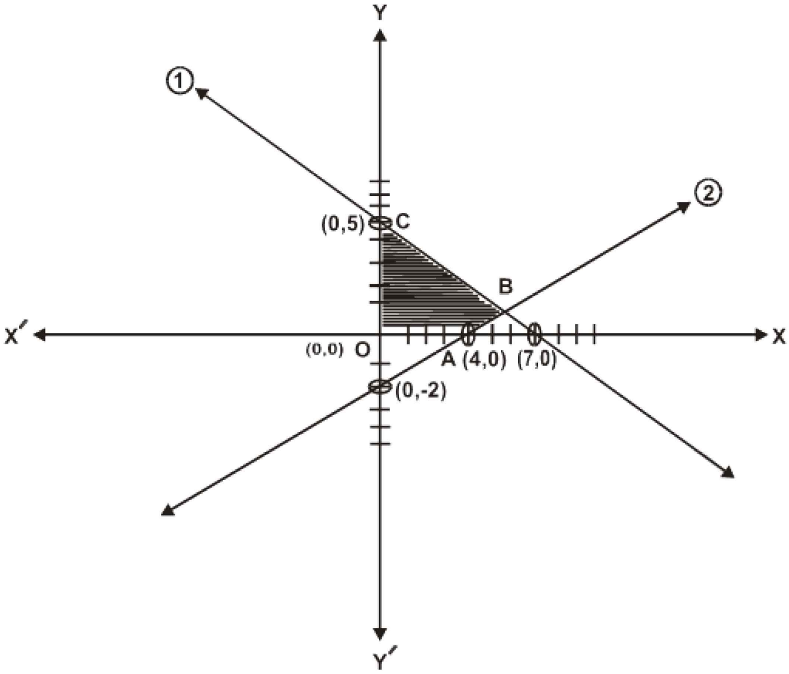
$x - 2y < 4$

$0 - 2(0) < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $x - 2y \leq 4$ will be towards the origin.



$\therefore$ OABC is the feasible solution region so corner points are

O (0, 0), A (4, 0), C (0, 5)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 5, we get

$$\begin{array}{rcl} 5x + 7y & = & 35 \\ \underline{-\ 5x \mp 10y} & = & \underline{-20} \\ 17y & = & 15 \\ y & = & \frac{15}{17} \end{array}$$

Put $y = \frac{15}{17}$ in eq. (2)

$$x - 2\left(\frac{15}{17}\right) = 4$$

$$x - \frac{30}{17} = 4$$

$$x = 4 + \frac{30}{17}$$

$$x = \frac{68 + 30}{17}$$

$$x = \frac{98}{17}$$

$$\therefore \quad B = \left(\frac{98}{17}, \frac{15}{17}\right)$$

Q.2: Graph the feasible region of the following system of linear inequalities and find the corner points in each case.

(i) $2x + y \leq 10$
$x + 4y \leq 12$
$x + 2y \leq 10$
$x \geq 0, y \geq 0$

(ii) $2x + 3y \leq 18$
$2x + y \leq 10$
$x + 4y \leq 12$
$x \geq 0, y \geq 0$

(iii) $2x + 3y \leq 18$
$x + 4y \leq 12$
$3x + y \leq 12$
$x \geq 0, y \geq 0$

(iv) $x + 2y \leq 14$
$3x + 4y \leq 36$
$2x + y \leq 10$
$x \geq 0, y \geq 0$

(v) $x + 3y \leq 15$
$2x + y \leq 12$
$4x + 3y \leq 24$
$x \geq 0, y \geq 0$

(vi) $2x + y \leq 20$
$8x + 15y \leq 120$
$x + y \leq 11$
$x \geq 0, y \geq 0$

Solution:

(i) $2x + y \leq 10$
$x + 4y \leq 12$
$x + 2y \leq 10$
$x \geq 0, y \geq 0$

The associated eqs. are

$2x + y = 10$ ...... (1)
$x + 4y = 12$ ...... (2)
$x + 2y = 10$ ...... (3)

<u>x-intercept</u>

Put $y = 0$ in eqs. (1), (2) and (3)

$2x + 0 = 10$	$x + 4(0) = 12$	$x + 2(0) = 10$
$2x = 10$	$x = 12$	$x = 10$
$x = \frac{10}{2} = 5$	$\therefore$ Point is (12, 0)	$\therefore$ Point is (10, 0)
$\therefore$ Point is (5, 0)		

<u>y-intercept</u>

Put $x = 0$ in eqs. (1), (2) and (3)

$2(0) + y = 10$	$0 + 4y = 12$	$0 + 2y = 10$
$y = 10$	$4y = 12$	$2y = 10$

∴ Point is (0, 10)

$y = \frac{12}{4} = 3$

∴ Point is (0, 3)

$y = \frac{10}{2} = 5$

∴ Point is (0, 5)

Test Point

Put (0, 0)

$2x + y < 10$

$2(0) + 0 < 10$

$0 < 10$

Which is true.

∴ Graph of an inequality $2x + y \leq 10$ will be towards the origin side.

Put (0, 0) in

$x + 4y < 12$

$0 + 4(0) < 12$

$0 < 12$

Which is true.

∴ Graph of an inequality $x + 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

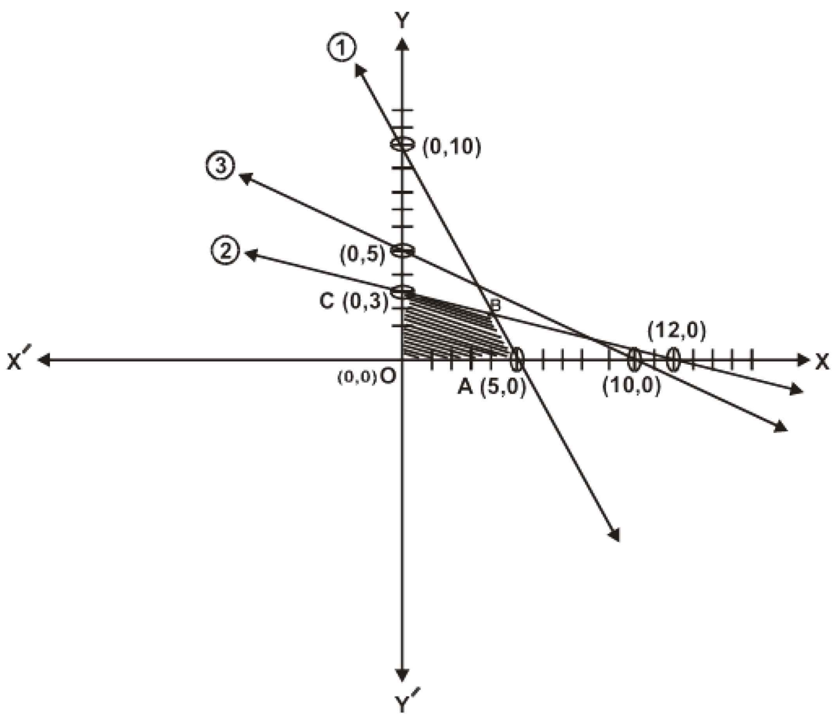
$x + 2y < 10$

$0 + 2(0) < 10$

$0 < 10$

Which is true.

∴ Graph of an inequality $x + 2y \leq 10$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so the corner points are

O (0, 0), A (5, 0), C (0, 3)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 2, we get

$$\begin{array}{rcl} 2x + y & = & 10 \\ \underline{-\ 2x \pm 8y} & = & \underline{-24} \\ -7y & = & -14 \\ y & = & \frac{14}{7} = 2 \end{array}$$

Put $y = 2$ in eq. (2)

$$x + 4(2) = 12$$

$$x + 8 = 12$$

$$x = 12 - 8 = 4$$

$\therefore$ B = (4, 2)

(ii) $\mathbf{2x + 3y \le 18}$ **(Guj. Board 2005) (Lhr. Board 2008)**

$\mathbf{2x + y \le 10}$

$\mathbf{x + 4y \le 12}$

$\mathbf{x \ge 0\ ,\ y \ge 0}$

The associated equations are

$2x + 3y = 18$ .... (1)

$2x + y = 10$ .... (2)

$x + 4y = 12$ .... (3)

<u>x-intercept</u>

Put $y = 0$ in eqs. (1), (2) and (3)

$2x + 3(0) = 18$	$2x + 0 = 10$	$x + 4(0) = 12$
$2x = 18$	$2x = 10$	$x = 12$
$x = \frac{18}{2} = 9$	$x = \frac{10}{2} = 5$	$\therefore$ Point is (12, 0)
$\therefore$ Point is (9, 0)	$\therefore$ Point is (5, 0)	

<u>y-intercept</u>

Put $x = 0$ in eqs. (1), (2) and (3)

$2(0) + 3y = 18$

$3y = 18$

$y = \frac{18}{3} = 6$

$\therefore$ Point is (0, 6)

$2(0) + y = 10$

$y = 10$

$\therefore$ Point is (0, 10)

$0 + 4y = 12$

$4y = 12$

$y = \frac{12}{4} = 3$

$\therefore$ Point is (0, 3)

<u>Test Point</u>

Put (0, 0)

$2x + 3y < 18$

$2(0) + 3(0) < 18$

$0 < 18$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 18$ will be towards the origin side.

Put (0, 0) in

$2x + y < 10$

$2(0) + 0 < 10$

$0 < 10$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 10$ will be towards the origin side.

Put (0, 0) in

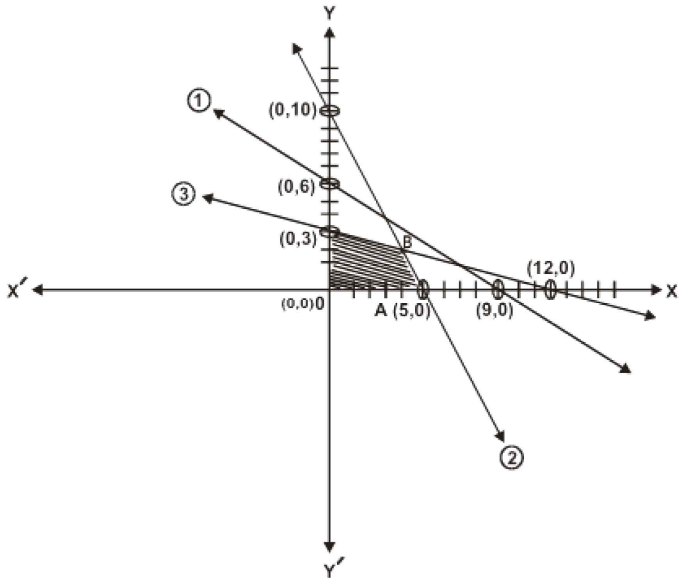
$x + 4y < 12$

$0 + 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $x + 4y \leq 12$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so the corner points are

O (0, 0), A (5, 0), C (0, 3)

To find B solving eq. (2) & eq. (3)

Eq. (2) – Eq. (3) × 2, we get

$$\begin{array}{rcl} 2x + y &=& 10 \\ -\ 2x \pm 8y &=& -24 \\ \hline -7y &=& -14 \\ y &=& \dfrac{-14}{-7} = 2 \end{array}$$

Put $y = 2$ in eq. (3)

$x + 4(2) = 12$

$x + 8 = 12$

$x = 12 - 8 = 4$

$\therefore$ B = (4, 2)

(iii) $\mathbf{2x + 3y \le 18}$

$\mathbf{x + 4y \le 12}$

$\mathbf{3x + y \le 12}$

$\mathbf{x \ge 0 \quad , \quad y \ge 0}$

The associated equations are

$2x + 3y = 18$ .... (1)

$x + 4y = 12$ .... (2)

$3x + y = 12$ .... (3)

x-intercept

Put $y = 0$ in eqs. (1), (2) and (3)

$$2x + 3(0) = 18$$
$$2x = 18$$
$$x = \frac{18}{2} = 9$$

$\therefore$ Point is (9, 0)

$$x + 4(0) = 12$$
$$x = 12$$

$\therefore$ Point is (12, 0)

$$3x + 0 = 12$$
$$3x = 12$$
$$x = \frac{12}{3} = 4$$

$\therefore$ Point is (4, 0)

y-intercept

Put $x = 0$ in eqs. (1), (2) and (3)

$$2(0) + 3y = 18$$
$$3y = 18$$
$$x = \frac{18}{3} = 6$$

$\therefore$ Point is (0, 6)

$$0 + 4y = 12$$
$$y = \frac{12}{4} = 3$$

$\therefore$ Point is (0, 3)

$$3(0) + y = 12$$
$$y = 12$$

$\therefore$ Point is (0, 12)

Test Point

Put (0, 0) in

$2x + 3y < 18$

$2(0) + 3(0) < 18$

$0 < 18$

Which is true.

$\therefore$ Graph of an inequality $2x + 3y \leq 18$ will be towards the origin side.

Put (0, 0) in

$x + 4y < 12$

$0 + 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $x + 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

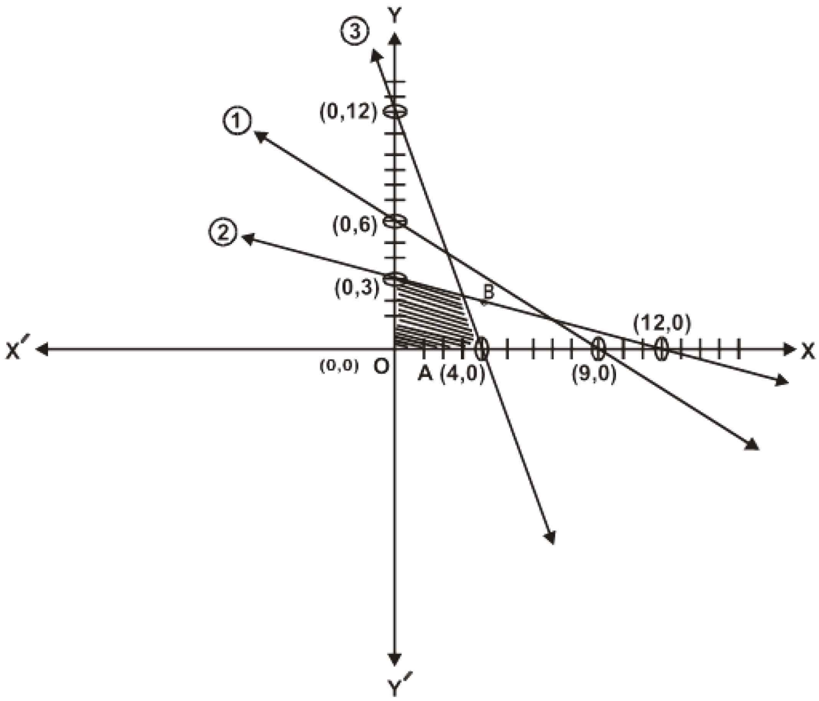
$3x + y < 12$

$3(0) + 0 < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $3x + y \leq 12$ will be towards the origin side.


Y
③
(0,12)
①
(0,6)
②
(0,3)
B
(12,0)
X′
X
(0,0)
O
A (4,0)
(9,0)
Y′

$\therefore$ OABC is the feasible solution region so the corner points are

O (0, 0), A (4, 0), C (0, 3)

To find B solving eq. (2) & eq. (3)

Eq. (2) × 3 – Eq. (3), we get

$$\begin{array}{rcl} 3x + 12y &=& 36 \\ -\ 3x \pm\ \ y &=& -12 \\ \hline 11y &=& 24 \end{array}$$

$$y = \frac{24}{11}$$

Put $y = \dfrac{24}{11}$ in eq. (3)

$$3x + \frac{24}{11} = 12$$

$$3x = 12 - \frac{24}{11}$$

$$3x = \frac{132 - 24}{11}$$

$$x = \frac{108}{33} = \frac{36}{11}$$

$$\therefore \quad B\left(\frac{36}{11}, \frac{24}{11}\right)$$

(iv) $\mathbf{x + 2y \le 14}$

$\mathbf{3x + 4y \le 36}$

$\mathbf{2x + y \le 10}$

$\mathbf{x \ge 0 \quad , \quad y \ge 0}$

The associated equations are

$x + 2y = 14$ .... (1)

$3x + 4y = 36$ .... (2)

$2x + y = 10$ .... (3)

x-intercept

Put y = 0 in eqs. (1), (2) and (3)

$x + 2(0) = 14$

$x = 14$

$\therefore$ Point is (14, 0)

$3x + 4(0) = 36$

$3x = 36$

$x = \dfrac{36}{3} = 12$

$\therefore$ Point is (12, 0)

$2x + 0 = 10$

$2x = 10$

$x = \dfrac{10}{2} = 5$

$\therefore$ Point is (5, 0)

y-intercept

Put $x = 0$ in eqs. (1), (2) and (3)

$$0 + 2y = 14$$
$$y = \frac{14}{2} = 7$$
$\therefore$ Point is (0, 7)

$$3(0) + 4y = 36$$
$$4y = 36$$
$$x = \frac{36}{4} = 9$$
$\therefore$ Point is (0, 9)

$$2(0) + y = 10$$
$$y = 10$$
$\therefore$ Point is (0, 10)

Test Point

Put (0, 0) in

$x + 2y < 14$

$0 + 2(0) < 14$

$0 < 14$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \le 14$ will be towards the origin side.

Put (0, 0) in

$3x + 4y < 36$

$3(0) + 4(0) < 36$

$0 < 36$

Which is true.

$\therefore$ Graph of an inequality $3x + 4y \le 36$ will be towards the origin side.

Put (0, 0) in

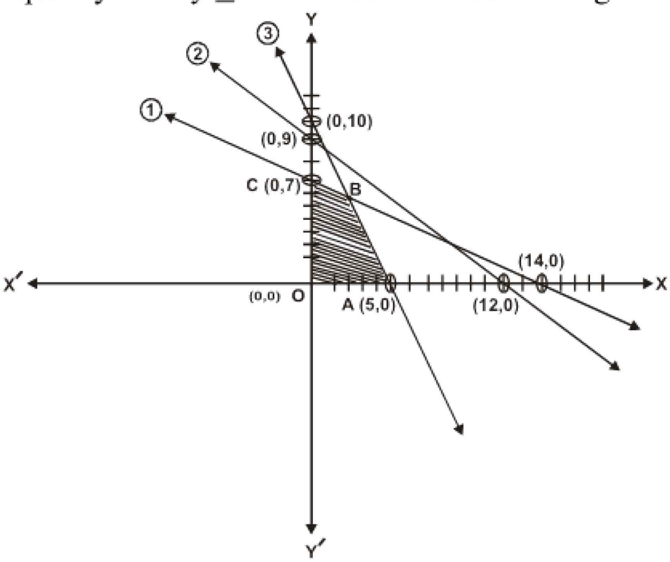
$2x + y < 10$

$2(0) + 0 < 10$

$0 < 10$

Which is true.

$\therefore$ Graph of an inequality $2x + y \le 10$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so the corner points are

O (0, 0), A (5, 0), C (0, 7)

To find B solving eq. (1) & eq. (3)

Eq. (1) × 2 – Eq. (3), we get

$$\begin{array}{rcl} 2x + 4y &=& 28 \\ \underline{-\ 2x \pm\ y} &=& \underline{-10} \\ 3y &=& 18 \\ y &=& \dfrac{18}{3} = 6 \end{array}$$

Put $y = 6$ in eq. (1)

$$\begin{array}{rcl} x + 2(6) &=& 14 \\ x + 12 &=& 14 \\ x &=& 14 - 12 \\ x &=& 2 \end{array}$$

$\therefore$ B (2, 6)

(v) $\mathbf{x + 3y \leq 15}$

$\mathbf{2x + y \leq 12}$

$\mathbf{4x + 3y \leq 24}$

$\mathbf{x \geq 0,\quad y \geq 0}$

The associated equations are

$x + 3y = 15$ .... (1)

$2x + y = 12$ .... (2)

$4x + 3y = 24$ .... (3)

<u>x-intercept</u>

Put $y = 0$ in eqs. (1), (2) and (3)

$x + 3(0) = 15$ $x = 15$ $\therefore$ Point is (15, 0)	$2x + 0 = 12$ $2x = 12$ $x = \dfrac{12}{2} = 6$ $\therefore$ Point is (6, 0)	$4x + 3(0) = 24$ $4x = 24$ $x = \dfrac{24}{4} = 6$ $\therefore$ Point is (6, 0)

<u>y-intercept</u>

Put $x = 0$ in eqs. (1), (2) and (3)

$0 + 3y = 15$ $y = \dfrac{15}{3} = 5$ $\therefore$ Point is (0, 5)	$2(0) + y = 12$ $y = 12$ $\therefore$ Point is (0, 12)	$4(0) + 3y = 24$ $3y = 24$ $y = \dfrac{24}{3} = 8$ $\therefore$ Point is (0, 8)

Test Point

Put (0, 0) in

$x + 3y < 15$

$0 + 3(0) < 15$

$0 < 15$

Which is true.

∴ Graph of an inequality $x + 3y \leq 15$ will be towards the origin side.

Put (0, 0) in

$2x + y < 12$

$2(0) + 0 < 12$

$0 < 12$

Which is true.

∴ Graph of an inequality $2x + y \leq 12$ will be towards the origin side.

Put (0, 0) in

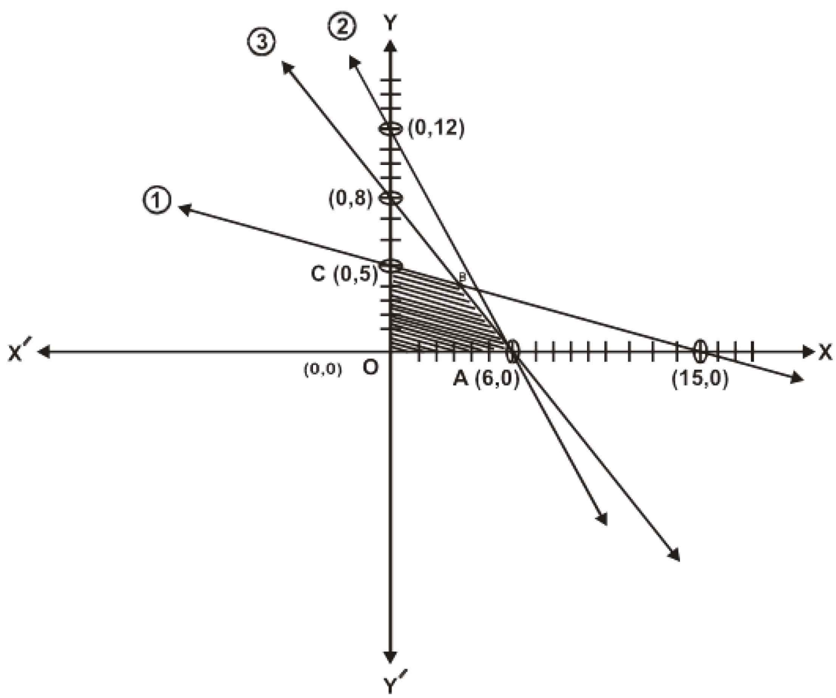
$4x + 3y < 24$

$4(0) + 3(0) < 24$

$0 < 24$

Which is true.

∴ Graph of an inequality $4x + 3y \leq 24$ will be towards the origin side.



∴ OABC is the feasible solution region so the corner points are
O (0, 0), A (6, 0), C (0, 5)
To find B solving eq. (1) & eq. (3)
Eq. (1) – Eq. (3), we get

$$\begin{array}{rcl} x + 3y &=& 15 \\ \underline{-\ 4x \pm 3y} &=& \underline{-24} \\ -3x &=& -9 \\ y &=& \frac{-9}{-3} = 3 \end{array}$$

$$\begin{array}{rcl} \text{Put } x &=& 3 \text{ in eq. (1)} \\ 3 + 3y &=& 15 \\ 3y &=& 15 - 3 \\ 3y &=& 12 \\ y &=& \frac{12}{3} = 4 \end{array}$$

∴ B (3, 4)

(vi) $\mathbf{2x + y \leq 20}$
$\mathbf{8x + 15y \leq 120}$
$\mathbf{x + y \leq 11}$
$\mathbf{x \geq 0 ,\quad y \geq 0}$

The associated equations are

$$2x + y = 20 \quad .... (1)$$
$$8x + 15y = 120 \quad .... (2)$$
$$x + y = 11 \quad .... (3)$$

<u>x-intercept</u>

Put y = 0 in eqs. (1), (2) and (3)

$2x + 0 = 20$	$8x + 15(0) = 120$	$x + 0 = 11$
$2x = 20$	$8x = 120$	$x = 11$
$x = \frac{20}{2} = 10$	$x = \frac{120}{8} = 15$	∴ Point is (11, 0)
∴ Point is (10, 0)	∴ Point is (15, 0)	

<u>y-intercept</u>

Put x = 0 in eqs. (1), (2) and (3)

$2(0) + y = 20$	$8(0) + 15y = 120$	$0 + y = 11$
$y = 20$	$15y = 120$	$y = 11$
∴ Point is (0, 20)		∴ Point is (0, 11)

$$y = \frac{120}{15} = 8$$

$\therefore$ Point is (0, 8)

Test Point

Put (0, 0) in

$2x + y < 20$

$2(0) + 0 < 20$

$0 < 20$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 20$ will be towards the origin side.

Put (0, 0) in

$8x + 15y < 120$

$8(0) + 15(0) < 120$

$0 < 120$

Which is true.

$\therefore$ Graph of an inequality $8x + 15y \leq 120$ will be towards the origin side.

Put (0, 0) in

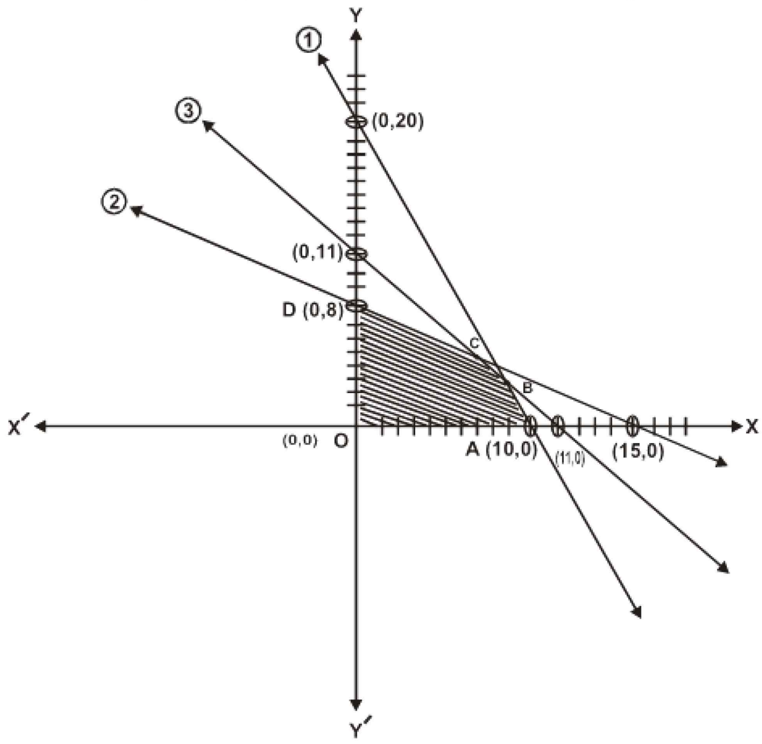
$x + y < 11$

$0 + 0 < 11$

$0 < 11$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 11$ will be towards the origin side.



$\therefore$ OABCD is the feasible solution region so the corner points are

O (0, 0), A (10, 0), D (0, 8)

To find B solving eq. (1) & eq. (3)

Eq. (1) – Eq. (3), we get

$$\begin{array}{rcrcl} & 2x & + & y & = 20 \\ - & x & \pm & y & = -11 \\ \hline & & & x & = 9 \end{array}$$

Put $x = 9$ in eq. (3)

$$9 + y = 11$$
$$y = 11 - 9$$
$$y = 2$$

$\therefore$ B (9, 2)

To find C solving eq. (2) & eq. (3)

Eq. (2) – Eq. (3) × 8, we get

$$\begin{array}{rcrcl} & 8x & + & 15y & = 120 \\ - & 8x & \pm & 8y & = -88 \\ \hline & & & 7y & = 32 \end{array}$$

$$y = \frac{32}{7}$$

Put $y = \frac{32}{7}$ in eq. (3)

$$x + \frac{32}{7} = 11$$
$$x = 11 - \frac{32}{7}$$
$$= \frac{77 - 32}{7}$$
$$= \frac{45}{7}$$

$\therefore$ $C\left(\frac{45}{7}, \frac{32}{7}\right)$

EXERCISE 5.3

Q.1 Maximize $f(x, y) = 2x + 5y$ (Lhr. Board 2007)

Subject to the constraints

$2y - x \leq 8$; $x - y \leq 4$; $x \geq 0$; $y \geq 0$

Solution:

The associated equations are

$2y - x = 8$ ...... (1)

$x - y = 4$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$2(0) - x = 8$

$x = -8$

$\therefore$ Point is $(-8, 0)$

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$2y - 0 = 8$

$2y = 8$

$$y = \frac{8}{2} = 4$$

$\therefore$ Point is $(0, 4)$

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x - 0 = 4$

$x = 4$

$\therefore$ Point is $(4, 0)$

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 - y = 4$

$y = -4$

$\therefore$ Point is $(0, -4)$

<u>Test Point</u>

Put $(0, 0)$ in

$2y - x < 8$

$2(0) - 0 < 8$

$0 < 8$

Which is true.

$\therefore$ Graph of an inequality $2y - x < 8$ will be towards the origin side.

Put (0, 0) in

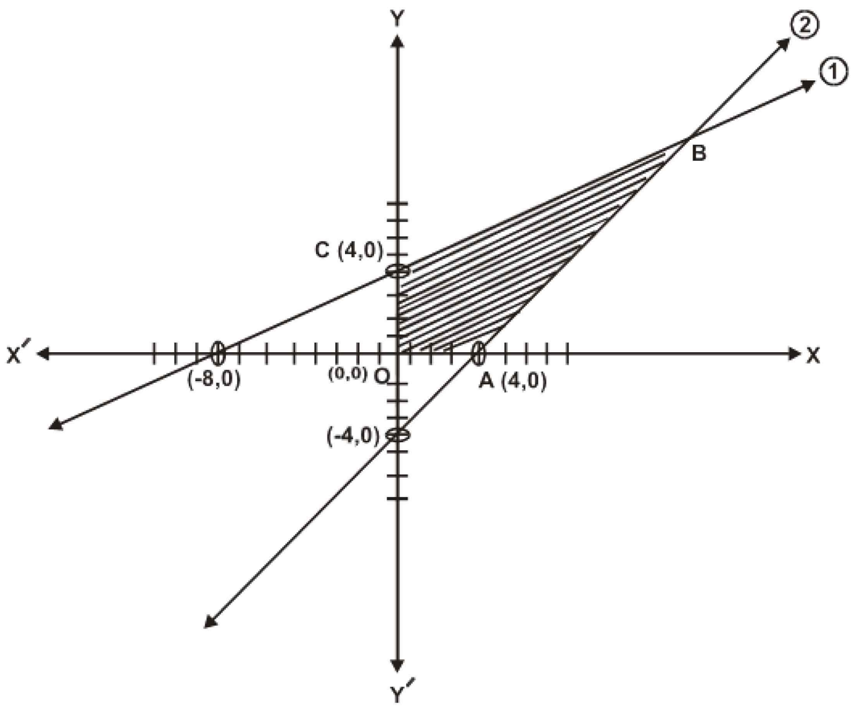
$x - y < 4$

$0 - 0 < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $x - y \leq 4$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region so corner points are

O (0, 0), A (4, 0), C (0, 4)

To find B solving eq. (1) & eq. (2)

Adding eq. (1) & eq. (2)

$$\begin{aligned} 2y - x &= 8 \\ \underline{x - y} &\underline{= 4} \\ y &= 12 \end{aligned}$$

Put $y = 12$ in eq. (2)

$$x - 12 = 4$$
$$x = 4 + 12 = 16$$

$\therefore$ B (16, 12)

Now

$$f(x, y) = 2x + 5y \quad \text{......... (3)}$$

Put O (0, 0) in eq. (3)

$$f(0, 0) = 2(0) + 5(0) = 0$$

Put A (4, 0) in eq. (3)

$$f(4, 0) = 2(4) + 5(0) = 8 + 0 = 8$$

Put B (16, 12) in eq. (3)

$$f(16, 12) = 2(16) + 5(12) = 32 + 60 = 92$$

Put C (0, 4) in eq. (3)

$$f(0, 4) = 2(0) + 5(4) = 20$$

The maximum value of $f(x, y) = 2x + 5y$ is 92 at the corner point B (16, 12).

Q.2 Maximize $f(x, y) = x + 3y$ (Lhr. Board 2006) (Guj. Board 2007, 2008)
Subject to the constraints
$2x + 5y \leq 30$; $5x + 4y \leq 20$; $x \geq 0$; $y \geq 0$

Solution:

The associated equation are

$$2x + 5y = 30 \quad \text{...... (1)}$$
$$5x + 4y = 20 \quad \text{...... (2)}$$

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$$2(x) + 5(0) = 30$$
$$2x = 30$$
$$x = \frac{30}{2} = 15$$

$\therefore$ Point is (15, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$2(0) + 5y = 30$$
$$5y = 30$$
$$y = \frac{30}{5} = 6$$

$\therefore$ Point is (0, 6)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$5x + 4(0) = 20$$
$$5x = 20$$

$$x = \frac{20}{5} = 4$$

$\therefore$ Point is (4, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$5(0) + 4y = 20$$

$$y = \frac{20}{4} = 5$$

$\therefore$ Point is (0, 5)

<u>Test Point</u>

Put (0, 0) in

$2x + 5y < 30$

$2(0) - 5(0) < 30$

$0 < 30$

Which is true.

$\therefore$ Graph of an inequality $2x + 5y \leq 30$ will be towards the origin side.

Put (0, 0) in

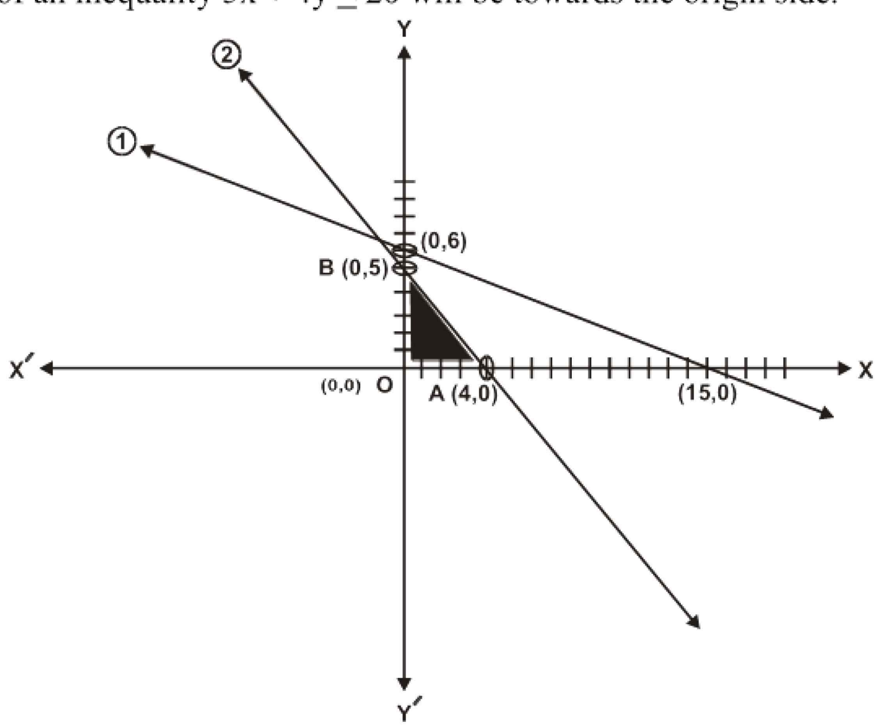
$5x + 4y < 20$

$5(0) + 4(0) < 20$

$0 < 20$

Which is true.

$\therefore$ Graph of an inequality $5x + 4y \leq 20$ will be towards the origin side.



$\therefore$ OAB is the feasible solution region so corner points are

O (0, 0), A (4, 0), B (0, 5)

$$f(x, y) = x + 3y \quad .... (3)$$

Put O (0, 0) in eq. (3)

$$f(0, 0) = 0 + 3(0) = 0$$

Put A (4, 0) in eq. (3)

$$f(4, 0) = 4 + 3(0) = 4$$

Put A (0, 5) in eq. (3)

$$f(0, 5) = 0 + 3(5) = 15$$

The maximum value of f (x, y) = x + 3y is 15 at corner point B (0, 5).

Q.3 Maximize Z = 2x + 3y

Subject to the constraints

$3x + 4y \le 12$; $2x + y \le 4$; $4x - y \le 4$; $x \ge 0$; $y \ge 0$

Solution:

The associated eqs. are

$$3x + 4y = 12 \quad ...... (1)$$

$$2x + y = 4 \quad ...... (2)$$

$$4x - y = 4 \quad .... (3)$$

<u>x-intercept</u>

Put $y = 0$ in eqs. (1), (2) and (3)

$3x + 4(0) = 12$ $3x = 12$ $x = \frac{12}{3} = 4$ $\therefore$ Point is (4, 0)	$2x + 0 = 4$ $x = \frac{4}{2} = 2$ $\therefore$ Point is (2, 0)	$4x - 0 = 4$ $x = \frac{4}{4}$ $x = 1$ $\therefore$ Point is (1, 0)

<u>y-intercept</u>

Put $x = 0$ in eqs. (1), (2) and (3)

$3(0) + 4y = 12$ $y = \frac{12}{4}$ $y = 3$ $\therefore$ Point is (0, 3)	$2(0) = 4$ $y = 4$ $\therefore$ Point is (0, 4)	$4(0) - y = 4$ $y = -4$ $\therefore$ Point is (0, – 4)

<u>Test Point</u>

Put (0, 0) in

$3x + 4y < 12$

$3(0) + 4(0) < 12$

$0 < 12$

Which is true.

$\therefore$ Graph of an inequality $3x + 4y \leq 12$ will be towards the origin side.

Put (0, 0) in

$2x + y < 4$

$2(0) + 0 < 4$

$0 < 4$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 4$ will be towards the origin side.

Put (0, 0) in

$4x - y < 4$

$4(0) - 0 < 4$

$4(0) - 0 < 4$

$0 < 4$

which is true

$\therefore$ Graph of an inequality $4x - y \leq 4$ will be towards the origin side.

$\therefore$ OABCD is the feasible solution region so corner points are

O (0, 0), A (1, 0), D (0, 3)

To find B solving eq. (2) and eq. (3)

Eq. (2) – Eq. (3), we get

$$\begin{aligned} 2x + y &= 4 \\ 4x - y &= 4 \\ \hline 6x &= 8 \\ x = \frac{8}{6} &= \frac{4}{3} \end{aligned}$$

Put $x = \frac{4}{3}$ in eq. (2)

$$2\left(\frac{4}{3}\right) + y = 4$$

$$y = 4 - \frac{8}{3}$$

$$= \frac{12-8}{3} = \frac{4}{3}$$

$\therefore$ $B\left(\frac{4}{3}, \frac{4}{3}\right)$

To find C solving eq. (1) and eq. (2)

Eq. (1) – Eq. (2) × 4, we get

$$\begin{aligned} 3x + 4y &= 12 \\ 8x + 4y &= 16 \\ - \quad - \quad & \quad - \\ \hline -5x &= -4 \\ x &= \frac{-4}{-5} \end{aligned}$$

Put $x = \frac{4}{5}$ in eq. (2)

$$\begin{aligned} 2\left(\frac{4}{5}\right) + y &= 4 \\ y &= 4 - \frac{8}{5} \\ &= \frac{20-8}{5} \\ &= \frac{12}{5} \end{aligned}$$

$\therefore \quad C\left(\frac{4}{5}, \frac{8}{5}\right)$

$z = 2x + 3y$ .... (3)

Put O (0, 0) in eq. (3)

$z = 2(0) + 3(0) = 0$

Put A (1, 0) in eq. (3)

$z = 2(1) + 3(0) = 2$

Put $B\left(\frac{4}{3}, \frac{4}{3}\right)$ in eq. (3)

$$\begin{aligned} z &= 2\left(\frac{4}{3}\right) + 3\left(\frac{4}{3}\right) \\ &= \frac{8}{3} + \frac{12}{3} \\ &= \frac{20}{3} \end{aligned}$$

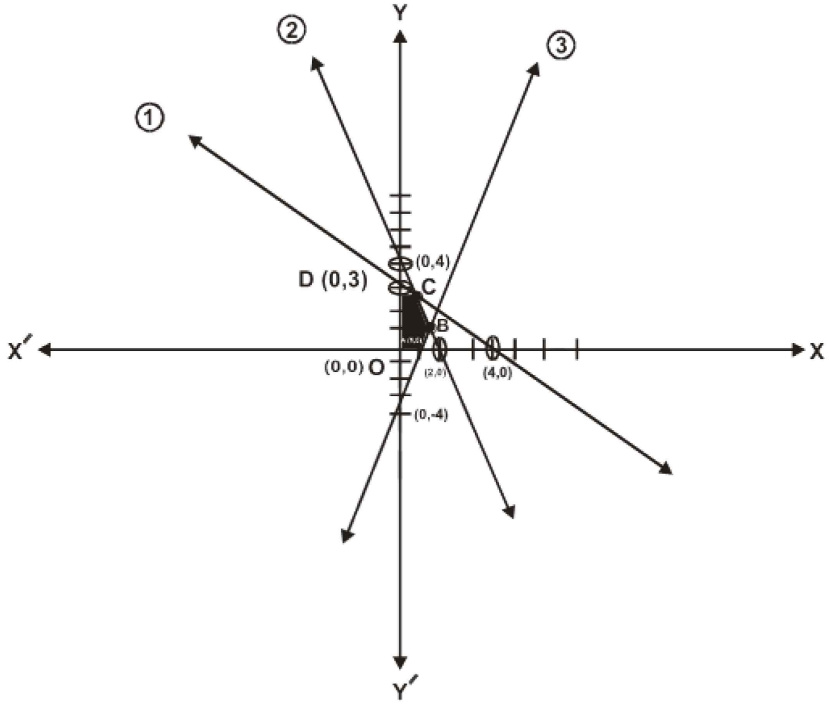
Put $C\left(\frac{4}{5}, \frac{12}{5}\right)$ in eq. (3)

$$z = 2\left(\frac{4}{5}\right) + 3\left(\frac{12}{5}\right)$$
$$= \frac{8}{5} + \frac{36}{5}$$
$$= \frac{44}{5}$$

Put D (0, 3) in eq. (3)

$z = 2(0) + 3(3) = 9$

The maximum value of $z = 2x + 3y$ is 9 at corner point D (0, 3).



Q.4 Minimize $z = 2x + y$

Subject to the constraints

$x + y \geq 3$; $7x + 5y \leq 35$; $x \geq 0$; $y \geq 0$ (Guj. Board 2005)

Solution:

The associated eqs. are

$x + y = 3$ ...... (1)

$7x + 5y = 35$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$x + 0 = 3$

$x = 3$

$\therefore$ Point is (3, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$0 + y = 3$$
$$y = 3$$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$7x + 5(0) = 35$$
$$7x = 35$$
$$x = \frac{35}{7} = 5$$

$\therefore$ Point is (5, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$7(0) + 5y = 35$$
$$y = \frac{35}{5} = 7$$

$\therefore$ Point is (0, 7)

<u>Test Point</u>

Put (0, 0) in

$x + y > 3$

$0 + 0 > 3$

$0 > 3$

Which is false.

$\therefore$ Graph of an inequality $x + y \geq 3$ will not towards the origin side.

Put (0, 0) in

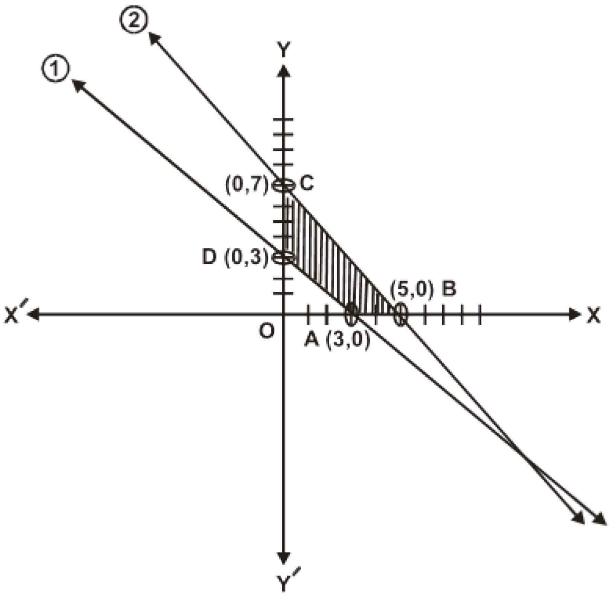
$7x + 5y < 35$

$7(0) + 5(0) < 35$

$0 < 35$

Which is true.

$\therefore$ Graph of an inequality $7x + 5y \leq 35$ will be towards the origin side.



∴ ABCD is the feasible solution region so corner points are

A (3, 0), B (5, 0), C (0, 7), D (0, 3)

$z = 2x + y$ .... (3)

Put A (3, 0) in eq. (3)

$z = 2(3) + 0 = 6$

Put B (5, 0) in eq. (3)

$z = 2(5) + 0 = 10$

Put C (0, 7) in eq. (3)

$z = 2(0) + 7 = 7$

Put D (0, 3) in eq. (3)

$z = 2(0) + 3 = 3$

The minimum value of $z = 2x + y$ is 3 at corner point D (0, 3).

Q.5 Maximize the function defined as f (x, y) = 2x + 3y subject to the constraints

$2x + y \leq 8$; $x + 2y \leq 14$; $x \geq 0$; $y \geq 0$ (Lhr. Board 2009)

Solution:

The associated eqs. are

$2x + y = 8$ ...... (1)

$x + 2y = 14$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$2x + 0 = 8$

$$2x = 8$$

$$x = \frac{8}{2} = 4$$

$\therefore$ Point is (4, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$$2(0) + y = 8$$

$$y = 8$$

$\therefore$ Point is (0, 8)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$$x + 2(0) = 14$$

$$x = 14$$

$\therefore$ Point is (14, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$$0 + 2y = 14$$

$$y = \frac{14}{2} = 7$$

$\therefore$ Point is (0, 7)

<u>Test Point</u>

Put (0, 0) in

$2x + y < 8$

$2(0) + 0 < 8$

$0 < 8$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 8$ will be towards the origin side.

Put (0, 0) in

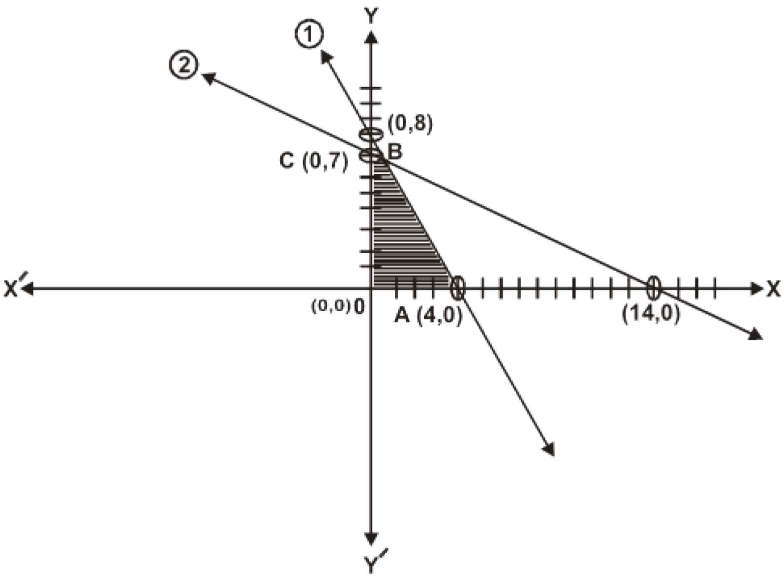
$x + 2y < 14$

$0 + 2(0) < 14$

$0 < 14$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \leq 14$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region. So corner points are

O (0, 0), A (4, 0), C (0, 7)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 2, we get

$$\begin{array}{rcl} 2x + y & = & 8 \\ -2x \pm 4y & = & -28 \\ \hline -3y & = & -20 \end{array}$$

$$y = \frac{20}{3}$$

Put $y = \frac{20}{3}$ in eq. (2)

$$x + 2\left(\frac{20}{3}\right) = 14$$

$$x + \frac{40}{3} = 14$$

$$x = 14 - \frac{40}{3}$$

$$x = \frac{42 - 40}{3}$$

$$x = \frac{2}{3}$$

$\therefore$ $B\left(\frac{2}{3}, \frac{20}{3}\right)$

$f(x, y) = 2x + 3y$ .... (3)

Put O (0, 0) in eq. (3)

$f(0, 0) = 2(0) + 3(0) = 0$

Put A (4, 0) in eq. (3)

$f(4, 0) = 2(4) + 3(0) = 8$

Put $B\left(\frac{2}{3}, \frac{20}{3}\right)$ in eq. (3)

$$f\left(\frac{2}{3}, \frac{20}{3}\right) = 2\left(\frac{2}{3}\right) + 3\left(\frac{20}{3}\right)$$

$$= \frac{4}{3} + \frac{60}{3}$$

$$= \frac{4 + 60}{3} = \frac{64}{3}$$

Put C (0, 7) in eq. (3)

$f(0, 7) = 2(0) + 3(7) = 21$

The maximum value of $f(x, y) = 2x + 3y$ is $\frac{64}{3}$ at corner point $B\left(\frac{2}{3}, \frac{20}{3}\right)$.

Q.6: Minimize $z = 3x + y$ subject to the constraints

$3x + 5y \geq 15$; $x + 6y \geq 9$; $x \geq 0$; $y \geq 0$ (Lhr. 2005, 2011)

Solution:

The associated eqs. are

$3x + 5y = 15$ ...... (1)

$x + 6y = 9$ ...... (2)

<u>x-intercept</u>

Put $y = 0$ in eq. (1)

$3x + 5(0) = 15$

$3x = 15$

$$x = \frac{15}{3} = 5$$

$\therefore$ Point is (5, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$3(0) + 5y = 15$

$5y = 15$

$$y = \frac{15}{5} = 3$$

$\therefore$ Point is (0, 3)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x + 3(0) = 9$

$x = 9$

$\therefore$ Point is (9, 0)

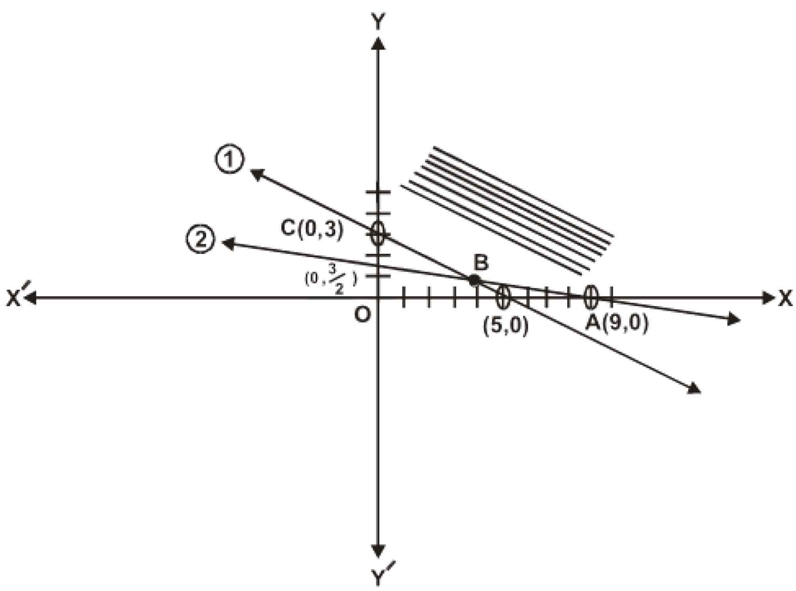
<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 + 6y = 9$

$y = \frac{3}{2} = 3$

$\therefore$ Point is $(0, \frac{3}{2})$



<u>Test Point</u>

Put (0, 0) in

$3x + 5y > 15$

$3(0) + 5(0) > 15$

$0 > 15$

Which is false.

$\therefore$ Graph of an inequality $3x + 5y \geq 15$ will not be towards the origin side.

Put (0, 0) in

$x + 6y > 9$

$0 + 6(0) > 9$

$0 > 9$

Which is true.

$\therefore$ Graph of an inequality $x + 3y \leq 9$ will not be towards the origin side.

$\therefore$ ABC is the feasible solution region. So corner points are

A (9, 0), C (0, 3)

$z = 3x + y \quad (3)$

Put A (9, 0) in eq. (3)

$z = 3(9) + 0 = 27$

Put $B\left(\frac{45}{13}, \frac{12}{13}\right)$ in eq. (3)

$$z = 3\left(\frac{45}{13}\right) + \frac{12}{13}$$

$$z = \frac{135}{13} + \frac{12}{13} = \frac{147}{13}$$

Put C (0, 3) in eq. (3)

$z = 3(0) + 3 = 3$

To find B solving eq. (1) & eq. (2)

eq. (1) – eq. × 3, we get

$$\begin{aligned} 3x + 5y &= 15 \\ 3x + 18y &= 27 \\ - \quad - \quad &\quad - \\ \hline -13y &= -12 \end{aligned}$$

$$y = \frac{12}{13}$$

Put $y = \frac{12}{13}$ in eq. (2)

$$x + 6\left(\frac{12}{13}\right) = 9$$

$$x + \frac{72}{13} = 9$$

$$x = 9 - \frac{72}{13}$$

$$x = \frac{117 - 72}{13}$$

$$= \frac{45}{13}$$

$$\therefore B\left(\frac{45}{13}, \frac{12}{13}\right)$$

The minimum value of $z = 3x + y$ is 3 at corner point C (0, 3).

Q.7: Each unit of food x costs Rs. 25 and contains 2 units of protein and 4 units of iron while each unit of food Y costs Rs. 30 and contains 3 units of protein and 2 units of iron. Each animal must receive at least 12 units of protein and 16 units of iron each day. How many units of each food should be fed to each animal at the smallest possible cost?

Solution:

Let x be the unit of food X and y be the unit of food Y.

Minimize $f(x, y) = 25x + 30y$

Subject to the constraints

$2x + 3y \geq 12$

$4x + 2y \geq 16$

$x \geq 0 \quad , \; y \geq 0$

The associated eqs. are

$2x + 3y = 12 \quad \text{.......} (1)$

$4x + 2y = 16 \quad \text{.......} (2)$

x-intercept

Put $y = 0$ in eq. (1)

$2x + 3(0) = 12$

$2x = 12$

$x = \frac{12}{2} = 6$

$\therefore$ Point is (6, 0)

y-intercept

Put $x = 0$ in eq. (1)

$2(0) + 3y = 12$

$3y = 12$

$y = \frac{12}{3} = 4$

$\therefore$ Point is (0, 4)

x-intercept

Put $y = 0$ in eq. (2)

$4x + 2(0) = 16$

$4x = 16$

$x = \frac{16}{4} = 4$

$\therefore$ Point is (4, 0)

y-intercept

Put $x = 0$ in eq. (2)

$4(0) + 2y = 16$

$2y = 16$

$y = \frac{16}{2} = 8$

$\therefore$ Point is (0, 8)

Test Point

Put (0, 0) in

$2x + 3y > 12$

$2(0) + 3(0) > 12$

$0 > 12$

Which is false.

$\therefore$ Graph of an inequality $2x + 3y \geq 12$ will not be towards the origin side.

Put (0, 0) in

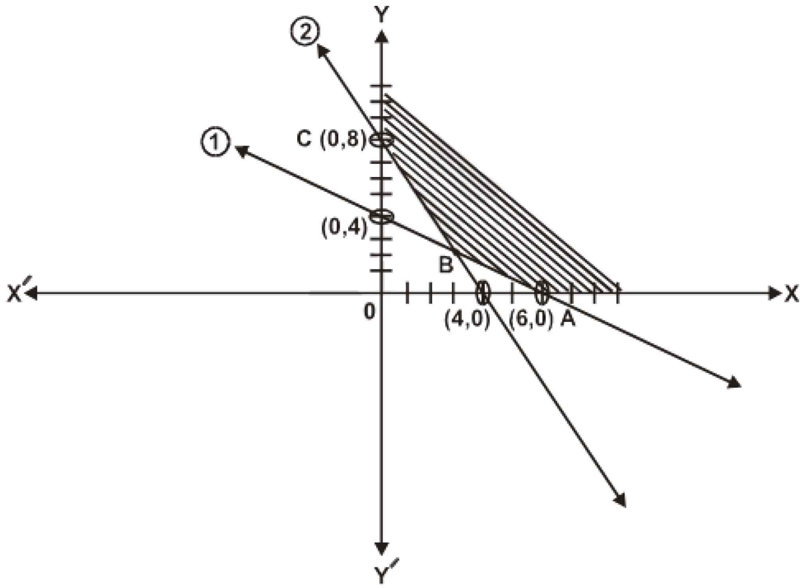
$4x + 2y > 16$

$4(0) + 2(0) > 16$

$0 > 16$

Which is false.

$\therefore$ Graph of an inequality $4x + 2y \geq 16$ will not be towards the origin side.



$\therefore$ ABC is the feasible solution region. So corner points are

A (6, 0), C (0, 8)

To find B solving eq. (1) & eq. (2)

Eq. (1) × 2 – Eq. (2), we get

$$\begin{aligned} 4x + 6y &= 24 \\ \underline{-4x \pm 2y} &= \underline{-16} \\ 4y &= 8 \\ y &= \frac{8}{4} = 2 \end{aligned}$$

Put $y = 2$ in eq. (1)

$$2x + 3(2) = 12$$
$$2x + 6 = 12$$
$$2x = 12 - 6$$
$$2x = 6$$
$$x = \frac{6}{2} = 3$$

$\therefore$ B (3, 2)

$f(x, y) = 25x + 30y$ .... (3)

Put A (6, 0) in eq. (3)

$f(6, 0) = 25(6) + 30(0) = 150$

Put B (3, 2) in eq. (3)

$f(3, 2) = 25(3) + 30(2) = 75 + 60 = 135$

Put C (0, 8) in eq. (3)

$f(0, 8) = 25(0) + 30(8) = 240$

The smallest cost of $f(x, y) = 25x + 30y$ is 135 at corner point B (3, 2)

Q.8: A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space at most for 20 items. A fan costs him Rs. 360 and a sewing machine costs Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

Solution:

Let x be the number of fans and y be the number of sewing machines.

Maximize $f(x, y) = 22x + 18y$

Subject to the constraints

$360x + 240y \leq 5760$

$x + y \leq 20$

$x \geq 0 \quad , \quad y \geq 0$

The associated eqs. are

$360x + 240y = 5760$ ...... (1)

$x + y = 20$ ...... (2)

x-intercept

Put $y = 0$ in eq. (1)

$360x + 240(0) = 5760$

$$x = \frac{5760}{360} = 16$$

$\therefore$ Point is (16, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (1)

$360(0) + 240y = 5760$

$$y = \frac{5760}{240} = 24$$

$\therefore$ Point is (0, 24)

<u>x-intercept</u>

Put $y = 0$ in eq. (2)

$x + 0 = 20$

$x = 20$

$\therefore$ Point is (20, 0)

<u>y-intercept</u>

Put $x = 0$ in eq. (2)

$0 + y = 20$

$y = 20$

$\therefore$ Point is (0, 20)

<u>Test Point</u>

Put (0, 0) in

$360x + 240y < 5760$

$360(0) + 240(0) < 5760$

$0 < 5760$

Which is true.

$\therefore$ Graph of an inequality $360x + 240y \leq 5760$ will be towards the origin side.

Put (0, 0) in

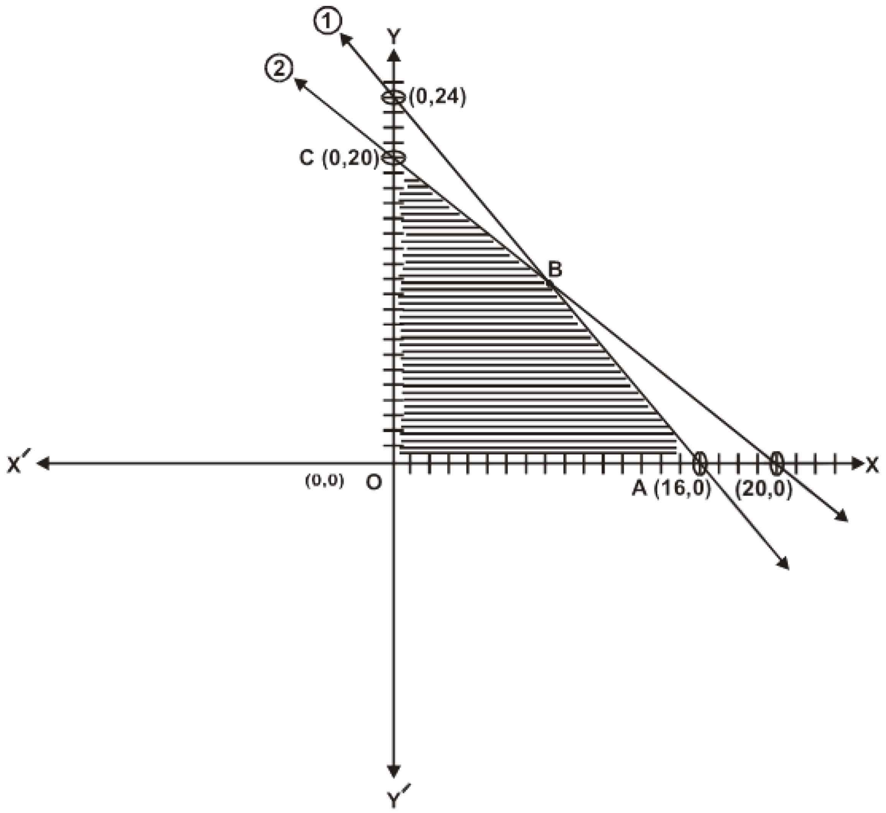
$x + y < 20$

$0 + 0 < 20$

$0 < 20$

Which is true.

$\therefore$ Graph of an inequality $x + y \leq 20$ will be towards the origin side.



$\therefore$ OABC is the feasible solution region.

So corner points are

O (0, 0), A (16, 0), C (0, 20)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 240, we get

$$\begin{aligned} 360x + 240y &= 5760 \\ \underline{-240x \pm 240y} &= \underline{-4800} \\ 120x &= 8 \\ x &= \frac{690}{120} = 8 \end{aligned}$$

Put $x = 8$ in eq. (2)

$$8 + y = 20$$

$$y = 20 - 8 = 12$$

$\therefore$ B (8, 12)

$$f(x, y) = 22x + 18y \quad .... (3)$$

Put O (0, 0) in eq. (3)

$$f(0, 0) = 22(0) + 18(0) = 0$$

Put A (16, 0) in eq. (3)

$f(16, 0) = 22(16) + 18(0) = 352$

Put B (8, 12) in eq. (3)

$f(8, 12) = 22(8) + 18(12) = 176 + 216 = 392$

Put C (0, 20) in eq. (3)

$f(0, 20) = 22(0) + 18(20) = 360$

The maximum profit of (x, y) = 22x + 18 y is 392 at corner point B (8, 12).

Q.9: A machine can produce product A by using 2 units of chemical and 1 unit of a compound or can produce product B by using 1 unit of chemical and 2 units of the compound. Only 800 units of chemical and 1000 units of the compound are available. The profits per unit of A and B are Rs. 30 and Rs. 20 respectively, maximize the profit function.

Solution:

Let x be the units of product A and y be the units of product B.

Maximize $f(x, y) = 30x + 20y$

Subject to the constraints

$2x + y \leq 800$

$x + 2y \leq 1000$

$x \geq 0 \quad , \quad y \geq 0$

The associated eqs. are

$2x + y = 800 \quad ...... (1)$

$x + 2y = 1000 \quad ...... (2)$

x-intercept

Put $y = 0$ in eq. (1)

$2x + 0 = 800$

$$x = \frac{800}{2} = 400$$

$\therefore$ Point is (400, 0)

y-intercept

Put $x = 0$ in eq. (1)

$2(0) + y = 800$

$y = 800$

$\therefore$ Point is (0, 800)

x-intercept

Put $y = 0$ in eq. (2)

$x + 2(0) = 1000$

$x = 1000$

$\therefore$ Point is (1000, 0)

y-intercept

Put $x = 0$ in eq. (2)

$$0 + 2y = 1000$$

$$y = \frac{1000}{2} = 500$$

$\therefore$ Point is (0, 500)

Test Point

Put (0, 0) in

$2x + y < 800$

$2(0) + 0 < 800$

$0 < 800$

Which is true.

$\therefore$ Graph of an inequality $2x + y \leq 800$ will be towards the origin side.

Put (0, 0) in

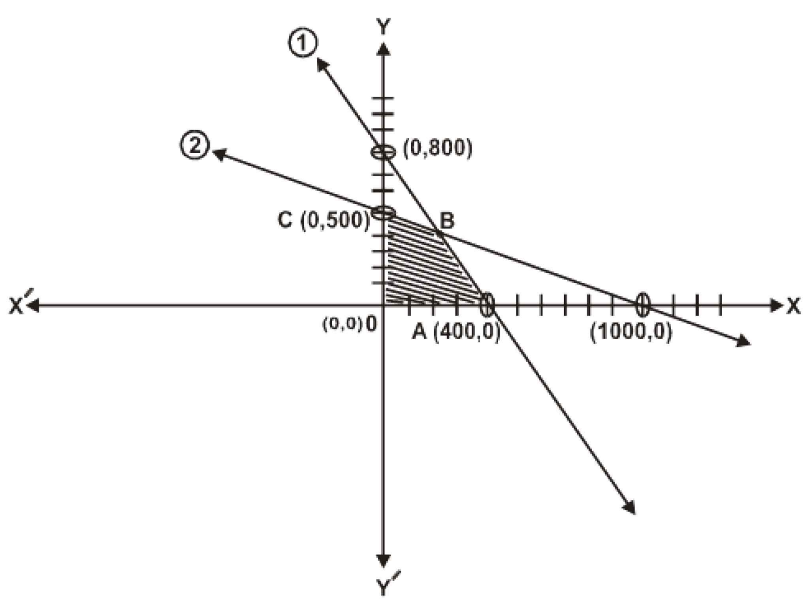
$x + 2y < 1000$

$0 + 2(0) < 1000$

$0 < 1000$

Which is true.

$\therefore$ Graph of an inequality $x + 2y \leq 1000$ will be towards the origin side.



OABC is the feasible solution region. So corner points are

O (0, 0), A (400, 0), C (0, 500)

To find B solving eq. (1) & eq. (2)

Eq. (1) – Eq. (2) × 2, we get

$$\begin{array}{rcl} 2x + y & = & 800 \\ \underline{-2x \pm 4y} & \underline{=} & \underline{-2000} \\ -3y & = & -1200 \\ y & = & \dfrac{1200}{3} = 400 \end{array}$$

Put $y = 400$ in eq. (2)

$$x + 2(400) = 1000$$
$$x + 800 = 1000$$
$$x = 1000 - 800$$
$$x = 200$$

$\therefore$ B (200, 400)

$f(x, y) = 30x + 20y$ .... (3)

Put O (0, 0) in eq. (3)

$f(0, 0) = 30(0) + 20(0) = 0$

Put A (400, 0) in eq. (3)

$f(400, 0) = 30(400) + 20(0) = 12000$

Put B (200, 400) in eq. (3)

$f(200, 400) = 30(200) + 20(400)$

$= 6000 + 8000 = 14000$

Put C (0, 500) in eq. (3)

$f(0, 500) = 30(0) + 20(500) = 10000$

The maximum profit of $f(x, y) = 30x + 20y$ is 14000 at corner point B (200, 400).

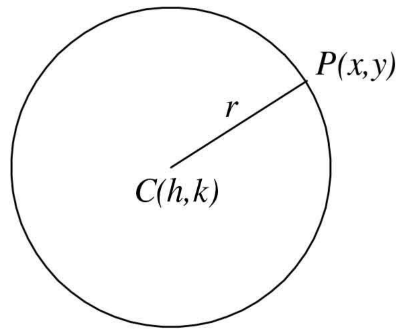
Circle

The set of all point in the plane that are equally distant from a fixed point is called a *circle.*

The fixed point is called *centre* of the circle and the distance from the centre of the circle to any point on the circle is called the *radius* of circle.

Equation of Circle

Let r be radius and $C(h,k)$ be centre of circle. Let $P(x,y)$ be any point on circle then



$$|PC| = r$$

$$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$$

On squaring

$$(x-h)^2+(y-k)^2 = r^2$$

This is equation of circle in standard form.

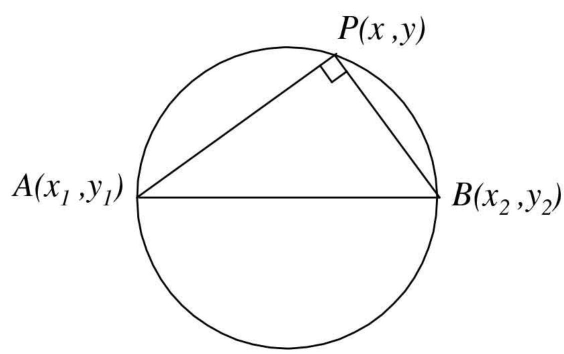
If centre of circle is at origin i.e. $C(h,k)=C(0,0)$ then equation of circle becomes

$$x^2+y^2 = r^2$$

Equation of circle with end points of diameter

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be end points of diameter.

Let $P(x,y)$ be any point on circle then



$m\angle APB = 90^\circ$

(**Note**: *An angle in a semi circle is a right angle – see Theorem 4 at page 270*)

Thus the line AP and BP are $\perp$ar to each other and we have

$$(\text{Slope of } AP)\ (\text{Slope of } BP) = -1$$

$$\Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

is the required equation of circle with end points of diameter $A(x_1,y_1)$ & $B(x_2,y_2)$.

General form of an equation of a circle

The equation

$$x^2+y^2+2gx+2fy+c = 0$$

represents a circle.

$$\Rightarrow x^2+2gx+g^2+y^2+2fy+f^2+c = g^2+f^2$$

$$\Rightarrow (x+g)^2+(y+f)^2 = g^2+f^2-c$$

$$\Rightarrow \quad (x-(-g))^2+(y-(-f))^2 = \left(\sqrt{g^2+f^2-c}\right)^2$$

This is equation of circle in standard form with

centre at $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$

Question # 1

In each of the following, find an equation of the circle with

(a) centre $(5,-2)$ and radius 4 (b) centre $\left(\sqrt{2},-3\sqrt{3}\right)$ and radius $2\sqrt{2}$

(c) ends of a diameter at $(-3,2)$ and $(5,-6)$.

Solution

(a) Given: centre $C(h,k) = (5,-2)$, radius $=r=4$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow (x-5)^2+(y+2)^2 = (4)^2$$
$$\Rightarrow x^2-10x+25+y^2+4y+4 = 16$$
$$\Rightarrow x^2+y^2-10x+4y+25+4-16 = 0$$
$$\Rightarrow x^2+y^2-10x+4y+13 = 0$$

(b) Given: centre $C(h,k) = \left(\sqrt{2},-3\sqrt{3}\right)$, radius $=r=2\sqrt{2}$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow \left(x-\sqrt{2}\right)^2+\left(y+3\sqrt{3}\right)^2 = \left(2\sqrt{2}\right)^2$$
$$\Rightarrow x^2-2\sqrt{2}x+2+y^2+6\sqrt{3}y+27 = 8$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+2+27-8 = 0$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21 = 0$$

(c) Given end points of diameter:

$$A(x_1,y_1) = (-3,2) \quad , \quad B(x_2,y_2) = (5,-6)$$

Equation of circle with ends of diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$
$$\Rightarrow (x-(-3))(x-5)+(y-2)(y-(-6)) = 0$$
$$\Rightarrow (x+3)(x-5)+(y-2)(y+6) = 0$$
$$\Rightarrow x^2+3x-4x-15+y^2-2y+6y-12 = 0$$
$$\Rightarrow x^2+y^2-2x+4y-27 = 0$$

Question # 2

Find the centre and radius of the circle with the given equation

(a) $x^2+y^2+12x-10y = 0$ (b) $5x^2+5y^2+14x+12y-10 = 0$

(c) $x^2+y^2-6x+4y+13 = 0$ (d) $4x^2+4y^2-8x+12y-25 = 0$

Solution

(a) $x^2+y^2+12x-10y = 0$

Here $2g=12$, $2f=-10$, $c=0$

$\Rightarrow g=6$, $f=-5$

So centre $= (-g,-f) = (-6,5)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(6)^2+(-5)^2-0}$

$= \sqrt{36+25} = \sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10 = 0$

$\Rightarrow x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2 = 0$ ÷ing by 5

Here $2g=\frac{14}{5}$, $2f=\frac{12}{5}$, $c=-2$

$\Rightarrow g=\frac{7}{5}$, $f=\frac{6}{5}$

Centre $= (-g,-f) = \left(-\frac{7}{5},-\frac{6}{5}\right)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{\left(\frac{7}{5}\right)^2+\left(\frac{6}{5}\right)^2-(-2)}$

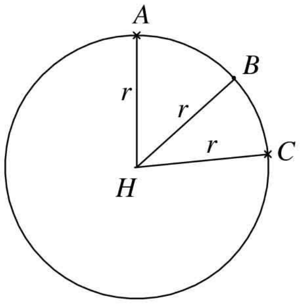
$= \sqrt{\frac{49}{25}+\frac{36}{25}+2} = \sqrt{\frac{27}{5}} = 3\sqrt{\frac{3}{5}}$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 3

Written an equation of the circle that passes through the given points



(a) $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

(b) $A(-7,7)$, $B(5,-1)$, $C(10,0)$

(c) $A(a,0), B(0,b), C(0,0)$ (d) $A(5,6), B(-3,2), C(3,-4)$

Solution

Given: $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Let $H(h,k)$ be centre and r be radius of circle, then

$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$

$$\Rightarrow \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$

$$\Rightarrow (h-4)^2+(k-5)^2=(h+4)^2+(k+3)^2=(h-8)^2+(k+3)^2=r^2 \quad \ldots\ldots \text{(i)}$$

From eq. (i)

$$(h-4)^2+(k-5)^2 = (h+4)^2+(k+3)^2$$
$$\Rightarrow h^2-8h+16+k^2-10k+25 = h^2+8h+16+k^2+6k+9$$
$$\Rightarrow h^2-8h+16+k^2-10k+25-h^2-8h-16-k^2-6k-9 = 0$$
$$\Rightarrow -16h-16k+16 = 0 \qquad \Rightarrow h+k-1 = 0 \text{ .... (ii)}$$

Again from (i)

$$(h+4)^2+(k+3)^2 = (h-8)^2+(k+3)^2$$
$$\Rightarrow (h+4)^2 = (h-8)^2 \Rightarrow h^2+8h+16 = h^2-16h+64$$
$$\Rightarrow h^2+8h+16-h^2+16h-64 = 0$$
$$\Rightarrow 24h-48 = 0 \qquad \Rightarrow 24h = 48 \qquad \Rightarrow \boxed{h = 2}$$

Putting value of h in (ii)

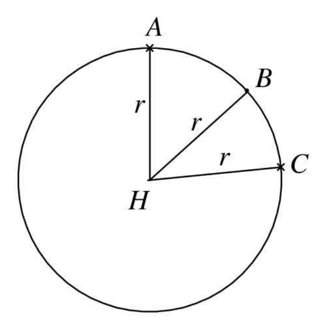
$$2+k-1 = 0 \qquad \Rightarrow k+1 = 0 \qquad \Rightarrow \boxed{k = -1}$$

Again from (i)

$$r^2 = (h-4)^2+(k-5)^2$$
$$= (2-4)^2+(-1-5)^2 \qquad \because h=2,\ k=-1$$
$$= (-2)^2+(-6)^2 = 4+36 = 40 \qquad \Rightarrow r = \sqrt{40}$$

Now equation of circle with centre at $H(2,-1)$ & $r=\sqrt{40}$

$$(x-2)^2+(y+1)^2 = \left(\sqrt{40}\right)^2$$
$$\Rightarrow x^2-4x+4+y^2+2y+1 = 40 \Rightarrow x^2+y^2-4x+2y+4+1-40 = 0$$
$$\Rightarrow x^2+y^2-4x+2y-35 = 0 \qquad \textit{Ans.}$$



(b) Given: $A(-7,7)$, $B(5,-1)$, $C(10,0)$

Let $H(h,k)$ be centre and r be radius of circle, then

$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$
$$\Rightarrow \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$
$$\Rightarrow (h+7)^2+(k-7)^2=(h-5)^2+(k+1)^2=(h-10)^2+(k-0)^2=r^2 \text{ .... (i)}$$

From equation (i) we have

$$(h+7)^2+(k-7)^2 = (h-10)^2+(k-0)^2$$
$$\Rightarrow h^2+14h+49+k^2-14k+49 = h^2-20h+100+k^2$$
$$\Rightarrow h^2+14h+49+k^2-14k+49-h^2+20h-100-k^2 = 0$$
$$\Rightarrow 34h-14k-2 = 0 \qquad \Rightarrow 17h-7k-1 = 0 \text{ .... (ii)}$$

Again from (i)

$$(h-5)^2+(k+1)^2 = (h-10)^2+(k-0)^2$$
$$\Rightarrow h^2-10h+25+k^2+2k+1 = h^2-20h+100+k^2$$

$$\Rightarrow \quad h^2-10h+25+k^2+2k+1-h^2+20h-100-k^2 = 0$$
$$\Rightarrow \quad 10h+2k-74 = 0$$
$$\Rightarrow \quad 5h+k-37 = 0 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Multiplying eq. (iii) by 7 and subtracting from (ii)

$$\begin{array}{rrrl} 17h & -7k & -1 & = 0 \\ 35h & +7k & -259 & = 0 \\ \hline 52h & & -260 & = 0 \end{array}$$

$$\Rightarrow \quad 52h = 260 \quad \Rightarrow \quad \boxed{h = 5}$$

Putting value of h in eq. (iii)

$$5(5)+k-37 = 0 \Rightarrow 25+k-37 = 0 \Rightarrow k-12 = 0 \Rightarrow \boxed{k = 12}$$

Again from eq. (i), we have

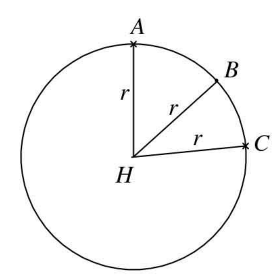
$$r^2 = (h+7)^2+(k-7)^2$$
$$= (5+7)^2+(12-7)^2 = (12)^2+(5)^2 = 144+25 = 169$$
$$\Rightarrow \quad r = 13$$

Now equation of circle with centre $(5,12)$ and radius 13:

$$(x-5)^2+(y-12)^2 = (13)^2$$
$$\Rightarrow \quad x^2-10x+25+y^2-24y+144 = 169$$
$$\Rightarrow \quad x^2+y^2-10x-24y+25+144-169 = 0$$
$$\Rightarrow \quad x^2+y^2-10x-24y = 0$$

(c) Given: $A(a,0)$, $B(0,b)$, $C(0,0)$

Let $H(h,k)$ be centre and r be radius of circle, then



$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$
$$\Rightarrow \quad \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$

$$\Rightarrow \quad (h-a)^2+(k-0)^2 = (h-0)^2+(k-b)^2 = (h-0)^2+(k-0)^2 = r^2$$
$$\Rightarrow \quad (h-a)^2+k^2 = h^2+(k-b)^2 = h^2+k^2 = r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From equation (i)

$$(h-a)^2+k^2 = h^2+k^2 \Rightarrow h^2-2ha+a^2+k^2 = h^2+k^2$$
$$\Rightarrow \quad -2ha+a^2 = 0 \quad \Rightarrow \quad -2ha = -a^2 \quad \Rightarrow \quad h = \frac{a^2}{2a} \quad \Rightarrow \quad \boxed{h = \frac{a}{2}}$$

Again from equation (i)

$$h^2+(k-b)^2 = h^2+k^2$$
$$\Rightarrow \quad h^2+k^2-2bk+b^2 = h^2+k^2 \quad \Rightarrow \quad -2bk+b^2 = 0$$
$$\Rightarrow \quad 2bk = b^2 \quad \Rightarrow \quad k = \frac{b^2}{2b} \quad \Rightarrow \quad \boxed{k = \frac{b}{2}}$$

Again from equation (i)

$$r^2 = h^2 + k^2$$

$$= \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} \quad \Rightarrow r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

Now equation of circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{\sqrt{a^2+b^2}}{2}$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4} = 0 \Rightarrow x^2 + y^2 - ax - by = 0$$

Alternative Method

Given point on circle: $A(a,0)$, $B(0,b)$, $C(0,0)$

Consider an equation of circle in standard form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ .... (i)}$$

Since $A(a,0)$ lies on circle, therefore

$$(a)^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$$

$$\Rightarrow a^2 + 2ga + c = 0 \text{ .... (ii)}$$

Also $B(0,b)$ lies on the circle, then

$$(0)^2 + (b)^2 + 2g(0) + 2f(b) + c = 0$$

$$\Rightarrow b^2 + 2fb + c = 0 \text{ .... (iii)}$$

Also $C(0,0)$ lies on the circle, therefore

$$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0 \quad \Rightarrow c = 0$$

Putting value of c in (ii)

$$a^2 + 2ga + 0 = 0 \quad \Rightarrow 2ga = -a^2 \quad \Rightarrow g = -\frac{a^2}{2a} \quad \Rightarrow g = -\frac{a}{2}$$

Putting value of c in (iii)

$$b^2 + 2fb + 0 = 0 \quad \Rightarrow 2fb = -b^2$$

$$\Rightarrow f = -\frac{b^2}{2b} \quad \Rightarrow f = -\frac{b}{2}$$

Putting value of g, f and c in (i)

$$x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 = 0$$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

Question # 4

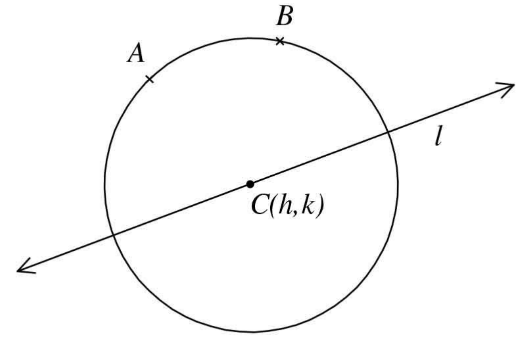
In each of the following, find an equation of the circle passes through

(a) $A(3,-1)$, $B(0,1)$ and have centre at $4x-3y-3=0$

(b) $A(-3,1)$ with radius 2 and centre at $2x-3y+3=0$

(c) $A(5,1)$ and tangent to the line $2x-y-10=0$ at $B(3,-4)$

(d) $A(1,4), B(-1,8)$ and tangent to the line $x+3y-3=0$



Solution

(a) Given: $A(3,-1)$, $B(0,1)$

$l:\ 4x-3y-3 = 0$

Let $C(h,k)$ be centre and r be radius of circle

$\because$ A & B lies on circle

$\therefore$ $|\overline{CA}| = |\overline{CB}| = r$

$\Rightarrow \sqrt{(h-3)^2+(k+1)^2} = \sqrt{(h-0)^2+(k-1)^2} = r$ …….. (i)

$\Rightarrow (h-3)^2+(k+1)^2 = h^2+(k-1)^2$ on squaring

$\Rightarrow h^2-6h+9+k^2+2k+1 = h^2+k^2-2k+1$

$\Rightarrow h^2-6h+9+k^2+2k+1-h^2-k^2+2k-1 = 0$

$\Rightarrow -6h+4k+9 = 0$ ……… (ii)

Now since $C(h,k)$ lies on given equation l

$\therefore$ $4h-3k-3 = 0$ ……… (iii)

$\times$ing equation (ii) by 3 & (iii) by 4 then adding

$$\begin{array}{rrrl} -18h & +12k & +27 & = 0 \\ 16h & -12k & -12 & = 0 \\ \hline -2h & & +15 & = 0 \end{array}$$

$\Rightarrow 2h = 15 \Rightarrow \boxed{h = \frac{15}{2}}$

Putting in (iii)

$4\left(\frac{15}{2}\right)-3k-3 = 0 \Rightarrow 30-3k-3 = 0$

$\Rightarrow -3k+27 = 0 \Rightarrow 3k = 27 \Rightarrow \boxed{k = 9}$

Now from eq. (i)

$r = \sqrt{h^2+(k-1)^2}$

$= \sqrt{\left(\frac{15}{2}\right)^2+(9-1)^2} = \sqrt{\frac{225}{4}+64} = \sqrt{\frac{448}{4}}$

Now equation of circle with centre at $C(h,k)=\left(\frac{15}{2},9\right)$ and radius $\sqrt{\frac{481}{4}}$

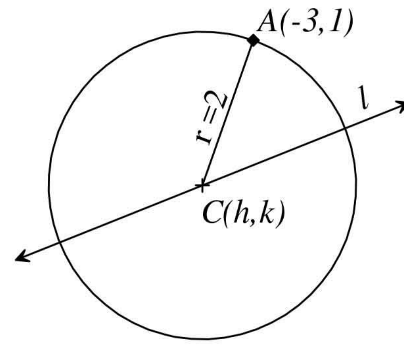
$$\left(x-\frac{15}{2}\right)^2+(y-9)^2 = \left(\sqrt{\frac{481}{4}}\right)^2$$

$$\Rightarrow\ x^2-15x+\frac{225}{4}+y^2-18y+81-\frac{481}{4} = 0$$

$$\Rightarrow\ x^2+y^2-15x-18y+17 = 0$$

(b) Given: $A(-3,1)$ lies on circle , radius $= r = 2$

$l:\ 2x-3y+3 = 0$



Let $C(h,k)$ be centre of circle.

Since $A(-3,1)$ lies on circle

$$\therefore\ r = \left|\overline{AC}\right|$$

$$\Rightarrow\ 2 = \sqrt{(h+3)^2+(k-1)^2}$$

$$\Rightarrow\ 4 = (h+3)^2+(k-1)^2$$

$$\Rightarrow\ 4 = h^2+6h+9+k^2-2k+1 \Rightarrow\ h^2+6h+9+k^2-2k+1-4 = 0$$

$$\Rightarrow\ h^2+6h+9+k^2-2k-3 = 0\ \ldots\ldots\ \text{(i)}$$

Since centre $C(h,k)$ lies on l

$$\therefore\ 2h-3k+3 = 0$$

$$\Rightarrow\ 2h = 3k-3 \quad \Rightarrow\ h = \frac{3k-3}{2}\ \ldots\ldots\ldots\ \text{(ii)}$$

Putting value of h in (i)

$$\left(\frac{3k-3}{2}\right)^2+k^2+6\left(\frac{3k-3}{2}\right)-2k+6 = 0$$

$$\Rightarrow\ \frac{9k^2-18k+9}{4}+k^2+9k-9-2k+6 = 0$$

$$\Rightarrow\ 9k^2-18k+9+4k^2+36k-36-8k+24 = 0 \qquad \times\text{ing by 4}$$

$$\Rightarrow\ 13k^2+10k-3 = 0 \quad \Rightarrow\ 13k^2+13k-3k-3 = 0$$

$$\Rightarrow\ 13k(k+1)-3(k+1) = 0 \Rightarrow\ (k+1)(13k-3) = 0$$

$$\Rightarrow\ k = -1 \quad \text{or} \quad k = \frac{3}{13},$$

Putting value of k in (ii)

$$h = \frac{3(-1)-3}{2} = \frac{-6}{2} = -3$$

$\Rightarrow\ (-3,-1)$ is centre of circle

$$h = \frac{3\left(3/13\right)-3}{2} = \frac{9/13-3}{2} = \frac{-30/13}{2} = \frac{-15}{13}$$

$\Rightarrow\ \left(-\frac{15}{13},\frac{3}{13}\right)$ is centre of circle.

Now equation of circle with centre at $(-3,1)$ and radius 2

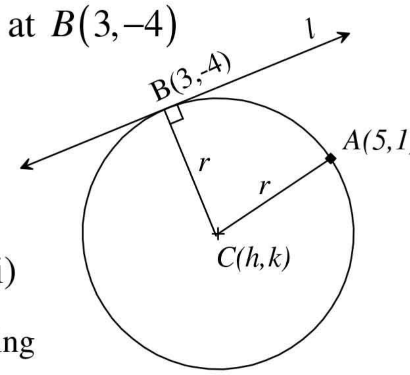
$$(x+3)^2+(y+1)^2 = (2)^2 \quad \Rightarrow \quad (x+3)^2+(y+1)^2 = 4$$

Now equation of circle with centre at $\left(-\frac{15}{13},\frac{3}{13}\right)$ and radius 2

$$\left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = (2)^2$$

$$\Rightarrow \left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = 4$$

(c) Given: $A(5,1)$ and l: $2x-y-10=0$ is tangent at $B(3,-4)$



Let $C(h,k)$ be centre and r be radius of circle.

$\because$ $A(5,1)$ and $B(3,-4)$ lies on circle

$\therefore$ $|AC| = |BC| = r$

$\Rightarrow \sqrt{(h-5)^2+(k-1)^2} = \sqrt{(h-3)^2+(k+4)^2} = r$ ..... (i)

$\Rightarrow (h-5)^2+(k-1)^2 = (h-3)^2+(k+4)^2$ On squaring

$\Rightarrow h^2-10h+25+k^2-2k+1 = h^2-6h+9+k^2+8k+16$

$\Rightarrow h^2-10h+25+k^2-2k+1-h^2+6h-9-k^2-8k-16 = 0$

$\Rightarrow -4h-10k+1 = 0$ ……….. (ii)

Now slope of tangent $l = m_1 = -\frac{a}{b} = -\frac{2}{-1} = 2$

And slope of radial segment $\overline{CB} = m_2 = \frac{k+4}{h-3}$

Since radial segment is perpendicular to tangent therefore

$$m_1m_2 = -1$$

$\Rightarrow 2\left(\frac{k+4}{h-3}\right) = -1 \quad \Rightarrow \quad 2k+8 = -h+3$

$\Rightarrow h-3+2k+8 = 0$

$\Rightarrow h+2k+5 = 0$ ……….. (iii)

Multiplying eq. (iii) by 4 and adding in (ii)

$$\begin{array}{rrrl} 4h & +\ 8k & +\ 20 & = 0 \\ -4h & -10k & +1 & = 0 \\ \hline & -\ 2k & +21 & = 0 \end{array}$$

$\Rightarrow 2k = 21 \quad \Rightarrow \quad \boxed{k = \frac{21}{2}}$

Putting value of k in (iii)

$h+2\left(\frac{21}{2}\right)+5 = 0 \quad \Rightarrow \quad h+21+5 = 0$

$$\Rightarrow\ h+26=0 \quad \Rightarrow \quad \boxed{h=-26}$$

Now from eq. (i)

$$r=\sqrt{(h-3)^2+(k+4)^2}$$

$$=\sqrt{(-26-3)^2+\left(\frac{21}{2}+4\right)^2} \quad =\sqrt{(-29)^2+\left(\frac{29}{2}\right)^2}$$

$$=\sqrt{841+\frac{841}{4}} \quad =\sqrt{\frac{4205}{4}}$$

Now equation of circle with centre at $\left(-26,\dfrac{21}{2}\right)$ and radius $\sqrt{\dfrac{4205}{4}}$

$$(x+26)^2+\left(y-\frac{21}{2}\right)^2=\left(\sqrt{\frac{4205}{4}}\right)^2$$

$$\Rightarrow\ x^2+52x+676+y^2-21y+\frac{441}{4}-\frac{4205}{4}=0$$

$$\Rightarrow\ x^2+y^2+52x-21y-265=0$$

(d) Given; $A(1,4)$, $B(-1,8)$

$$l:\ x+3y-3=0$$

Let $C(h,k)$ be centre and r be radius of circle then

$$\left|\overline{AC}\right|=\left|\overline{BC}\right|=r$$

$$\Rightarrow\ \sqrt{(h-1)^2+(k-4)^2}=\sqrt{(h+1)^2+(k+8)^2}=r \quad \ldots\ldots\ldots\ (i)$$

Also l is tangent to circle

$\therefore$ radius of circle $=$ $\perp$ar distance of $C(h,k)$ form l

$$\Rightarrow\ r=\frac{|h+3k-3|}{\sqrt{(1)^2+(3)^2}}$$

$$\Rightarrow\ r=\frac{|h+3k-3|}{\sqrt{10}} \quad \ldots\ldots\ldots\ (ii)$$

Now from (i)

$$\sqrt{(h-1)^2+(k-4)^2}=\sqrt{(h+1)^2+(k-8)^2}$$

On squaring

$$(h-1)^2+(k-4)^2=(h+1)^2+(k-8)^2$$

$$\Rightarrow\ h^2-2h+1+k^2-8k+16=h^2+2h+1+k^2-16k+64$$

$$\Rightarrow\ h^2-2h+1+k^2-8k+16-h^2-2h-1-k^2+16k-64=0$$

$$\Rightarrow\ -4h+8k-48=0$$

$$\Rightarrow\ h-2k+12=0 \quad \ldots\ldots\ldots\ldots\ (iii)$$

Now from (i) & (ii)

$$\sqrt{(h-1)^2+(k-4)^2} = \frac{|h+3k-3|}{\sqrt{10}}$$

On squaring

$$(h-1)^2+(k-4)^2 = \frac{|h+3k-3|^2}{10}$$

$$\Rightarrow\ 10\left[(h-1)^2+(k-4)^2\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow\ 10\left[h^2-2h+1+k^2-8k+16\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow\ 10h^2-20h+10+10k^2-80k+160-h^2-9k^2-9-6hk+18k+6h = 0$$

$$\Rightarrow\ 9h^2+k^2-14h-62k-6hk+161 = 0 \ \ldots\ldots\ldots\ (\text{iv})$$

From (iii)

$$h = 2k-12 \ \ldots\ldots\ (\text{v})$$

Putting in (iv)

$$9(2k-12)^2+k^2-14(2k-12)-62k-6(2k-12)k+161 = 0$$

$$\Rightarrow\ 9\left(4k^2-48k+144\right)+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow\ 36k^2-432k+1296+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow\ 25k^2-450k+1625 = 0$$

$$\Rightarrow\ k^2-18k+65 = 0 \qquad \div\text{ing by } 25$$

$$\Rightarrow\ k^2-13k-5k+65 = 0 \qquad \Rightarrow\ k(k-13)-5(k-13) = 0$$

$$\Rightarrow\ (k-13)(k-5) = 0$$

$$\Rightarrow\ k=13 \quad \text{or} \quad k = 5$$

Putting in eq. (v)

$$h = 2(13)-12$$
$$= 26-12 = 14$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$

$$\Rightarrow\ r = \sqrt{(14-1)^2+(13-4)^2}$$

$$= \sqrt{(13)^2+(9)^2} = \sqrt{169+81}$$

$$= \sqrt{250}$$

Now eq. of circle with centre $(14,13)$ and radius $\sqrt{170}$

$$(x-14)^2+(y-13)^2 = \left(\sqrt{250}\right)^2$$

$$\Rightarrow\ (x-14)^2+(y-13)^2 = 250$$

Putting in (v)

$$h = 2(5)-12$$
$$= 10-12 = -2$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$

$$= \sqrt{(2-1)^2+(5-4)^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

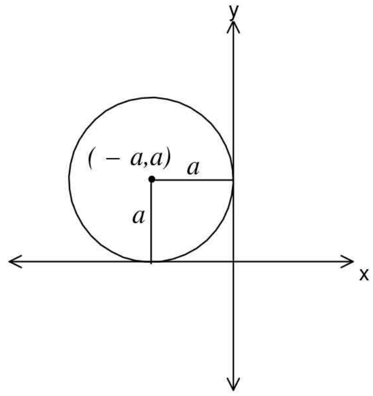
Now eq. of circle with centre $(2,5)$ and radius $\sqrt{2}$

$$(x-2)^2+(y-5)^2 = \left(\sqrt{2}\right)^2$$

$$\Rightarrow\ (x-2)^2+(y-5)^2 = 2$$

Question # 5

Find an equation of a circle of radius a and lying in the second quadrant such that it is tangent to both the axes.



Solution

Radius of circle $= r = a$

$\because$ circle lies in second quadrant and touching both the axis therefore centre of circle is $(-a, a)$

So equation of circle

$$(x-(-a))^2+(y-a)^2 = (a)^2$$

$$\Rightarrow\ x^2-2ax+a^2+y^2-2ay+a^2-a^2 = 0$$

$$\Rightarrow\ x^2+y^2-2ax-2ay+a^2 = 0$$

Question # 6

Show that the lines $3x-2y=0$ and $2x+3y-13=0$ are tangents to the circle $x^2+y^2+6x-4y=0$

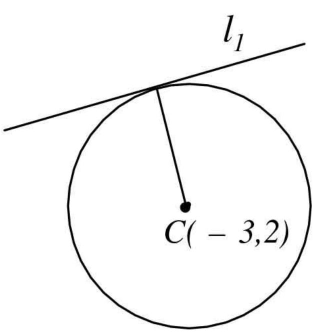
Solution

Suppose

$l_1:\ 3x-2y = 0$

$l_2:\ 2x+3y-13 = 0$

$S:\ x^2+y^2+6x-4y = 0$



From S

$2g = 6\ ,\ \ 2f = -4\ ,\ \ c = 0$

$\Rightarrow\ \ g = 3\ ,\ \ f = -2\ ,$

Centre $C(-g,-f) = C(-3,2)$

$$\text{Radius} = r = \sqrt{g^2+f^2-c}$$

$$= \sqrt{(3)^2+(-2)^2-0}$$

$$= \sqrt{9+4=0} = \sqrt{13}$$

Now to check l_1 is tangent to circle, we find

$$\perp\text{ar distance of } l_1 \text{ from centre} = \frac{|3(-3)-2(2)+0|}{\sqrt{(3)^2+(-2)^2}}$$

$$= \frac{|-9-4|}{\sqrt{9+4}} = \frac{|-13|}{\sqrt{13}} = \frac{13}{\sqrt{13}}$$

$$= \sqrt{13} = \text{radius of circle}$$

$\Rightarrow\ l_1$ is tangent to given circle.

Now to check l_2 is tangent to circle, let

$$\perp\text{ar distance of } l_2 \text{ from centre} = \frac{|2(-3)+3(2)-13|}{\sqrt{(2)^2+(3)^2}}$$

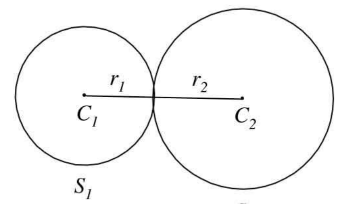
$$= \frac{|-6+6-13|}{\sqrt{4+9}} = \frac{|-13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}} = \sqrt{13} = \text{Radius of circle}$$

$\Rightarrow$ l_2 is also tangent to given circle.

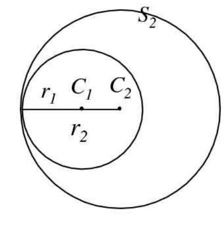
Circles touching each other externally or internally

Let C_1 be centre and r_1 be radius of circle S_1 and C_2 be centre and r_2 be radius of circle S_2.



Then they touch each other externally if

$$\left|\overline{C_1C_2}\right| = r_1 + r_2$$



And they touch each other internally if

$$\left|\overline{C_1C_2}\right| = |r_2 - r_1|$$

Question # 7

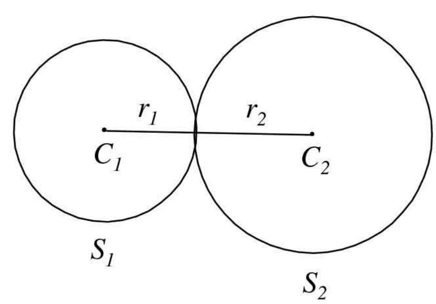
Show that the circles

$x^2+y^2+2x-2y-7=0$ and $x^2+y^2-6x+4y+9=0$ touch externally.

Solution

$$\text{Let } S_1:\ x^2+y^2+2x-2y-7=0$$

$$S_2:\ x^2+y^2-6x+4y+9=0$$



For S_1:

$$2g=2 \quad , \quad 2f=-2 \quad , \quad c=-7$$

$$\Rightarrow g=1 \quad , \quad f=-1 \ ,$$

Let C_1 be centre and r_1 be radius of circle S_1, then

$$C_1(-g,-f) = C_1(-1,1)$$

$$\text{Radius} = r_1 = \sqrt{g^2+f^2-c}$$

$$= \sqrt{(1)^2+(-1)^2-(-7)} = \sqrt{1+1+7} = \sqrt{9} = 3$$

For S_2:

$$2g=-6 \quad , \quad 2f=4 \quad , \quad c=9$$

$$\Rightarrow g=-3 \quad , \quad f=2$$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g,-f) = C_2(3,-2)$$

$$\text{Radius} = r_2 = \sqrt{g^2+f^2-c}$$

$$= \sqrt{(-3)^2+(2)^2-9} = \sqrt{9+4-9} = \sqrt{4} = 2$$

Now circles touch each other externally if

$$|C_1C_2| = r_1 + r_2$$

$$\Rightarrow \sqrt{(3+1)^2+(-2-1)^2} = 3+2$$

$$\Rightarrow \sqrt{16+9} = 5 \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

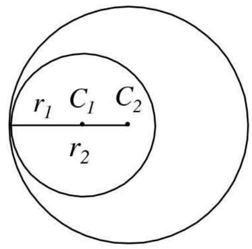
Hence both circles touch each other externally.

Question # 8

Show that the circles

$x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touches internally.

Solution



Suppose $S_1:\ x^2 + y^2 + 2x - 8 = 0$

$S_2:\ x^2 + y^2 - 6x + 6y - 46 = 0$

For S_1: $2g = 2$, $2f = 0$, $c = -8$

$\Rightarrow g = 1$, $f = 0$

Let C_1 be centre and r_1 be radius of circle S_1 then

$$C_1(-g,-f) = C(-1,0)$$

Radius $= r_1 = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(1)^2 + (0)^2 + 8} = \sqrt{9} = 3$$

For S_2: $2g = -6$, $2f = 6$, $c = -46$

$\Rightarrow g = -3$, $f = 3$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g,-f) = C_2(3,-3)$$

Radius $= r_2 = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(3)^2 + (-3)^2 - (-46)} = \sqrt{9+9+46} = \sqrt{64} = 8$$

Now circles touch each other internally if

$$\left|\overline{C_1C_2}\right| = |r_2 - r_1| \Rightarrow \sqrt{(3+1)^2 + (-3-0)^2} = |8-3|$$

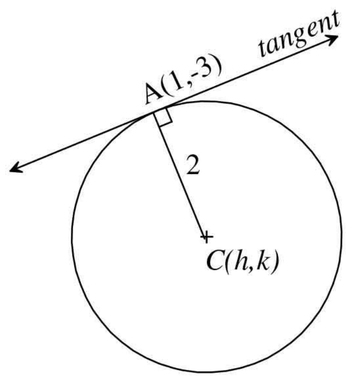
$$\Rightarrow \sqrt{16+9} = |5| \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

Hence circles are touching each other internally.

Question # 9

Find an equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at $A(1,-3)$.

Solution



Given: Radius $r = 2$,

Tangent: $x - y - 4 = 0$ at $A(1,-3)$

Suppose $C(h,k)$ be the centre then

$$|AC| = 2$$

$$\Rightarrow \sqrt{(h-1)^2 + (k+3)^2} = 2$$

On squaring

$$(h-1)^2 + (k+3)^2 = 4$$

$$\Rightarrow h^2 - 2h + 1 + k^2 + 6k + 9 - 4 = 0 \Rightarrow h^2 + k^2 - 2h + 6k + 6 = 0 \quad \text{........ (i)}$$

Now slope of radial line $AC = \dfrac{k+3}{h-1}$

Slope of line tangent $= -\dfrac{1}{-1} = 1$

Since radial line is $\perp$ar to tangent, therefore

(Slope of radial line) (Slope of tangent) $= -1$

$$\Rightarrow \left(\frac{k+3}{h-1}\right)(1) = -1$$

$$\Rightarrow k+3 = -(h-1) \quad \Rightarrow k = -h+1-3 \quad \Rightarrow k = -h-2 \text{ …….. (ii)}$$

Putting in (i) $h^2 + (-h-2)^2 - 2h + 6(-h-2) + 6 = 0$

$$\Rightarrow h^2 + h^2 + 4h + 4 - 2h - 6h - 12 + 6 = 0 \Rightarrow 2h^2 - 4h - 2 = 0 \quad \Rightarrow h^2 - 2h - 1 = 0$$

$$\Rightarrow h = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Putting $h = 1+\sqrt{2}$ in (ii)

$k = -1-\sqrt{2}-2$

$\Rightarrow k = -3-\sqrt{2}$

Now equation of circle with centre $\left(1+\sqrt{2}, -3-\sqrt{2}\right)$ and radius 2.

$$\left(x-(1+\sqrt{2})\right)^2 - \left(y-(-3-\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow \left(x-1-\sqrt{2}\right)^2 - \left(y+3+\sqrt{2})\right)^2 = 4$$

Putting $h = 1-\sqrt{2}$ in (ii)

$k = -1+\sqrt{2}-2 \Rightarrow k = -3+\sqrt{2}$

Now equation of circle with centre $\left(1-\sqrt{2}, -3+\sqrt{2}\right)$ and radius 2.

$$\left(x-(1-\sqrt{2})\right)^2 - \left(y-(-3+\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow \left(x-1+\sqrt{2}\right)^2 - \left(y+3-\sqrt{2})\right)^2 = 4$$

Equation of tangent and normal to the circle

Consider an equation of circle

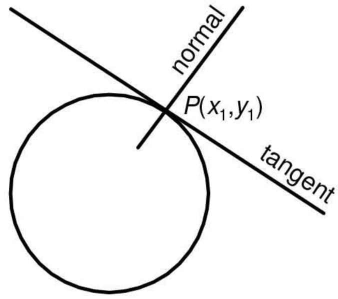
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then equation of tangent at (x_1, y_1) is

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

The equation of normal at (x_1, y_1) is

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$



(*See proof at page 257*)

Question # 1

Write down equations of tangent and normal to the circle

(i) $x^2 + y^2 = 25$ at $(4,3)$ and at $(5\cos\theta, 5\sin\theta)$

(ii) $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ at $\left(1, \frac{10}{3}\right)$

Solution $x^2 + y^2 = 25$

Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ 2y\frac{dy}{dx} = -2x \qquad \Rightarrow\ \frac{dy}{dx} = -\frac{x}{y}$$

At $(4,3)$:

Slope of tangent at $(4,3)$ $= m = \left.\frac{dy}{dx}\right|_{(4,3)} = -\frac{4}{3}$

Now equation of tangent at $(4,3)$ having slope $-\frac{4}{3}$

$$y - 3 = -\frac{4}{3}(x - 4) \qquad \Rightarrow\ 3y - 9 = -4x + 16$$

$$\Rightarrow\ 4x - 16 + 3y - 9 = 0 \qquad \Rightarrow\ \boxed{4x + 3y - 25 = 0}$$

Since normal is $\perp$ar to tangent therefore

Slope of normal at $(4,3)$ $= -\frac{1}{m} = -\frac{1}{-4/3} = \frac{3}{4}$

Now equation of normal at $(4,3)$ having slope $\frac{3}{4}$

$$y - 3 = \frac{3}{4}(x - 4)$$

$$\Rightarrow\ 4y - 12 = 3x - 12 \qquad \Rightarrow\ 3x - 12 - 4y + 12 = 0$$

$$\Rightarrow\ \boxed{3x - 4y = 0}$$

At $(5\cos\theta, 5\sin\theta)$

Slope of tangent at $(5\cos\theta, 5\sin\theta) = m = \left.\dfrac{dy}{dx}\right|_{(5\cos\theta,5\sin\theta)} = -\dfrac{5\cos\theta}{5\sin\theta} = -\dfrac{\cos\theta}{\sin\theta}$

Now equation of tangent at $(5\cos\theta, 5\sin\theta)$ having slope $-\dfrac{\cos\theta}{\sin\theta}$

$$y - 5\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - 5\cos\theta)$$
$$\Rightarrow \quad y\sin\theta - 5\sin^2\theta = -x\cos\theta + 5\cos^2\theta$$
$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5\sin^2\theta + 5\cos^2\theta$$
$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5\left(\sin^2\theta + \cos^2\theta\right)$$
$$\Rightarrow \quad x\cos\theta + y\sin\theta = 5(1)$$
$$\Rightarrow \quad \boxed{x\cos\theta + y\sin\theta = 5}$$

Since normal are $\perp$ ar to tangent therefore

Slope of normal $= -\dfrac{1}{m} = \dfrac{\sin\theta}{\cos\theta}$

Now equation of normal at $(5\cos\theta, 5\sin\theta)$ having slope $\dfrac{\sin\theta}{\cos\theta}$

$$y - 5\sin\theta = \frac{\sin\theta}{\cos\theta}(x - 5\cos\theta)$$
$$\Rightarrow \quad y\cos\theta - 5\sin\theta\cos\theta = x\sin\theta - 5\sin\theta\cos\theta$$
$$\Rightarrow \quad x\sin\theta - 5\sin\theta\cos\theta - y\cos\theta + 5\sin\theta\cos\theta = 0$$
$$\Rightarrow \quad \boxed{x\sin\theta - y\cos\theta = 0}$$

(ii) **[Alternative Method]**

$$3x^2 + 3y^2 + 5x - 13y + 2 = 0$$
$$\Rightarrow \quad x^2 + y^2 + \frac{5}{3}x - \frac{13}{3}y + \frac{2}{3} = 0$$

Comparing it with general equation of circle

$$2g = \frac{5}{3} \quad , \quad 2f = -\frac{13}{3} \quad , \quad c = \frac{2}{3}$$
$$\Rightarrow \quad g = \frac{5}{6} \quad , \quad f = -\frac{13}{6}$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

Here $(x_1, y_1) = \left(1, \dfrac{10}{3}\right)$

$$\Rightarrow \quad 1\cdot x + \frac{10}{3}\cdot y + \frac{5}{6}(x+1) - \frac{13}{6}\left(y + \frac{10}{3}\right) + \frac{2}{3} = 0$$
$$\Rightarrow \quad x + \frac{10}{3}y + \frac{5}{6}x + \frac{5}{6} - \frac{13}{6}y - \frac{130}{18} + \frac{2}{3} = 0$$

$$\Rightarrow \frac{11}{6}x+\frac{7}{6}y+\frac{157}{18} = 0$$
$$\Rightarrow 33x+21y+157 = 0$$

is the required tangent.

Now equation of normal at (x_1, y_1)

$$(y-y_1)(x_1+g) = (x-x_1)(y_1+f)$$

Here $(x_1, y_1) = \left(1, \frac{10}{3}\right)$

$$\Rightarrow \left(y-\frac{10}{3}\right)\left(1+\frac{5}{6}\right) = (x-1)\left(\frac{10}{3}-\frac{13}{6}\right)$$
$$\Rightarrow \left(y-\frac{10}{3}\right)\left(\frac{11}{6}\right) = (x-1)\left(\frac{7}{6}\right) \qquad \Rightarrow 11y-\frac{110}{3} = 7x-7$$
$$\Rightarrow 7x-7-11y+\frac{110}{3} = 0 \qquad \Rightarrow 7x-11y+\frac{89}{3} = 0$$
$$\Rightarrow 21x-33y+89 = 0$$

is required equation of normal.

Question # 2

Write down equations of tangent and normal to the circle

$4x^2+4y^2-16x+24y-117 = 0$

at the point on the circle whose abscissa is -4.

Solution

$$4x^2+4y^2-16x+24y-117 = 0 \quad \ldots\ldots\ldots \text{(i)}$$

Since abscissa $= -4$, so putting $x=-4$ in given eq.

$$4(-4)^2+4y^2-16(-4)+24y-117 = 0$$
$$\Rightarrow 64+4y^2+64+24y-117 = 0$$
$$\Rightarrow 4y^2+24y+11 = 0$$
$$\Rightarrow y = \frac{-24\pm\sqrt{(24)^2-4(4)(11)}}{2(4)}$$
$$= \frac{-24\pm\sqrt{576-176}}{8} \qquad = \frac{-24\pm\sqrt{400}}{8}$$
$$\Rightarrow y = \frac{-24\pm 20}{8}$$
$$\Rightarrow y = \frac{-24+20}{8} \qquad \text{or} \qquad y = \frac{-24-20}{8}$$
$$\Rightarrow y = -\frac{1}{2} \qquad \text{or} \qquad y = -\frac{11}{2}$$

So we have points $\left(-4,-\frac{1}{2}\right)$ & $\left(-4,-\frac{11}{2}\right)$

$$\text{(i)} \Rightarrow x^2+y^2-4x+6y-\frac{117}{4} = 0$$

Comparing it with general equation of circle

$$2g = -4 \quad , \quad 2f = 6 \quad , \quad c = -\frac{117}{4}$$

$$\Rightarrow g = -2 \quad , \quad f = 3$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

For $(x_1, y_1) = \left(-4, -\frac{1}{2}\right)$

Solve yourself as Q # 1(ii)

For $(x_1, y_1) = \left(-4, -\frac{11}{2}\right)$

Solve yourself as Q # 1(ii)

Position of the point with a circle

Consider the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The point $P(x_1, y_1)$ lies on the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

The point $P(x_1, y_1)$ lies outside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

And the point $P(x_1, y_1)$ lies inside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$$

Question # 3

Check the position of the point $(5,6)$ with respect to the circle

(i) $x^2 + y^2 = 81$ (ii) $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

Solution

(i) $$x^2 + y^2 = 81$$

$$\Rightarrow x^2 + y^2 - 81 = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

To check the position of point $(5,6)$, Putting $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 - 81 = 25 + 36 - 81$$
$$= -20 < 0$$

Hence $(5,6)$ lies inside the circle.

(ii) $$2x^2 + 2y^2 + 12x - 8y + 1 = 0$$

$$\Rightarrow x^2 + y^2 + 6x - 4y + \frac{1}{2} = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

To check the position of point put $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 + 6(5) - 4(6) + \frac{1}{2}$$

$$= 25+36+30-24+\frac{1}{2}$$

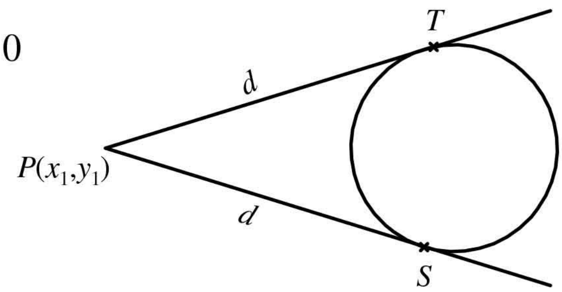
$$= \frac{135}{2} > 0$$

Hence $(5,6)$ lies outside the circle.

Length of tangent to the circle

Consider equation of circle

$$x^2+y^2+2gx+2fy+c = 0$$



If d denotes length of tangent from point $P(x_1, y_1)$ to the circle then

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Question # 4

Find the length of the tangent drawn from the point $(-5,4)$ to the circle $5x^2+5y^2-10x+15y-131 = 0$

Solution $\quad 5x^2+5y^2-10x+15y-131 = 0$

$$\Rightarrow x^2+y^2-2x+3y-\frac{131}{3} = 0$$

Now length of tangent from point $P(x_1, y_1)$ is

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

For $(x_1, y_1) = (-5,4)$

$$d = \sqrt{(-5)^2+(4)^2-2(-5)+3(4)-\frac{131}{5}}$$

$$= \sqrt{25+16+10+12-\frac{131}{5}}$$

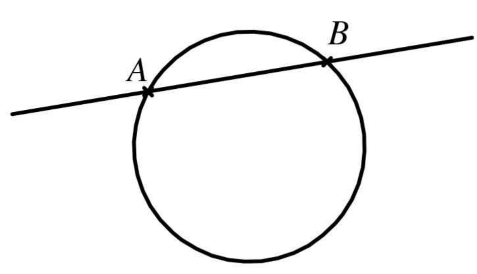
$$= \sqrt{\frac{184}{5}} = 2\sqrt{\frac{46}{5}} \text{ units}$$

Question # 5

Find the length of the chord cut off from the line $2x+3y = 13$ by the circle $x^2+y^2 = 26$

Solution $\quad 2x+3y = 13$ ……….. (i)

$x^2+y^2 = 26$ ……….. (ii)



From (i)

$$2x = 13-3y$$

$$\Rightarrow x = \frac{13-3y}{2} \text{ ………….. (iii)}$$

Putting in (ii)

$$\left(\frac{13-3y}{2}\right)^2 + y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2}{4} + y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2+4y^2}{4} = 26$$

$$\Rightarrow 13y^2 - 78y + 169 = 104$$

$$\Rightarrow 13y^2 - 78y + 169 - 104 = 0$$

$$\Rightarrow 13y^2 - 78y + 65 = 0$$

$$\Rightarrow y^2 - 6y + 5 = 0$$

$$\Rightarrow y^2 - 5y - y + 5 = 0$$

$$\Rightarrow y(y-5) - 1(y-5) = 0$$

$$\Rightarrow (y-5)(y-1) = 0$$

$$\Rightarrow y = 5 \quad \text{or} \quad y = 1$$

Putting in (iii)

$$x = \frac{13-3(5)}{2} = -1 \qquad\qquad x = \frac{13-3(1)}{2} = 5$$

$\Rightarrow (-1,5)$ and $(5,1)$ are end points of chord intercepted.

So length of chord $= \sqrt{(5+1)^2 + (1-5)^2}$

$= \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

Question # 6

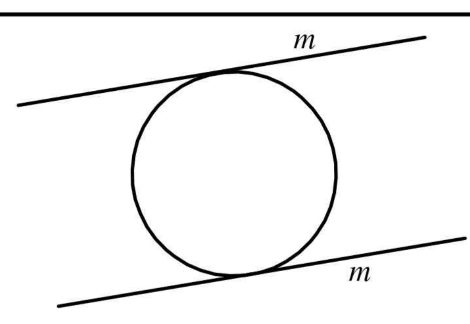
Find the coordinates of the points of intersection of the line $x + 2y = 6$ with the circle : $x^2 + y^2 - 2x - 2y - 39 = 0$

Solution $x^2 + y^2 - 2x - 2y - 39 = 0$ .... (i)

$x + 2y = 6$ .... (ii)

Just solve (i) & (ii) *to get the points*

Equation of tangent to the circle having slope m



Consider an equation of circle

$$x^2 + y^2 = a^2$$

Then equations of tangents parallel to the line having slope m are

$$y = mx \pm a\sqrt{1+m^2}$$

Question # 7

Find equations of the tangents to the circle $x^2 + y^2 = 2$

(i) parallel to the $x - 2y + 1 = 0$

(ii) perpendicular to the line $3x + 2y = 6$

Solution $x^2 + y^2 = 2$

Centre of circle is at origin with radius $a = \sqrt{2}$

i) Let l: $x - 2y + 1 = 0$

Slope of $l = m = -\dfrac{1}{-2} = \dfrac{1}{2}$

Since required tangent is parallel to l

$\therefore$ Slope of tangent $= m = \dfrac{1}{2}$

Now equations of tangents are

$$y = mx \pm a\sqrt{1+m^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2} \qquad \Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\frac{1}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{\frac{5}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{\frac{10}{4}} \qquad \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2}$$

$$\Rightarrow 2y = x \pm \sqrt{10}$$

$\Rightarrow x - 2y \pm \sqrt{10} = 0$ are the req. tangents.

(ii) *Do yourself as above*

Question # 8

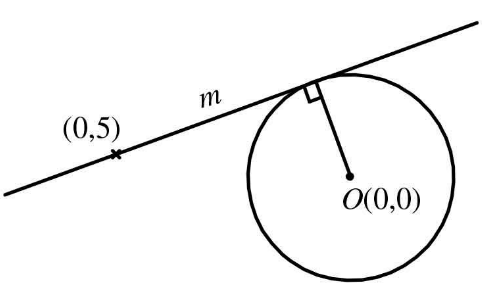
Find equations of the tangent drawn from

(i) $(0,5)$ to $x^2 + y^2 = 16$

(ii) $(-1,2)$ to $x^2 + y^2 + 4x + 2y = 0$

(iii) $(-1,2)$ to $(x+1)^2 + (y-2)^2 = 26$

Also find the point of contact.



Solution

(i) $x^2 + y^2 = 16$

$\Rightarrow$ radius of circle is 4 with centre $O(0,0)$

Let slope of required tangent be m, then eq. of tangent passing through $(0,5)$

$$y - 5 = m(x - 0)$$

$$\Rightarrow y - 5 = mx$$

$$\Rightarrow mx - y + 5 = 0 \;\ldots\ldots\ldots\ldots\; \text{(i)}$$

Now since (i) is tangent to circle, therefore

Radius of circle = $\perp$ar distance of (i) from centre $O(0,0)$

$$\Rightarrow 4 = \frac{|m(0)-0+5|}{\sqrt{m^2+(-1)^2}}$$

$$\Rightarrow 4 = \frac{|5|}{\sqrt{m^2+1}} \qquad \Rightarrow 4\sqrt{m^2+1} = |5|$$

On squaring

$$\left(4\sqrt{1+m^2}\right)^2 = |5|^2$$

$$\Rightarrow 16\left(m^2+1\right) = 25 \qquad \Rightarrow 16m^2+16 = 25$$

$$\Rightarrow 16m^2 = 25-16 \qquad \Rightarrow 16m^2 = 9$$

$$\Rightarrow m^2 = \frac{9}{16} \qquad \Rightarrow m = \pm\frac{3}{4}$$

When $m=\frac{3}{4}$, putting in (i)

$$\frac{3}{4}x-y+5 = 0$$

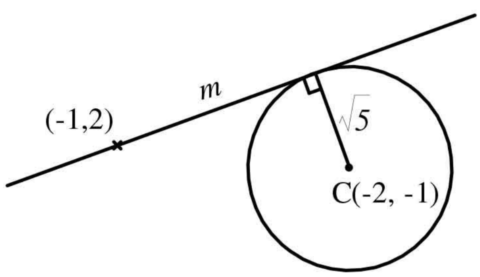
$$\Rightarrow 3x-4y+20 = 0$$

When $m=-\frac{3}{4}$, putting in (i)

$$-\frac{3}{4}x-y+5 = 0$$

$$\Rightarrow 3x+4y-2 = 0$$

(ii) $\qquad x^2+y^2+4x+2y = 0$



Comparing it with general equation of circle

We have $2g = 4$, $2f = 2$, $c = 0$

$$\Rightarrow g = 2 \quad , \quad f = 1$$

Centre $C(-g,-f) = C(-2,-1)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(2)^2+(1)^2-0}$

$= \sqrt{4+1} = \sqrt{5}$

Let m be a slope of required tangent, so equation of tangent passing thorough $(-1,2)$

$$y-2 = m(x+1)$$

$$\Rightarrow y-2 = mx+m$$

$$\Rightarrow mx-y+(m+2) = 0 \;\ldots\ldots\ldots\ldots\; \text{(i)}$$

$\because$ (i) is tangent ot circle,

$\therefore$ $\perp$ar distance of tangent from centre $(-2,-1)$ = Radius of circle

$$\Rightarrow \frac{|m(-2)-(-1)+(m+2)|}{\sqrt{(m)^2+(-1)^2}} = \sqrt{5}$$

$$\Rightarrow \frac{|-2m+1+m+2|}{\sqrt{m^2+1}} = \sqrt{5}$$

$$\Rightarrow |-m+3| = \sqrt{5}\cdot\sqrt{m^2+1}$$

On squaring

$$m^2-6m+9 = 5(m^2+1)$$
$$\Rightarrow 5m^2+5-m^2+6m-9 = 0$$
$$\Rightarrow 4m^2+6m-4 = 0$$
$$\Rightarrow 2m^2+3m-2 = 0$$
$$\Rightarrow 2m^2+4m-m-2 = 0$$
$$\Rightarrow 2m(m+2)-1(m+2) = 0$$
$$\Rightarrow (m+2)(2m-1) = 0$$
$$\Rightarrow m+2 = 0 \quad \text{or} \quad 2m-1 = 0$$
$$\Rightarrow m=-2 \quad \text{or} \quad m=\frac{1}{2}$$

Putting value of m in (i)

$-2x-y+(-2+2) = 0$	$\frac{1}{2}x-y+\left(\frac{1}{2}+2\right) = 0$
$\Rightarrow -2x-y+0 = 0$	$\Rightarrow \frac{1}{2}x-y+\frac{5}{2} = 0$
$\Rightarrow 2x+y = 0$	$\Rightarrow x-2y+5 = 0$

(iii) $$(x+1)^2+(y-2)^2 = 26$$
$$\Rightarrow (x-(-1))^2+(y-2)^2 = \left(\sqrt{26}\right)^2$$

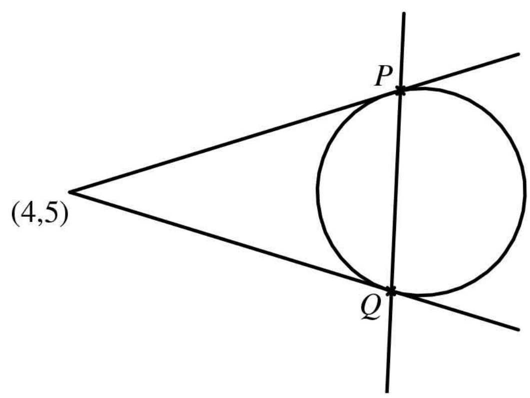
Centre of circle is $(-1,2)$ and radius $\sqrt{26}$

Now do yourself as above.

Note: *To find point of contact, solve equation of tangent and circle.*

Question # 9

Find an equation of the chord of contact of the tangents drawn from $(4,5)$ to the circle
$2x^2+2y^2-8x+12y+21 = 0$



Solution

Given: $2x^2+2y^2-8x+12y+21 = 0$

$$\Rightarrow x^2+y^2-4x+6y+\frac{21}{2} = 0$$

Comparing it with general equation of circle

$$2g = -4 \;,\quad 2f = 6 \;,\quad c = \frac{21}{2}$$
$$\Rightarrow g = -2 \;,\quad f = 3$$

Let the point of contact of two tangent be $P(x_1,y_1)$ and $Q(x_2,y_2)$

Eq. of tangent at $P(x_1, y_1)$

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

$$\Rightarrow x_1x + y_1y - 2(x + x_1) + 3(y + y_1) + \frac{21}{2} = 0$$

$$\Rightarrow x_1x + y_1y - 2x - 2x_1 + 3y + 3y_1 + \frac{21}{2} = 0$$

Since tangent is drawn from $(4,5)$, therefore

$$\Rightarrow x_1(4) + y_1(5) - 2(4) - 2x_1 + 3(5) + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 2x_1 + 8y_1 + \frac{35}{2} = 0$$

$$\Rightarrow 4x_1 + 16y_1 + 35 = 0 \quad \ldots\ldots\ldots \text{(i)}$$

Similarly equation of tangent passing through $Q(x_2, y_2)$ and $(4,5)$ gives

$$4x_2 + 16y_2 + 35 = 0 \quad \ldots\ldots\ldots \text{(ii)}$$

Eqs. (i) and (ii) show that both points P & Q lies on the line

$$4x + 16y + 35 = 0$$

So it is the required equation of chord of contact.

Question # 1

Prove that normal lines of a circle pass through the centre of the circle.

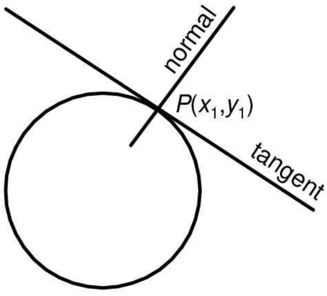
Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$

Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad 2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at (x_1, y_1) $= m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}$.



Since normal is $\perp$ ar to tangent therefore

Slope of normal at (x_1, y_1) $= -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}$.

Now equation of normal at (x_1, y_1) having slope $\frac{y_1}{x_1}$

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow \quad x_1 y - x_1 y_1 = y_1 x - y_1 x_1$$

$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

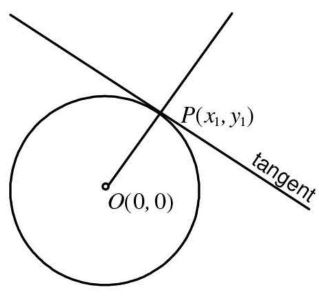
Clearly centre of circle $(0,0)$ satisfies (i), hence normal lines of the circles passing through the centre of the circle.

Question # 2

Prove that the straight line drawn from the centre of a circle perpendicular to a tangent passes through the point of tangency.

Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$



Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at (x_1, y_1) $= m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}$.

Slope of line $\perp$ ar to tangent $= -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}$.

Now equation of line perpendicular to tangent passing through centre $(0,0)$

$$y - 0 = \frac{y_1}{x_1}(x - 0)$$

$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Clearly the point of tangency (x_1, y_1) satisfy (i), hence the straight line drawn from the centre of circle perpendicular to a tangent passes through the point of tangency.

Question # 3

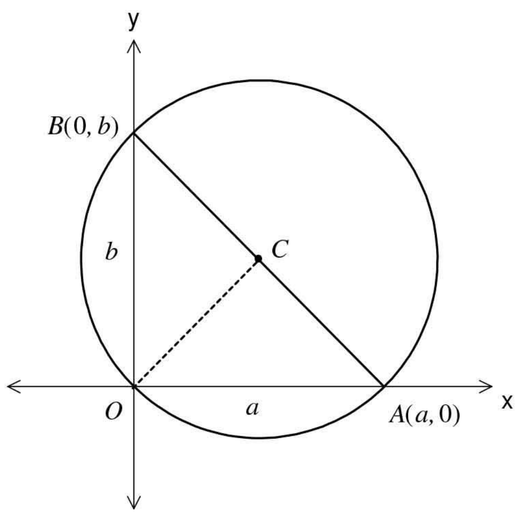
Prove that the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

Solution Let OAB be a right triangle with $|OA| = a$, $|OB| = b$.

Then the coordinates of A and B are $(a,0)$ and $(0,b)$ respectively.

Let C be the mid-point of hypotenuse AB. Then

$$\text{coordinate of } C = \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right).$$



Now
$$|CA| = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2}$$
$$= \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$
$$|CB| = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2}$$
$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$
$$|CO| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}.$$

Since $|CA| = |CB| = |CO|$, therefore C is the centre of the circumcircle.

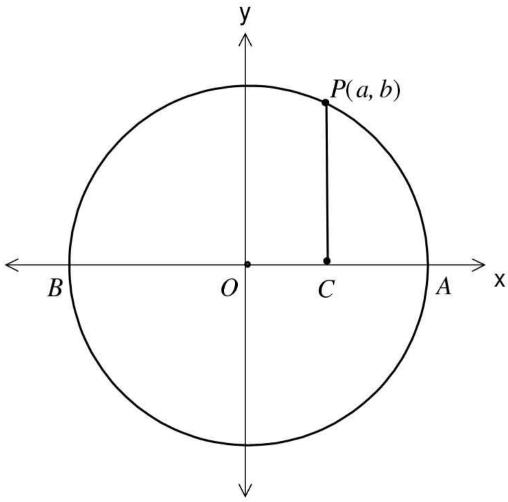
Hence the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

Mean proportional

Let a,b and c be three numbers. The number b is said to be *mean proportional* between a and b if a,b,c are in geometric means or

$$b^2 = ac \quad \text{or} \quad \frac{b}{a} = \frac{a}{c}.$$

Question # 4
Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.



Solution Consider a circle of radius r and centre $(0,0)$, then equation of circle

$$x^2 + y^2 = r^2$$

Let A and B are end-points of diameter of circle along x-axis, then coordinate of A and B are $(-r,0)$ and $(0,r)$ respectively.

Also let $P(a,b)$ be any point on circle and $\perp$ ar from P cuts diameter at C. Then coordinate of C are $(a,0)$.

Since $P(a,b)$ lies on a circle, therefore

$$a^2 + b^2 = r^2 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Now

$$|AC| = \sqrt{(r+a)^2 - (0-0)^2} = r+a.$$

$$|CB| = \sqrt{(r-a)^2 - (0-0)^2} = r-a.$$

$$|PC| = \sqrt{(a-a)^2 + (b-0)^2} = \sqrt{0+b^2} = b.$$

Now

$$\begin{aligned} |AC|\cdot|CB| &= (r+a)(r-a) \\ &= r^2 - a^2 \\ &= a^2 + b^2 - a^2 \qquad \text{from (i)} \\ &= b^2 = |PC|^2 \end{aligned}$$

$$\Rightarrow |AC|\cdot|CB| = |PC|\cdot|PC| \qquad \Rightarrow \frac{|AC|}{|PC|} = \frac{|PC|}{|CB|}$$

$\Rightarrow |PC|$ is a mean proportional to $|AC|$ and $|CB|$.

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Question # 1

Find the focus, vertex and directix of the parabola sketch its graph.

(i) $y^2 = 8x$ (ii) $x^2 = -16y$

(iii) $x^2 = 5y$ (iv) $y^2 = -12x$

(v) $x^2 = 4(y-1)$

(vi) $y^2 = -8(x-3)$

(vii) $(x-1)^2 = 8(y+2)$ (viii) $y = 6x^2 - 1$

(ix) $x+8-y^2+2y = 0$

(x) $x^2-4x-8y+4 = 0$

Solution

(i) $y^2 = 8x$

Here $4a = 8 \Rightarrow a = 2$

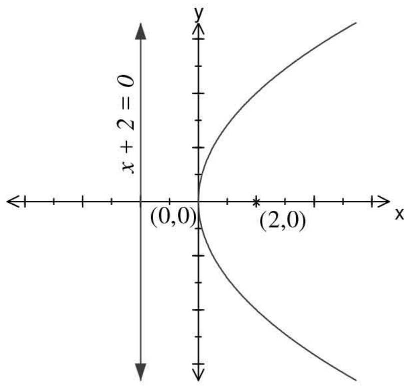
Vertex: $O(0,0)$

The axis of parabola is along x-axis and opening of parabola is to the right side.

Focus: $(a,0) = (2,0)$

Directrix: $x+a = 0$

$\Rightarrow x+2 = 0$



(ii) $x^2 = -16y$

Here $4a = 16 \Rightarrow a = 4$

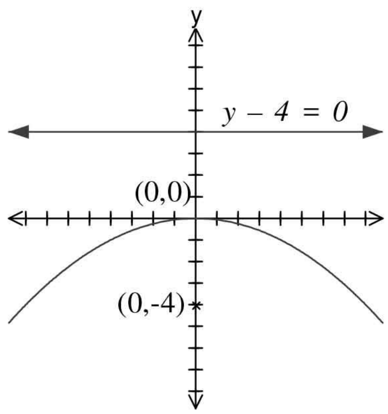
Vertex: $O(0,0)$

The axis of parabola is along $y-$axis and opening of the parabola is downward.

So Focus: $F(0,-a) = F(0,-4)$

Directrix: $y-a = 0$

$\Rightarrow y-4 = 0$



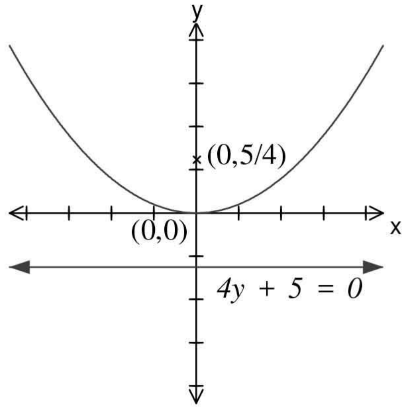
(ii) $x^2 = 5y$

Here $4a = 5 \Rightarrow a = \dfrac{5}{4}$

And vertex: $O(0,0)$

The axis of the parabola is along y-axis and opening of the parabola is upward.

Focus: $F(0,a) = F\left(0,\dfrac{5}{4}\right)$



Directrix: $y+a = 0 \quad \Rightarrow \quad y+\dfrac{5}{4} = 0$

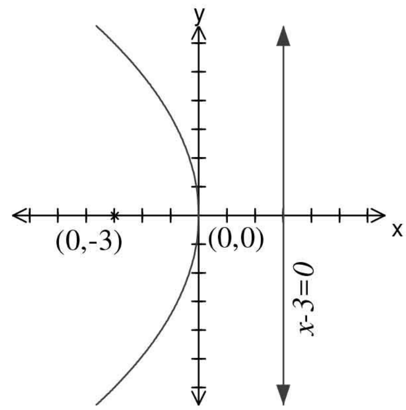
$\Rightarrow \quad 4y+5 = 0$

(iv) $y^2 = -12x$

Here $4a = 12 \quad \Rightarrow \quad a = 3$

And vertex: $O(0,0)$

The axis of the parabola is along x-axis and opening of the parabola is to the left side.



Focus: $F(-a,0) = (-3,0)$

Directrix: $x-a = 0$

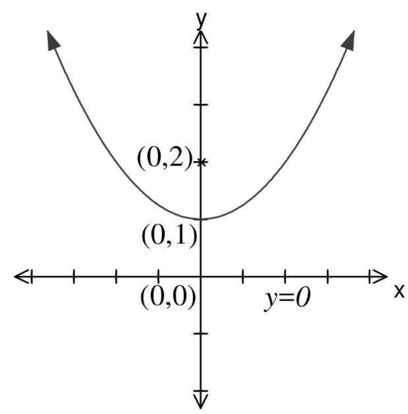
$\Rightarrow \quad x-3 = 0$

(v) $x^2 = 4(y-1)$ ……….. (i)

Put $X = x \quad , \quad Y = y-1$

$\Rightarrow \quad X^2 = 4Y$ ………… (ii)

Here $4a = 4 \quad \Rightarrow \quad a = 1$



And vertex of parabola (ii) is $O(0,0)$ with axis of parabola is along $Y-$axis open upward.

$\because$ Vertex: $O(0,0)$

$\Rightarrow \quad X = 0 \quad , \quad Y = 0$

$\Rightarrow \quad x = 0 \quad , \quad y-1=0 \quad \Rightarrow \quad y=1$

$\Rightarrow \quad (0,1)$ is vertex of parabola (i)

Now focus: $F(0,a) = F(0,1)$

$\Rightarrow \quad X = 0 \quad , \quad Y = 1$

$\Rightarrow \quad x = 0 \quad , \quad y-1 = 1$

$y = 1+1 \quad \Rightarrow \quad y = 2$

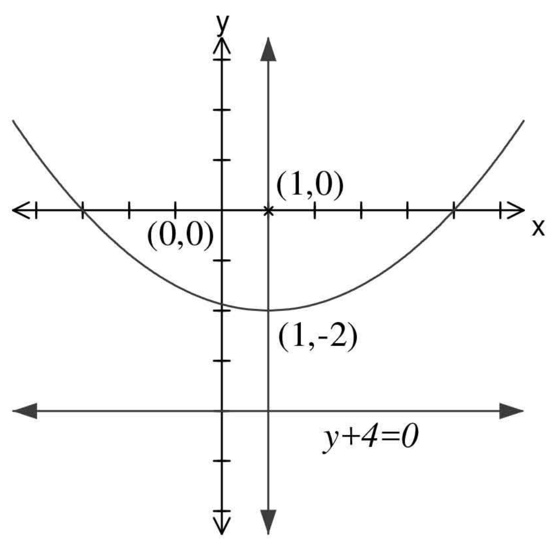
$\Rightarrow \quad (0,2)$ is focus of parabola (i)

Directrix of parabola (ii) is

$Y+a = 0 \quad \Rightarrow \quad Y+1 = 0$

$\Rightarrow \quad y-1+1 = 0$

$\Rightarrow \quad y = 0$ is directrix of parabola (i)



(vi) *Do yourself*

(vii) $(x-1)^2 = 8(y+2)$ ………… (i)

Put $X = x-1 \quad , \quad Y = y+2$ in (i)

$X^2 = 8Y$ ………… (ii)

Here $4a = 8 \Rightarrow a = 2$

Axis of parabola is along Y-axis open upward with vertex of $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 0$

$\Rightarrow x = 2 \quad , \quad y = -2$

$\Rightarrow (1,-2)$ is vertex of parabola (i)

Focus of (ii) is $(0,a) = (0,2)$

$\Rightarrow X = 0 \quad , \quad Y = 2$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 2$

$\Rightarrow x = 1 \quad , \quad y = 0$

$\Rightarrow (1,0)$ is the focus of given parabola (i)

Directrix of (ii)

$$Y + a = 0 \Rightarrow Y + 2 = 0$$

$\Rightarrow y+2+2 = 0 \Rightarrow y+4 = 0$ is directrix of given parabola.

(viii) $y = 6x^2 - 1$

$$\Rightarrow 6x^2 = y+1 \Rightarrow x^2 = \frac{1}{6}(y+1)$$

Now try yourself

(ix) $x+8-y^2+2y = 0$

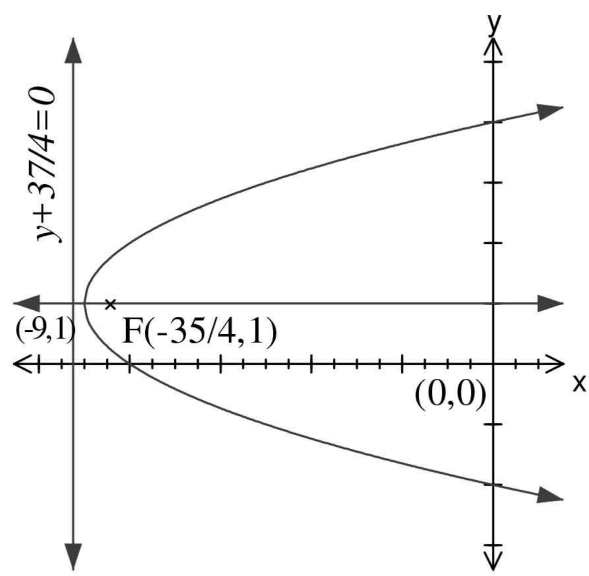
$\Rightarrow y^2 - 2y = x+8$

$\Rightarrow y^2 - 2y + 1 = x+8+1$

$\Rightarrow (y-1)^2 = x+9$

Put $X = x+9 \quad , \quad Y = y-1$

$Y^2 = X$



Here $4a = 1 \Rightarrow a = \frac{1}{4}$

The axis of parabola is along x-axis and it is opening to the right side.

Vertex of parabola (ii) is $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x+9 = 0 \quad , \quad y-1 = 0$

$\Rightarrow x = -9 \quad , \quad y = 1$

$\Rightarrow (-9,1)$ is vertex of the parabola (i)

Focus: $(a,0) = \left(\frac{1}{4},0\right)$

$$X = \frac{1}{4} \quad , \quad Y = 0$$

$$\Rightarrow \quad x+9 = \frac{1}{4} \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = \frac{1}{4}-9 \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = -\frac{35}{4} \quad , \quad y = 1$$

$\Rightarrow \left(-\frac{35}{4},1\right)$ is focus of parabola (i)

Directrix of parabola (ii)

$X + a = 0$

$$\Rightarrow \quad X+\frac{1}{4} = 0$$

$$\Rightarrow \quad x+9+\frac{1}{4} = 0$$

$\Rightarrow \quad x+\frac{37}{4} = 0$ is directrix of parabola (i)

(x) *Do yourself as above*

Question # 2

Write an equation of the parabola with given elements.

(i) Focus $(-3,1)$; directrix $x=3$
(ii) Focus $(2,5)$; directrix $y=1$
(iii) Focus $(-3,1)$; directrix $x-2y-3=0$
(iv) Focus $(1,2)$; vertex $(3,2)$
(v) Focus $(-1,0)$; vertex $(-1,2)$
(vi)Directrix $x=-2$,Focus $(2,2)$
(vii)Directrix $y=3$; vertex $(2,2)$
(viii) Directrix $y=1$, length of latusrectum is 8. Opens downward.
(ix)Axis $y=0$, through $(2,1)$ and $(11,-2)$
(x)Axis parallel to y-axis, the points $(0,3),(3,4)$ *and* $(4,11)$ lie on the graph.

Solution

(i) Focus: $F(-3,1)$

Directrix: $x = 3$ i.e. $x-3 = 0$

Let $P(x,y)$ be any point on parabola then by definition

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \quad \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-3|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \quad \sqrt{x^2+6x+9+y^2-2y+1} = |x-3|$$

On squaring

$$\Rightarrow \quad x^2+6x+9+y^2-2y+1 = x^2-6x+9$$

$\Rightarrow \; x^2+6x+9+y^2-2y+1-x^2+6x-9 = 0$

$\Rightarrow \; y^2+12x-2y+1 = 0$

is required equation of parabola.

(ii) *Do yourself as above.*

(iii) Focus: $F(-3,1)$

Directrix: $x-2y-3 = 0$

Let $P(x,y)$ be any point on parabola, then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix.

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}$$

$$\Rightarrow \sqrt{x^2+6x+9+y^2-2y+1} = \frac{|x-2y-3|}{\sqrt{5}}$$

$$\Rightarrow \sqrt{5}\sqrt{x^2+6x+9+y^2-2y+1} = |x-2y-3|$$

On squaring both sides

$$5\left(x^2+6x+9+y^2-2y+1\right) = x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5x^2+30x+45+5y^2-10y+5-x^2-4y^2-9+4xy-12y+6x = 0$$

$$\Rightarrow 4x^2+y^2+36x-22y+4xy+41 = 0$$

is required equation

(iv) Given: Focus $(1,2)$, Vertex $(3,2)$

Focus and vertex implies that axis of parabola is parallel to x-axis and opening to left side. Therefore eq. of parabola with vertex $(3,2)$

$$(y-2)^2 = -4a(x-3) \;\ldots\ldots\ldots\; \text{(i)}$$

Now $a =$ Distance between focus and vertex

$$= \sqrt{(3-1)^2+(2-2)^2} = \sqrt{4+0} = 2$$

Putting in (i)

$$(y-2)^2 = -4(2)(x-3) \quad \Rightarrow \quad y^2-4y+4 = -8x+24$$

$$\Rightarrow \; y^2-4y+4+8x-20 = 0 \quad \Rightarrow \quad y^2-4y+8x-20 = 0 \quad \text{is req. eq.}$$

(v) *Do yourself*

(vii) Directrix: $y = 3$ i.e. $y-3 = 0$

Vertex $(2,2)$

Since axis of parabola is parallel to y-axis (because directrix is parallel to x-axis). And opening is downward.

So equation of parabola with vertex $(h,k) = (2,2)$

$$(x-h)^2 = -4a(y-k)$$
$$\Rightarrow (x-2)^2 = -4a(y-2)$$

Now a = Distance of vertex $(2,2)$ from directrix

$$= \frac{|2-3|}{\sqrt{(0)^2+(1)^2}} = \frac{|-1|}{1} = 1$$

Putting in (i)

$$(x-2)^2 = -4(1)(y-2)$$
$$\Rightarrow x^2-4x+4 = -4y+8 \qquad \Rightarrow x^2-4x+4+4y-8 = 0$$
$$\Rightarrow x^2-4x+4y-4 = 0 \text{ is require equation.}$$

(viii) Directrix: $y = 1$

Latusractum $= 4a = 8 \Rightarrow a = 2$

$\because$ Parabola is open downward

$\therefore$ Consider vertex $= (h,-1)$

And equation of parabola

$$(x-h)^2 = -4a(y-k)$$
$$\Rightarrow (x-h)^2 = -4(2)(y+1) \quad \Rightarrow x^2-2hx+h^2 = -8y-8$$
$$\Rightarrow x^2-2hx+8y+h^2+8 = 0 \text{ is req. eq.}$$

(ix) Axis of parabola: $y = 0$

Let vertex is (h,k)

$\because$ it lies on x-axis $\therefore$ $k=0$

Now equation of parabola with vertex $(h,0)$

$$(y-0)^2 = 4a(x-h)$$
$$\Rightarrow y^2 = 4a(x-h) \text{ .... (i)}$$

$\because$ $(2,1)$ lies on parabola (i)

$\therefore$ $(1)^2 = 4a(2-h)$

$\Rightarrow 1 = 4a(2-h)$ .... (ii)

Also $(11,-2)$ lies on parabola (i)

$$(-2)^2 = 4a(11-h)$$

$\Rightarrow 4 = 4a(11-h)$

$\Rightarrow 1 = a(11-h)$ .... (iii)

Dividing (i) & (ii)

$$\frac{1}{1} = \frac{4a(2-h)}{a(11-h)}$$

$$\Rightarrow\ 1 = \frac{4(2-h)}{(11-h)} \qquad \Rightarrow\ 11-h = 8-4h$$

$$\Rightarrow\ 4h-h = 8-11 \qquad \Rightarrow\ 3h = -3 \qquad \Rightarrow\ h = -1$$

Putting in (ii)

$$1 = 4a(2-(-1))$$

$$\Rightarrow\ 1 = 4a(3) \qquad \Rightarrow\ 1 = 12a \qquad \Rightarrow\ a = \frac{1}{12}$$

Using in (i)

$$y^2 = 4\left(\frac{1}{12}\right)(x-(-1)) \qquad \Rightarrow\ 3y^2 = x+1$$

$$\Rightarrow\ 3y^2 - x - 1 = 0$$

is the required equation.

(x) Axis parallel to y-axis, then points (0,3), (3,4) and (4,11) lie on the graph.

Equation of parabola axis parallel to y-axis with vertex (h,k)

$$(x-h)^2 = 4a(y-k) \ \ldots\ldots\ldots\ldots \text{(i)}$$

$\because\ (0,3)$ lies on parabola (i)

$$\therefore\ (0-h)^2 = 4a(3-k)$$

$$\Rightarrow\ h^2 = 12a - 4ak \ \ldots\ldots\ldots\ldots \text{(ii)}$$

Also $(3,4)$ lies on parabola (i)

$$(3-h)^2 = 4a(4-k)$$

$$\Rightarrow\ 9-6h+h^2 = 16a-4ak$$

$$\Rightarrow\ h^2-6h+9 = 16a-4ak \ \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Also $(4,11)$ lies on parabola (i)

$$(4-h)^2 = 4a(11-k)$$

$$\Rightarrow\ 16-8h+h^2 = 44a-4ak$$

$$\Rightarrow\ h^2-8h+16 = 44a-4ak \ \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Subtracting (ii) & (iii)

$$\begin{array}{l} h^2 \qquad\qquad = 12a-4ak \\ h^2 - 6h + 9 = +16a - 4ak \\ \ \ {}_{+}\quad\ \ {}_{-}\qquad\ {}_{-}\qquad {}_{+} \\ \hline \qquad 6h-9 = -4a \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (v) \end{array}$$

Now subtracting (iii) & (iv)

$$h^2-6h+9 = 16a-4ak$$
$$h^2-8h+16=+44a-4ak$$
$$\underline{\quad - \quad + \qquad + \quad -\quad}$$
$$2h-7=-28a \text{ ....}(vi)$$

Multiplying (vi) with 3 and subtracting from (v)

$$6h-21 = -84a$$
$$+6h-9 = -4a$$
$$\underline{- \quad + \qquad +}$$
$$-12=-80a$$

$$\Rightarrow \boxed{a = \frac{12}{80} = \frac{3}{20}}$$

Putting in (v)

$$6h-9 = -40\left(\frac{3}{20}\right) \Rightarrow 6h = 9-\frac{3}{5} = \frac{42}{5} \Rightarrow \boxed{h = \frac{7}{5}}$$

Putting value of a and h in (ii), we get

$$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right)-4\left(\frac{3}{20}\right)k \quad \Rightarrow \frac{49}{25} = \frac{9}{5}-\frac{3}{5}k$$

$$\Rightarrow \frac{3}{5}k = \frac{9}{5}-\frac{49}{25} \Rightarrow \frac{3}{5}k = -\frac{4}{25} \Rightarrow \boxed{k = -\frac{4}{15}}$$

Now putting the value of a, h and k in (i), we get

$$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right) \quad \Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}\left(y+\frac{4}{15}\right)$$

$$\Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}y+\frac{4}{25} \quad \Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{49}{25}-\frac{4}{25} = 0$$

$$\Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{9}{5} = 0$$

$$\Rightarrow 5x^2-14x-3y+9 = 0$$

is the required equation.

Question # 3

Find an equation of the parabola having its focus at the origin and dirctrix parallel to the
(i) $x-axis$ (ii) $y-axis$

Solution

(i) When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$$y = h \quad \text{(parallel to x-axis)}$$

i.e. $y-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$$|PF| = \perp\text{ar distance of } P(x,y) \text{ from directrix}$$

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|y-h|}{\sqrt{(0)^2+(1)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|y-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = y^2-2hy+h^2 \quad \Rightarrow x^2+y^2-y^2+2hy-h^2 = 0$$

$$\Rightarrow x^2+2hy-h^2 = 0 \quad \text{is req. equation.}$$

ii) When directrix is parallel to y-axis.

When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$$x = h \quad \text{(parallel to } y\text{-axis)}$$

i.e. $x-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|x-h|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|x-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = x^2-2hx+h^2 \quad \Rightarrow x^2+y^2-x^2+2hx-h^2 = 0$$

$$\Rightarrow y^2+2hx-h^2 = 0 \text{ is req. equation}$$

Question # 4

Show that an equation of parabola with focus at $(a\cos\alpha, a\sin\alpha)$ and directrix $x\cos\alpha+y\sin\alpha+a=0$ is $(x\sin\alpha-y\cos\alpha)^2 = 4a(x\cos\alpha+y\sin\alpha)$

Solution Focus: $F(a\cos\alpha, a\sin\alpha)$

Directrix: $x\cos\alpha+y\sin\alpha+a = 0$

Let $P(x,y)$ be any point on parabola then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2} = \frac{|x\cos\alpha+y\sin\alpha+a|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$$

On squaring

$$(x-a\cos\alpha)^2+(y-\sin\alpha)^2 = \frac{|x\cos\alpha+y\sin\alpha+a|^2}{1}$$

$$\Rightarrow x^2-2ax\cos\alpha+a^2\cos^2\alpha+y^2-2ay\sin\alpha+a^2\sin^2\alpha$$
$$= x^2\cos^2\alpha+y^2\sin^2\alpha+a^2+2ax\cos\alpha+2ay\sin\alpha+2xy\sin\alpha\cos\alpha$$

$$\Rightarrow\ x^2 - x^2\cos\alpha + y^2 - y^2\sin\alpha + a^2\left(\cos^2\alpha + \sin^2\alpha\right)$$
$$= a^2 + 2ax\cos\alpha + 2ay\sin\alpha + 2xy\sin\alpha\cos\alpha + 2ax\cos\alpha + 2ay\sin\alpha$$
$$\Rightarrow\ x^2\left(1-\cos^2\alpha\right) + y^2\left(1-\sin^2\alpha\right) + a^2(1) - a^2 - 2xy\sin\alpha\cos\alpha$$
$$= 4ax\cos\alpha + 4ay\sin\alpha$$
$$\Rightarrow\ x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha = 4a\left(x\cos\alpha + y\sin\alpha\right)$$
$$\Rightarrow\ \left(x\sin\alpha - y\cos\alpha\right)^2 = 4a\left(x\cos\alpha + y\sin\alpha\right)$$

is equation of parabola which is given.

Question # 5

Show that the ordinate at any point P of the parabola is a mean propositional between the length of the latusrectum and the abscissa of P .

Solution

Consider equation of parabola

$$y^2 = 4ax$$
$$\Rightarrow\ y\cdot y = 4a\cdot x$$
$$\Rightarrow\ \frac{4a}{y} = \frac{y}{x}$$
$$\Rightarrow\ \frac{\textit{latus ractum}}{\textit{ordinate}} = \frac{\textit{ordinate}}{\textit{abscissa}}$$

$\Rightarrow$ ordinate is mean proportional between latus rectum and abscissa.

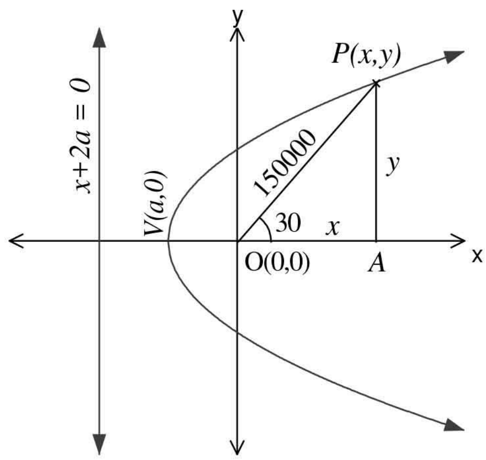
Question # 6

A comet has a parabolic orbit with the earth at the focus. When the comet is $150,000km$ from the earth, the line joining the comet and the earth makes an angle of 30 with the asix of the parabola. How close will the comet come to the earth

Solution Suppose earth be at focus which is origin and $V(-a,0)$ be vertex of parabola.

Then directrix of parabola;

$$x = -2a$$
$$\Rightarrow\ x + 2a = 0$$



Let comet be at a point $P(x,y)$ then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow\ \sqrt{(x-0)^2 + (y-0)^2} = \frac{|x+2a|}{\sqrt{(1)^2 + (0)^2}}$$
$$\Rightarrow\ \sqrt{x^2 + y^2} = |x+2a|$$

On squaring

$$x^2 + y^2 = (x+2a)^2 \ \ldots\ldots\ldots\ldots \text{(i)}$$

Also by Pythagoras theorem in $\triangle$ ABC

$$|OA|^2 + |AP|^2 = |OP|^2$$

$$\Rightarrow x^2 + y^2 = (150000)^2 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Comparing (i) and (ii)

$$(x+2a)^2 = (150000)^2$$

$$\Rightarrow x + 2a = \pm 150000 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Now from right triangle OAP

$$\cos 30^\circ = \frac{|OA|}{|OP|} \qquad \Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}(150000)$$

Using in (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

$$\Rightarrow 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000) \qquad \Rightarrow 2a = \pm 150000 - \sqrt{3}(75000)$$

$$\Rightarrow 2a = 75000\left(\pm 2 - \sqrt{3}\right) \qquad \Rightarrow a = 37500\left(\pm 2 - \sqrt{3}\right)$$

Since a is shortest distance and can't be negative

Therefore $\quad a = 37500\left(2 - \sqrt{3}\right) Km$

Question # 7

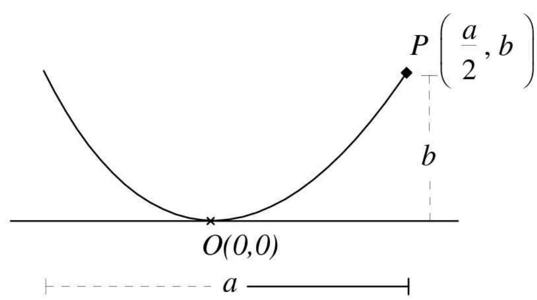
Find an equation of the parabola formed by the cables of a suspension bridge whose span is a m and the vertical height of the supporting towards is b m.

Solution Consider equation of parabola with vertex $O(0,0)$

$$x^2 = 4a'y \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Since $P\left(\frac{a}{2}, b\right)$ lies on parabola

$$\left(\frac{a}{2}\right)^2 = 4a'(b) \quad \Rightarrow a' = \frac{a^2}{16b}$$



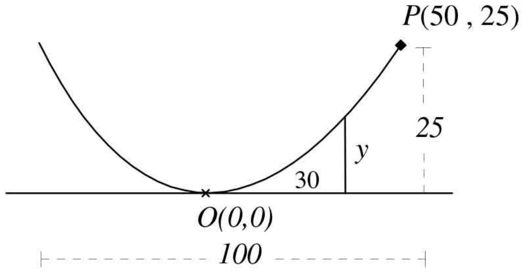
Putting in (i)

$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

$$\Rightarrow x^2 = \frac{a^2}{4b}y \quad \text{is required equation.}$$

Question # 8
A parabolic arch has a $100m$ base and height $25m$. Find the height of the arch at the point $30m$ from the centre of the base.

Solution Suppose equation of parabola with vertex $(0,0)$



$$x^2 = 4ay \quad \ldots\ldots\ldots\ldots\text{(i)}$$

From figure, we see that $P(50,25)$ lies on parabola.

$$(50)^2 = 4a(25)$$

$$\Rightarrow 2500 = 100a \qquad \Rightarrow a = 25$$

Putting in (i)

$$x^2 = 4(25)y \quad \Rightarrow x^2 = 100y$$

When $x = 30$

$$(30)^2 = 100y$$

$$\Rightarrow y = \frac{900}{100} \quad \Rightarrow y = 9$$

Hence the required height $=$ $9m$

Question # 9
Show that tangent at any point P of a parabola makes equal angles with the line PF and the line through P parallel to the axis of the parabola, F being focus. (These angles are called respectively angle of incidence and angle of reflection).

Solution

Suppose the parabola

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Let $P(x_1, y_1)$ be any point on parabola, then

$$y_1^2 = 4ax_1 \quad \ldots\ldots\ldots\text{(ii)}$$

Now differentiating (i) w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}4ax \quad \Rightarrow 2y\frac{dy}{dx} = 4a \quad \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent at $(x_1, y_1) = m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1}$

Now slope of PS $= m_2 = \frac{y_1 - 0}{x_1 - a}$

$$\Rightarrow m_2 = \frac{y_1}{x_1 - a}$$

Now slope of line parallel to axis of parabola $= m_3 = 0$
(because axis of parabola is along x-axis)

Let θ_1 be angle between tangent and line parallel to axis of parabola, then

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)} = \frac{2a/y_1}{1}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{2a}{y_1} \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Let θ_2 be angle between tangent and PS, then

$$\tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)} = \frac{\frac{y_1^2 - 2a(x_1 - a)}{y_1(x_1 - a)}}{\frac{x_1 - a + 2a}{x_1 - a}} = \frac{y_1^2 - 2ax_1 + 2a^2}{y_1(x_1 + a)}$$

$$= \frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 + a)} \qquad \text{from (ii)}$$

$$= \frac{2ax_1 + 2a^2}{y_1(x_1 + a)} = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$$

$$\Rightarrow \quad \tan\theta_2 = \frac{2a}{y_1} \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

From (iii) and (iv)

$$\tan\theta_1 = \tan\theta_2 \qquad \Rightarrow \quad \theta_1 = \theta_2$$

as required

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$

Special case of an Ellipse

Circle is a special case of an Ellipse. In circle ''e'' = 0

Parametric Equations of an Ellipse

$x = a\cos\theta$, $y = b\sin\theta$ are Parametric Equations of Ellipse.

Important points about an Ellipse

$\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1}$

(1) Eccentricity

$$e^2 = \frac{a^2-b^2}{a^2}$$
$$(ae)^2 = a^2-b^2$$
$$c^2 = a^2-b^2$$

(2) Foci $(\pm ae, 0)$ or $(\pm c, 0)$

(3) Length of major axis = 2a

(4) Length of minor axis = 2b

(5) Equations of directrix x : $x = \pm\frac{a}{e}$

(6) Length of latus rectum $= \frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(\pm a, 0)$

(9) Covertices $(0, \pm b)$

Note:

If center is other than (0, 0) then equations of ellipse be comes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1$$

$\mathbf{\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1}$

(1) Eccentricity

$$e^2 = \frac{a^2-b^2}{a^2}$$
$$(ae)^2 = a^2-b^2$$
$$c^2 = a^2-b^2 \text{ where } c = ae$$

(2) Foci $(0, \pm ae)$ or $(0, \pm c)$

(3) Length of major axis = 2a

(4) Length of minor axis = 2b

(5) Equations of directrix : $y = \pm\frac{a}{e}$

(6) Length of latus rectum $= \frac{2b^2}{a}$

(7) Center (0, 0)

(8) Vertices $(0, \pm a)$

(9) Covertices $(\pm b, 0)$

& $$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2} = 1$$

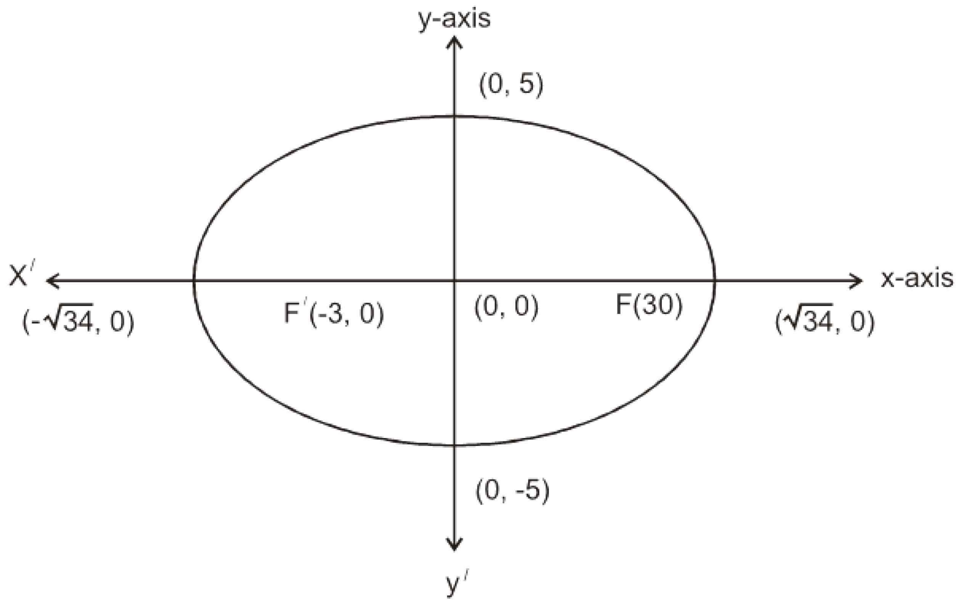
EXERCISE 6.5

Q.1: Find an equation of the Ellipse with given data and sketch its graph.

(i) Foci (±3, 0) and minor axis of length 10.
(Lahore Board 2009)

Solution:

Given $(\pm ae, 0) = (\pm 3, 0)$



$ae = 3$

$\Rightarrow \quad c = 3$ $\qquad$ Also $2b = 10 \Rightarrow \boxed{b = 5}$

We know that $c^2 = a^2 - b^2$

$$(3)^2 = a^2 - (5)^2$$

$$9 = a^2 - 25$$

$$9 + 25 = a^2$$

$$a^2 = 34 \Rightarrow \boxed{a = \pm\sqrt{34}}$$

Required equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{34} + \frac{y^2}{25} = 1$$

Here vertices are

$= (\pm a, 0)$

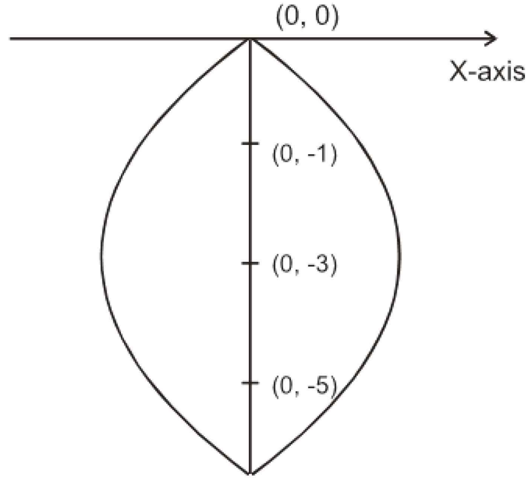
$= (\pm\sqrt{34}, 0)$

Covertices are $(0, \pm b)$

$= (0, \pm 5)$

(ii) Foci (0, – 1) & (0, – 5) and major axis of length 6.

Solution:



We know that center $=$ mid point of foci

$$= \left(\frac{0+0}{2}, \frac{-1-5}{2}\right)$$

$$= (0, -3)$$

Also we know that

C = distance between centre and focus

$$C = \sqrt{(0-0)^2 + (-3+1)^2}$$

$$C = \sqrt{4} = 2$$

Given $2a = 6$

$a = 3$

We know for Ellipse

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad (2)^2 = (3)^2 - b^2$$

$$4 - 9 = -b^2 \quad \Rightarrow \quad b^2 = 5$$

With center (0, – 3), required equation of the ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

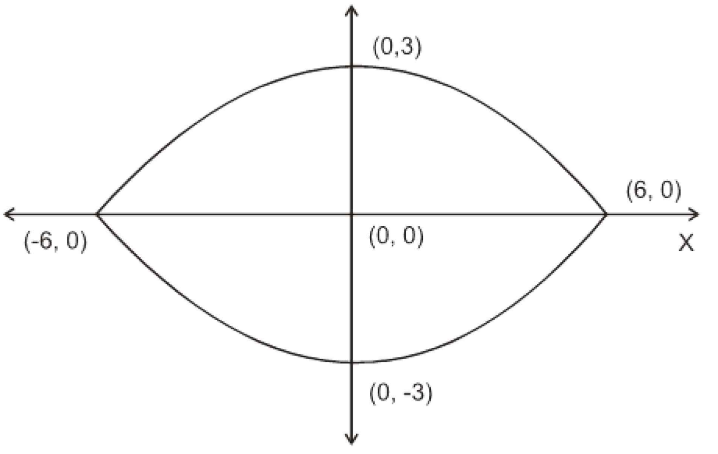
$$\frac{(x-0)^2}{5} + \frac{(y+3)^2}{(3)^2} = 1$$

$$\frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

(iii) **Foci $(-3\sqrt{3}, 0)$ & Vertices $(\pm 6, 0)$**

(Lahore Board 2009)

Solution:



$(\pm ae, 0) = (\pm 3\sqrt{3}, 0)$ & $(\pm a, 0) = (\pm 6, 0)$

$ae = 3\sqrt{3}$ $\quad a = 6$

$c = 3\sqrt{3}$ $\quad a = 6$

We know that $c^2 = a^2 - b^2$

$(3\sqrt{3})^2 = (6)^2 - b^2$

$27 = 36 - b^2 \quad \Rightarrow \quad b^2 = 9$

Required equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

Covertices are $= (0, \pm b)$

$= (0, \pm 3)$

(iv) **Vertices $(-1, 1), (5, 1)$ foci $(4, 1)$ & $(0, 1)$**

Solution:

$2a = |VV'|$ $\qquad 2c = |FF'|$

$2a = \sqrt{(5+1)^2 + (1-1)^2}$ $\qquad 2c = \sqrt{(0+4)^2 + (1-1)^2}$

$2a = \sqrt{36}$ $\qquad 2c = \sqrt{16}$

$2a = 6$ $\qquad 2c = 4$

$\boxed{a = 3}$ $\qquad \boxed{c = 2}$

We know that $c^2 = a^2 - b^2$

$(2)^2 = (3)^2 - b^2$

$$4 = 9 - b^2$$

$$\boxed{b^2 = 5}$$

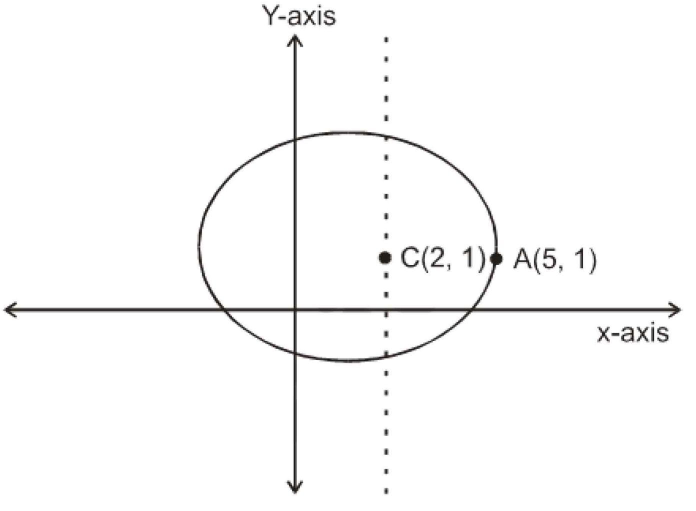
Now center of ellipse $= (h, k) =$ mid point of foci

$$= \left(\frac{4+0}{2}, \frac{1+1}{2}\right)$$

with this center required equation of ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

i.e; $$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$



(v) Foci $(\pm\sqrt{5}, 0)$ & passing through $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution:

Given $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

$c = \sqrt{5}$ we know that $c^2 = a^2 - b^2$

$(\sqrt{5})^2 = a^2 - b^2$

$5 = a^2 - b^2$

$\Rightarrow a^2 = 5 + b^2$ .... (1)

Since Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and it is passing through point $\left(\frac{3}{2}, \sqrt{3}\right)$ therefore it becomes $\frac{9}{4a^2} + \frac{3}{b^2} = 1$ ......(2)

from (1) we have $a^2 = 5 + b^2$ Put in (2)

$$\frac{9}{4(5+b^2)} + \frac{3}{b^2} = 1$$

$$\frac{9b^2 + 12(5+b^2)}{4b^2\,(5+b^2)} = 1$$

$$9b^2 + 60 + 12b^2 = 20b^2 + 4b^4$$

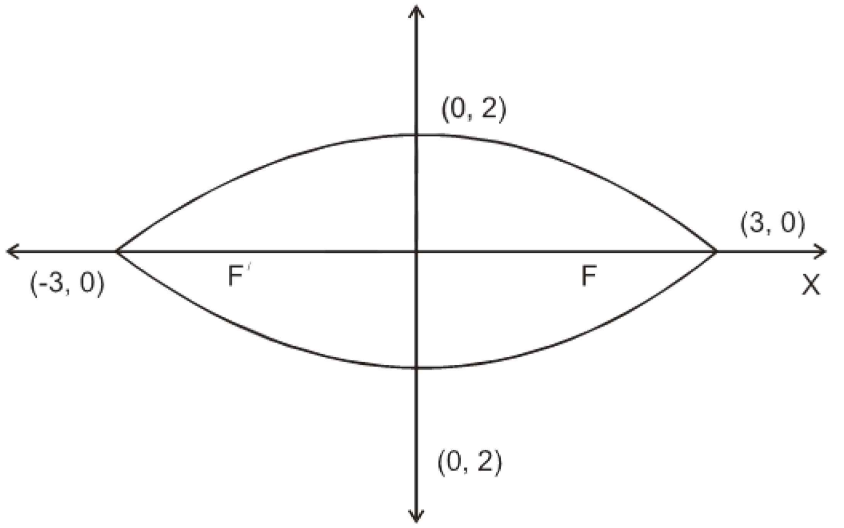
$$4b^4 - b^2 - 60 = 0$$

$$b^2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-60)}}{2(4)}$$

$$b^2 = \frac{1 \pm \sqrt{961}}{8}$$

$$= \frac{1 \pm 31}{8}$$

$$b^2 = \frac{1+31}{8}, \qquad \frac{1-31}{8}$$



$$b^2 = \frac{32}{8}, \qquad b^2 = \frac{-30}{8} \quad \text{(solution not possible)}$$

$$b^2 = 4$$

Required equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e; $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Vertices $(\pm a, 0) = (\pm 3, 0)$

Co-vertices $(0, \pm b) = (0, \pm 2)$

(vi) Vertices (0, ± 5), eccentricity = $\frac{3}{5}$

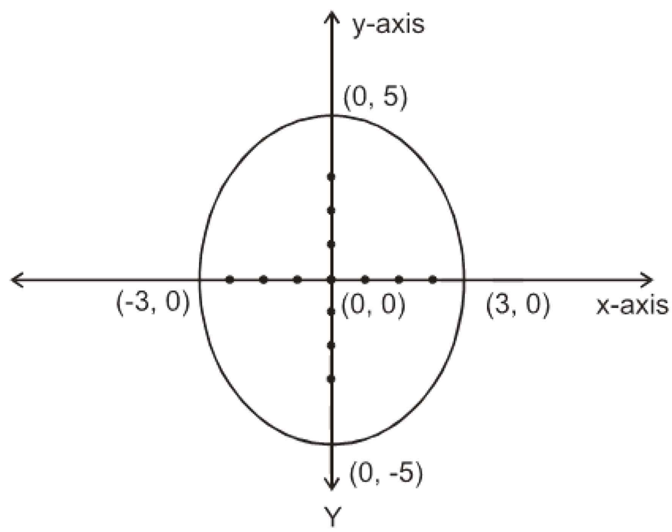
Solution:

$$\text{Vertices } (0, \pm a) = (0, \pm 5)$$

$$a = 5 \qquad e = \frac{3}{5}$$

$$ae = 5 \times \frac{3}{5}$$

$$\Rightarrow \quad c = 3$$



We know that

$$c^2 = a^2 - b^2$$

$$(3)^2 = (5)^2 - b^2$$

$$9 = 25 - b^2$$

$$b^2 = 16$$

Required equation of Ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Co-vertices

$$(\pm b, 0) = (\pm 4, 0)$$

$$\text{Foci} = (0, \pm c) = (0, \pm 3)$$

(vii) Centre (0, 0) focus (0, – 3), vertex (0, 4)
(Lahore Board 2011)

Solution:

$(0, -c) = (0, -3)$ $\quad a = 4$

$c = 3$

We know that $e^2 = \dfrac{a^2 - b^2}{a^2}$

$$(ae)^2 = a^2 - b^2$$
$$c^2 = a^2 - b^2$$
$$(3)^2 = 16 - b^2$$
$$9 = 16 - b^2$$
$$b^2 = 16 - 9 \quad \Rightarrow \quad b^2 = 7$$

Required equation of ellipse is $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{16} + \frac{x^2}{7} = 1$$

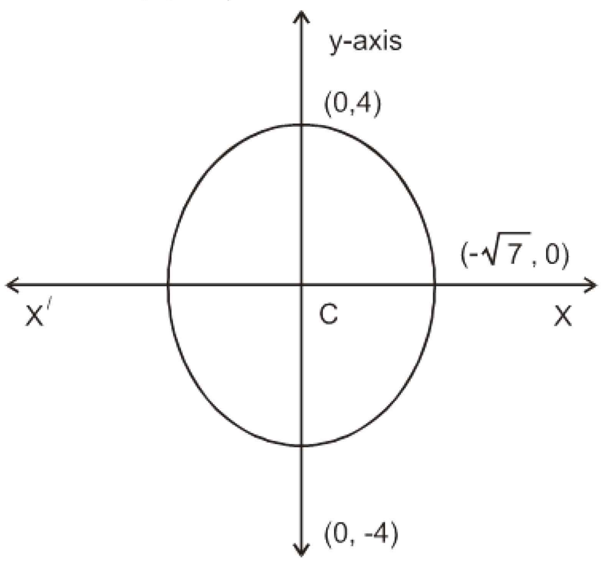
Its vertices are $(0, \pm a) = (0, \pm 4)$

Covertices $(\pm b, 0) = (\pm\sqrt{7}, 0)$

Coordinates of foci $= (0, \pm ae)$

$= (0, \pm 4\,\dfrac{3}{4})$

$= (0, \pm 3)$



(viii) Centre (2, 2) major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis and of length 6 units.

Solution:

Center (2, 2) also $2a = 8 \Rightarrow a = 4$

$2b = 6 \Rightarrow b = 3$

Equation of ellipse is $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$

$$\Rightarrow \frac{(y-2)^2}{16} + \frac{(x-2)^2}{9} = 1$$

Its vertices are $(0, \pm a) = (0, \pm 2)$

and $e^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{16-9}{16} = \dfrac{7}{16} \Rightarrow e = \dfrac{\sqrt{7}}{4}$

Coordinates of foci $= (\pm ae, 0) = (\pm \dfrac{4\sqrt{7}}{4}, 0) = (\pm\sqrt{7}, 0)$

$(y-2,\ x-2) = (\pm\sqrt{7}, 0)$

$y = 2 \pm \sqrt{7}$, $x = 2$

Thus foci are $(2, 2+\sqrt{7})$ & $(2, 2-\sqrt{7})$

For vertices, we have $x - 2 = 0$ $y - 2 = \pm a$

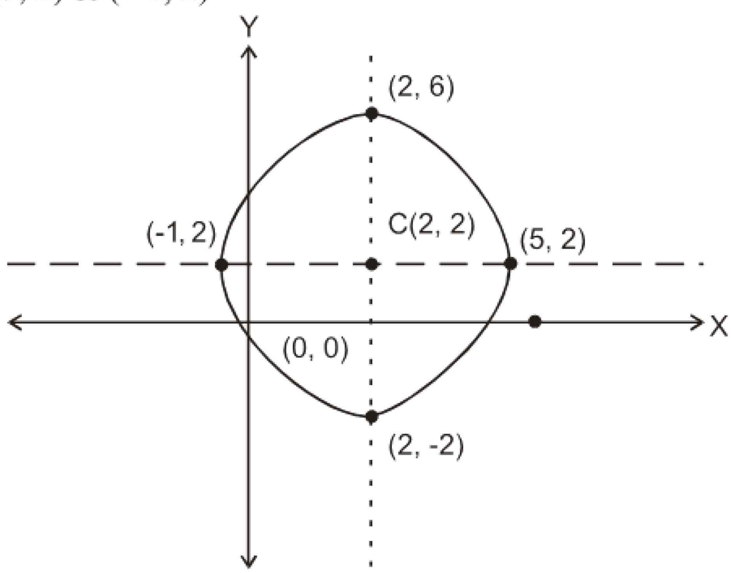
$x = 2$ $y = 2 \pm 4 \Rightarrow 6, -2$

So vertices are $(2, 6)$, $(2, -2)$

Next we have $x - 2 = \pm b$, $y - 2 = 0$

$x = 2 \pm 3$, $y = 2$

So covertices are $(5, 2)$ & $(-1, 2)$



(ix) Center (0, 0) symmetric with respect to both the axes and passing through the points (2, 3) and (6, 1).

Solution:

We know that equation of ellipse with center (0, 0) is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes through the points (2, 3) & (6, 1)

$$\frac{4^2}{a^2} + \frac{9^2}{b^2} = 1 \qquad \text{(I)} \qquad \frac{36}{a^2} + \frac{1}{b^2} = 1 \qquad \text{(II)}$$

Subtracting

$$\begin{array}{rcl} 4b^2 + 9a^2 & = & a^2b^2 \\ -36b^2 \pm a^2 & = & -a^2b^2 \\ \hline -32b^2 + 8a^2 & = & 0 \end{array}$$

$$8a^2 = 32b^2$$

$$a^2 = 4b^2 \qquad \text{Put in (I)}$$

$$\frac{4}{4b^2} + \frac{9}{b^2} = 1$$

$$\frac{1+9}{b^2} = 1$$

$$\boxed{10 = b^2} \qquad \boxed{a^2 = 40}$$

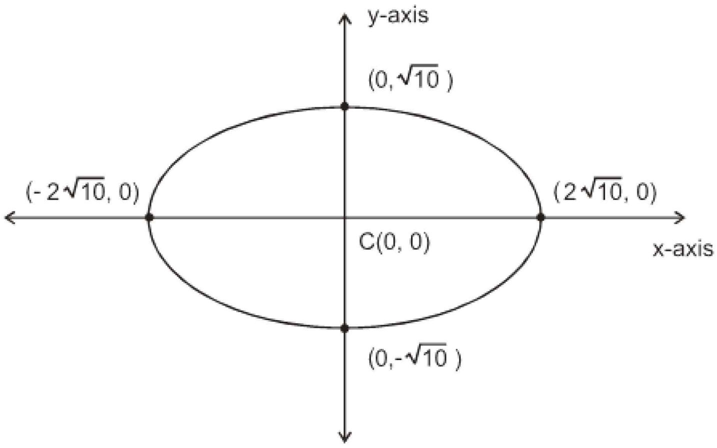
Required equation of ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1$$

Vertices are $(\pm a, 0) = (\pm\sqrt{40}, 0) = (\pm 2\sqrt{10}, 0)$

Covertices $(0, \pm b) = (0, \pm\sqrt{10})$

Foci $(\pm\sqrt{30}, 0)$



(x) Center (0, 0) major axis horizontal, the points (3, 1) (4, 0) lie on the graph.

Solution:

We know that equation of Ellipse with center (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes though the points (3, 1) & (4, 0)

For (3, 1)

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad (1)$$

For (4, 0)

$$\frac{16}{a^2} + \frac{0}{b^2} = 1 \qquad (2)$$

$\boxed{a^2 = 16}$ Put in (1)

$$\frac{9}{16} + \frac{1}{b^2} = 1$$

$$\frac{1}{b^2} = 1 - \frac{9}{16} = \frac{7}{16} \qquad \boxed{b^2 = \frac{16}{7}}$$

Required equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

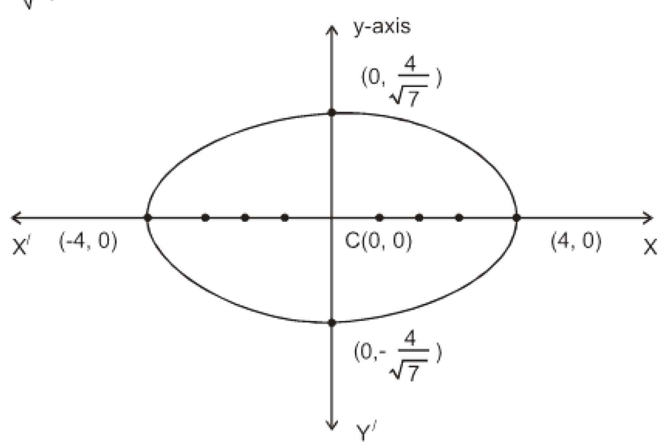
$$\frac{x^2}{16} + \frac{y^2}{16} = 1$$

$$\frac{x^2}{16} + \frac{7y^2}{16} = 1$$

Vertices $(\pm a, 0) = (\pm 4, 0)$

Covertices $(0, \pm b) = (0, \pm\frac{4}{\sqrt{7}})$

Foci $(\pm 4\sqrt{\frac{6}{7}}, 0)$



Q.2: Find the center, foci, eccentricity, vertices and directrix of the ellipse whose equation is given.

(i) $x^2 + 4y^2 = 16$ (Lahore Board 2009 (Supply))

Solution:

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Here $a^2 = 16$ $\quad b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Foci are $= (\pm c, 0) = (\pm 2\sqrt{3}, 0)$

Vertices are $= (\pm a, 0) = (\pm 4, 0)$

Directrix are $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{4}{\frac{\sqrt{3}}{2}} = \pm \frac{8}{\sqrt{3}}$

Clearly center of ellipse is (0, 0)

(ii) $9x^2 + y^2 = 18$

Solution:

$$9x^2 + y^2 = 18$$

$$\frac{x^2}{2} + \frac{y^2}{18} = 1$$

$a^2 = 18$, $b^2 = 2$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{18-2}{18} = \frac{16}{18} = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

foci are $= (0, \pm c) = (0, \pm 4)$

vertices are $= (\pm a, 0) = (\pm 3\sqrt{2}, 0)$

directrix are $= y = \pm \frac{a}{e} = \pm \frac{3\sqrt{2}}{\frac{2\sqrt{2}}{3}} = \pm \frac{9}{2}$

Clearly center is (0, 0)

(iii) $25x^2 + 9y^2 = 225$

Solution:

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \text{(I)}$$

$$a^2 = 25 \quad \& \quad b^2 = 9$$

Eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

foci $= (0, \pm c) = (0, \pm 4)$

vertices $= (0, \pm a) = (0, \pm 5)$

Center $= (0, 0)$

Directrix $y = \pm\frac{a}{e} = \pm\frac{5}{\frac{4}{5}} = \pm\frac{25}{4}$

(iv) $\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

Solution:

Let $2x - 1 = X$, $y + 2 = Y$

Given equation becomes

$$\frac{X^2}{4} + \frac{Y^2}{16} = 1 \qquad \text{Here } a^2 = 16 \;,\; b^2 = 4$$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16 - 4}{16} = \frac{12}{16} = \frac{3}{4}$$

Eccentricity

$$e = \frac{\sqrt{3}}{2}$$

Center:- For center put $X = 0$, $Y = 0$

$2x - 1 = 0$, $y + 2 = 0$

$x = \frac{1}{2}$, $y = -2$

Required center is $(\frac{1}{2}, -2)$

Foci $(0, \pm ae) = \left(0, \pm 4\left(\frac{\sqrt{3}}{2}\right)\right)$

$$(X, Y) = (0, \pm 2\sqrt{3})$$

$$(2x - 1, y + 2) = (0, \pm 2\sqrt{3})$$

$$2x - 1 = 0 \qquad y + 2 = \pm 2\sqrt{3}$$

$$x = \frac{1}{2} \qquad y = -2 \pm 2\sqrt{3}$$

Required foci are $\left(\frac{1}{2}, -2 \pm 2\sqrt{3}\right)$

Vertices $(0, \pm a) = (0, \pm 4)$

$$(X, Y) = (0, \pm 4)$$

$$(2x - 1, y + 2) = (0, \pm 4)$$

$$2x - 1 = 0 \qquad y + 2 = \pm 4$$

$$x = \frac{1}{2} \qquad y = -2 \pm 4$$

$$y = -6, \quad 2$$

Vertices are $\left(\frac{1}{2}, \ -6\right), \ \left(\frac{1}{2}, \ 2\right)$

Directrices $Y = \pm \frac{a}{e}$

$$y + 2 = \pm \frac{4}{\frac{\sqrt{3}}{2}}$$

$$y + 2 = \pm \frac{8}{\sqrt{3}} \quad \Rightarrow \quad y = -2 \pm \frac{8}{\sqrt{3}}$$

(v) $\mathbf{x^2 + 16x + 4y^2 - 16y + 76 = 0}$

Solution:

$$x^2 + 16x + 4(y^2 - 4y) = -76$$

$$(x^2 + 16x + 64) + 4(y^2 - 4y + 4) = -76 + 64 + 16$$

$$(x + 8)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x + 8)^2}{4} + \frac{4(y - 2)^2}{4} = \frac{4}{4}$$

$$\frac{(x + 8)^2}{4} + (y - 2)^2 = 1$$

Let

$$x + 8 = X, \qquad y - 2 = Y$$

$$\frac{X^2}{4} + \frac{Y^2}{1} = 1 \quad \text{(an ellipse)}$$

$a^2 = 4 \quad , \quad b^2 = 1 \qquad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4-1}{4} = \frac{3}{4}$

$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$

For the center Put $X = 0, \quad Y = 0$

$x + 8 = 0 \quad , \quad y - 2 = 0$

$x = -8 \quad , \quad y = 2$

Required centre $(-8, 2)$

Foci $= (\pm ae, 0)$

$(X, Y) = (\pm 2\left(\frac{\sqrt{3}}{2}\right), 0)$

$(x + 8, y - 2) = (\pm\sqrt{3}, 0)$

$x + 8 = \pm\sqrt{3} \qquad y - 2 = 0$

$x = -8 \pm \sqrt{3} \qquad y = 2$

Required foci are $(-8 \pm \sqrt{3}, 2)$

Vertices are $= (\pm a, 0)$

$(X, Y) = (\pm a, 0)$

$(x + 8, y - 2) = (\pm 2, 0)$

$x + 8 = \pm 2 \quad , \quad y - 2 = 0$

$x = -8 \pm 2 \qquad y = 2$

Required vertices are $(-6, 2)$ & $(-10, 2)$

Directrix $X = \pm\frac{a}{e}$

$$x + 8 = \pm\frac{2}{\frac{\sqrt{3}}{2}} = \pm\frac{4}{\sqrt{3}}$$

$$x = -8 \pm \frac{4}{\sqrt{3}}$$

(vi) $\mathbf{25x^2 + 4y^2 - 250x - 16y + 541 = 0}$

Solution:

$25x^2 - 250x + 4y^2 + 16y = -541$

$25(x^2 - 10x) + 4(y^2 - 4y) = -541$

$25(x^2 - 10x + 25) + 4(y^2 - 4y + 4) = -541 + 625 + 16$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

Dividing both sides by 100

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

Let

$x - 5 = X \quad , \quad y - 2 = Y$ above equation

Becomes

$$\frac{X^2}{4} + \frac{Y^2}{25} = 1 \qquad \text{(an Ellipse)}$$

$a^2 = 25 \quad , \quad b^2 = 4$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25-4}{25} = \frac{21}{25}$$

$$e = \frac{\sqrt{21}}{5}$$

For Center Put $X = 0$, $Y = 0$

$(x-5, y-2) = (0, 0)$

$x - 5 = 0 \quad , \quad y - 2 = 0$

$x = 5 \quad\quad y = 2$

Center is $(5, 2)$

Foci $= (0, \pm ae) = (0, \pm 5\frac{\sqrt{21}}{5})$

$(X, Y) = (0, \pm\sqrt{21})$

$(x-5, y-2) = (0, \pm\sqrt{21})$

$x - 5 = 0 \quad , \quad y - 2 = \pm\sqrt{21}$

$x = 5 \quad , \quad y = 2 \pm \sqrt{21}$

Foci are $(5, 2 \pm \sqrt{21})$

Vertices $(0, \pm a) = (0, \pm 5)$

$(X, Y) = (0, \pm 5)$

$(x-5, y-2) = (0, \pm 5)$

$x - 5 = 0 \quad , \quad y - 2 = \pm 5$

$x = 5 \quad , \quad y = 2 \pm 5$

$y = 7, -3$

Vertices are $(5, 7)$ & $(5, -3)$

Directrix are

$$Y = \pm \frac{a}{e}$$

$$y - 2 = \pm \frac{5}{\frac{\sqrt{21}}{5}}$$

$$y = 2 \pm \frac{25}{\sqrt{21}}$$

Q.3: Let a be a + ve number and 0 < c < a. Let F(– c, 0) & F′(c > 0) be two given points. Prove that the locus of points p(x, y) such that |PF| + |PF′| = 2a is an ellipse.

Solution:

Given P(x, y) , F(– c, 0) , F′(c , 0)

$$\therefore \quad |PF| + |PF| = 2a$$

$$\sqrt{(x + c)^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}$$

Squaring on both sides

$$[\sqrt{(x + c)^2 + y^2}\ ]^2 = [2a - \sqrt{(x - c)^2 + y^2}\ ]^2$$

$$(x + c)^2 + y^2 = 4a^2 + (x - c)^2 + y^2 - 4a\sqrt{(x + c)^2 + y^2}$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x - c)^2 + y^2}$$

$$4cx - 4a^2 = -4a\sqrt{(x - c)^2 + y^2}$$

(Dividing by 4)

$$(cx - a^2) = -a\sqrt{(x - c)^2 + y^2}$$

Again Squaring

$$(cx - a^2)^2 = [-a\sqrt{(x - c)^2 + y^2}\ ]^2$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2x^2 + a^2c^2 - 2cxa^2 + a^2y^2$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

Dividing throughout by $c^2 - a^2$

$$x^2 - \frac{a^2y^2}{c^2 - a^2} = a^2 \qquad (1)$$

We know that for ellipse $c^2 = a^2 - b^2 \quad \Rightarrow \quad c^2 - a^2 = -b^2$ put in (1)

$$x^2 - \frac{a^2y^2}{-b^2} = a^2$$

$$x^2 + \frac{a^2y^2}{b^2} = a^2$$

(Dividing throughout by a^2)

$$\frac{x^2}{a^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2}{a^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{which is an ellipse}$$

Q.4: Use problem 3 to find equation of the ellipse as locus of points P(x, y) such that the sum of the distances from P to the points (0, 0) & (1, 1) is 2.

Solution:

Given P(x, y), F(0, 0) & F′(1, 1) Also given that 2a = 2

For ellipse we know that

$$|PF| + |PF'| = 2a$$

$$\sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{(x-1)^2 + (y-1)^2} = 2 - \sqrt{x^2 + y^2} \quad \text{Squaring}$$

$$(x-1)^2 + (y-1)^2 = 4 + (x^2 + y^2) - 4\sqrt{x^2 + y^2}$$

$$x^2 + 1 - 2x + y^2 + 1 - 2y = 4 + x^2 + y^2 - 4\sqrt{x^2 + y^2}$$

$$-2x - 2y - 2 = -4\sqrt{x^2 + y^2}$$

$$x + y + 1 = 2\sqrt{x^2 + y^2} \qquad \text{(Dividing both sides by 2)}$$

Again squaring

$$(x + y + 1)^2 = (2\sqrt{x^2 + y^2})^2$$

$$x^2 + y^2 + 1 + 2xy + 2y + 2x = 4(x^2 + y^2)$$

$$4x^2 + 4y^2 - x^2 - y^2 - 1 - 2xy - 2y - 2x = 0$$

$$3x^2 + 3y^2 - 2xy - 2x - 2y - 1 = 0 \qquad \text{required ellipse}$$

Q.5: Prove that latusrectim of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Solution:

The given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

From the figure, the points A(c, h) & B(c, – h) lies on ellipse (1) therefore.

For A(c, h) equation (1) becomes

$$\frac{c^2}{a^2} + \frac{h^2}{b^2} = 1$$

$$\frac{h^2}{b^2} = 1 - \frac{c^2}{a^2}$$

$$\frac{h^2}{b^2} = \frac{a^2 - c^2}{a^2} \qquad (2)$$

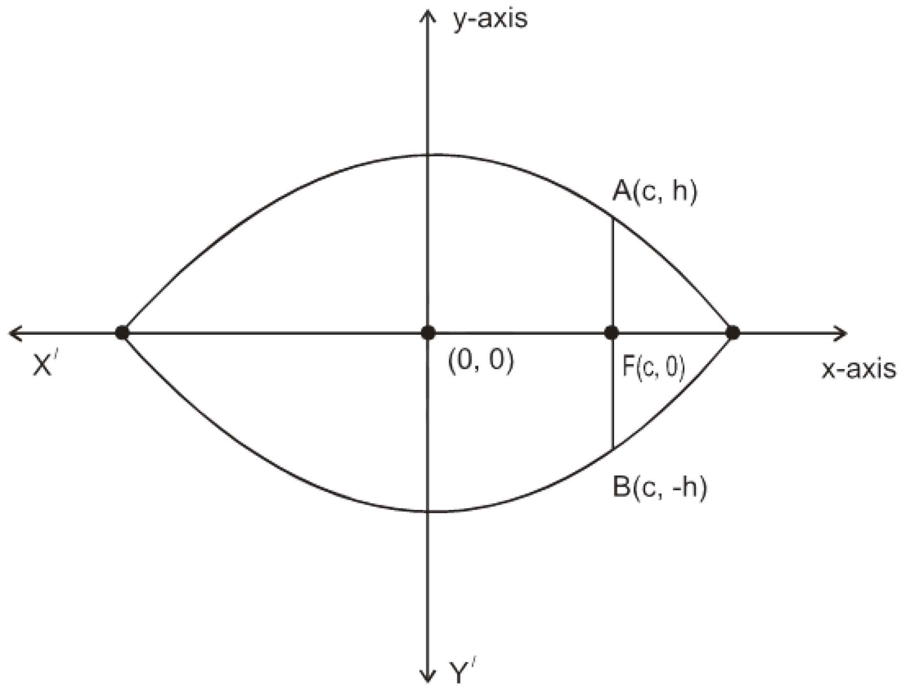
For ellipse, we know that

$$c^2 = a^2 - b^2$$

$$\Rightarrow \quad b^2 = a^2 - c^2 \qquad \text{Put in (2)}$$

$$\frac{h^2}{b^2} = \frac{b^2}{a^2}$$

$$h^2 = \frac{b^4}{a^2} \quad \Rightarrow \quad h = \pm\frac{b^2}{a}$$



Points A & B becomes

$$A(c, \frac{b^2}{a}) \text{ and } B(c, -\frac{b^2}{a})$$

Length of Latus rectum AB $= \sqrt{(c-c)^2 + \left(\frac{-b^2}{a} - \frac{b^2}{a}\right)^2}$

$$= \sqrt{\left(\frac{-2b^2}{a}\right)^2} = \sqrt{\frac{4b^4}{a^2}}$$

$$= \frac{2b^2}{a} \quad \text{Hence proved.}$$

Q.6: The major axis of an ellipse in standard form lies along the x-axis and has length $4\sqrt{2}$. The distance between the foci equals the length of minor axis. Write an equation of the ellipse.

Solution:

Since the major axis of the ellipse lies along x–axis so its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad .... \quad (1)$$

By the given condition $\quad 2a = 4\sqrt{2}$

$\Rightarrow \quad a = 2\sqrt{2}$

We know that

Distance between foci = length of minor axis

$2c = 2b$

$\Rightarrow \quad c = b$

Since for ellipse $\quad c^2 = a^2 - b^2$

$b^2 = a^2 - b^2$

$$2b^2 = a^2 \quad \Rightarrow \quad b^2 = \frac{a^2}{2}$$

$$\Rightarrow \quad b^2 = \frac{(2\sqrt{2})^2}{2} = \frac{8}{2} = 4$$

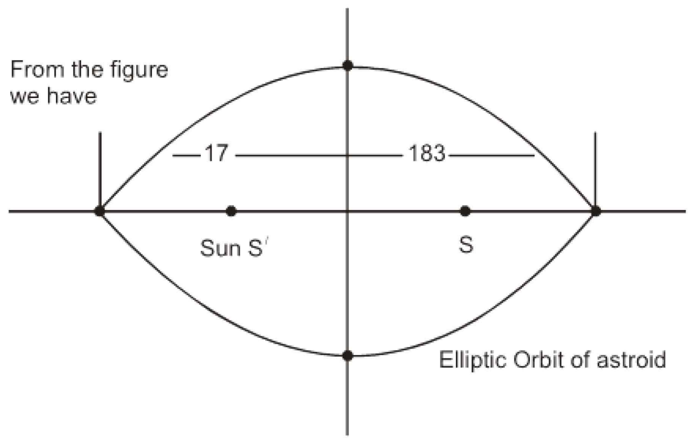
Putting values of a & b in (1)

$$\frac{x^2}{8} + \frac{y^2}{4} = 1 \qquad \text{Required equation of ellipse}$$

Q.7: An astroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges from 17 million miles to 183 million miles. Write on equation of the orbit of the astroid.

Solution: (Lahore Board 2004)

From the figure we have



Greatest distance of astroid from the sun is $a + c = 183$ (1)

And least distance of astroid from the sun is

$a - c = 17$ (2)

Adding (1) & (2)

$$\begin{aligned} a + c &= 183 \\ a - c &= 17 \\ \hline 2a &= 200 \\ a &= 100 \quad \text{Put in (1)} \\ 100 + c &= 183 \\ c &= 83 \end{aligned}$$

For ellipse, we know that

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad b^2 = a^2 - c^2$$

$$\Rightarrow \quad b^2 = (100)^2 - (83)^2$$

$$b^2 = 3111$$

Required equation of the orbit of astroid is

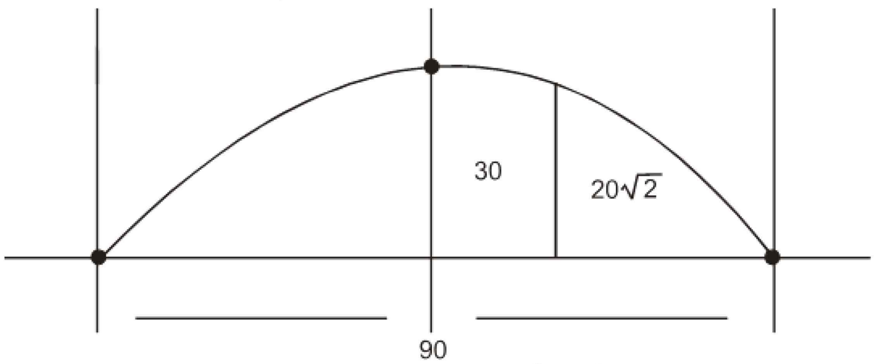
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(100)^2} + \frac{y^2}{3111} = 1$$

$$\frac{x^2}{10000} + \frac{y^2}{3111} = 1 \qquad \text{Ans.}$$

Q.8: An arch in the shape of semi ellipse is 90m wide at the base and 30 m high at the center. At what distance from the center is the arch $20\sqrt{2}$ m high?

Solution: (Lahore Board 2004)



Let the equation of the semi-elliptic are be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .... (1)

Since the arch is 90m wide at the base

so $2a = 90 \quad \Rightarrow \quad a = 45$

Since arch is 30 m high at the center, so

$b = 30$, putting a & b in (1)

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \quad ....(2)$$

Let d be the required distance from the center when arch is $20\sqrt{2}$ m high.

Putting $x = d$, $y = 20\sqrt{2}$ in (2)

$$\frac{d^2}{(45)^2} + \frac{(20\sqrt{2})^2}{(30)^2} = 1$$

$$\frac{d^2}{2025} + \frac{800}{900} = 1$$

$$\frac{d^2}{2025} = 1 - \frac{800}{900}$$

$$d^2 = \frac{100}{900} \times 2025$$

$$d^2 = \frac{(45)^2}{(3)^2}$$

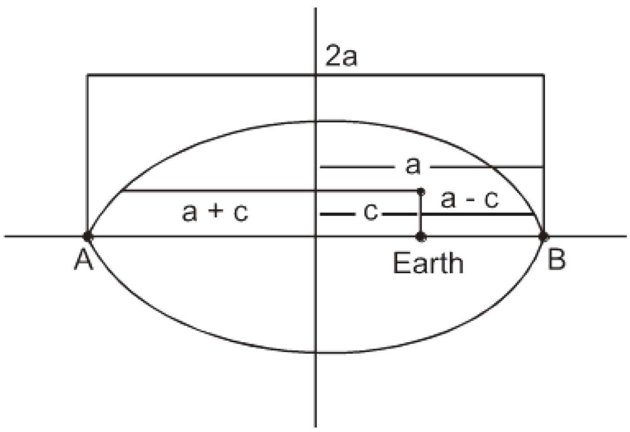
$$d = \frac{45}{3} = 15 \text{ meter}$$

Q.9: The moon orbits the earth in an elliptic path with earth at one focus. The major and minor axes of the orbit are 768, 806 km & 767, 746 km respectively. Find the greatest and least distances in astronomy called apogee & perigee) of the moon from the earth.

Solution:

By the given conditions

$2a = 768806 \quad \Rightarrow \quad a = 384403$



$2b = 767746 \Rightarrow b = 383874$

Since $\sqrt{c^2} = \sqrt{a^2 - b^2}$

$c = \sqrt{(384403)^2 - (383874)^2}$

$c = 20179$ km

The greatest distance

$a + c = 384403 + 20179$

$= 404582$ km

The least distance

$a - c = 384403 - 20179$

$= 364224$ km

Hyperbola

Let $e > 1$ and F be a fixed point and L be line not containing F. Also let P(x, y) be any point in the plane and |PM| be the perpendicular distance of P from L. The set of all the points P(x, y) such that $\frac{|PF|}{|PM|} = e > 1$ is called hyperbola.

Key points about Hyperbola

(1) Equation (Standard form)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2 \Rightarrow c^2 = a^2 + b^2$

(3) Foci $(\pm ae, 0)$

(4) Directrix $x = \pm \frac{a}{e}$

(5) Center (0, 0)

(6) Vertices $(\pm a, 0)$

(7) Covertices $(0, \pm b)$

(1) (Lahore Board 2009)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(2) Eccentricity $e^2 = \frac{a^2+b^2}{a^2}$

$(ae)^2 = a^2 + b^2$

$c^2 = a^2 + b^2$

(3) Foci $(0, \pm ae)$

(4) Directrix $y = \pm \frac{a}{e}$

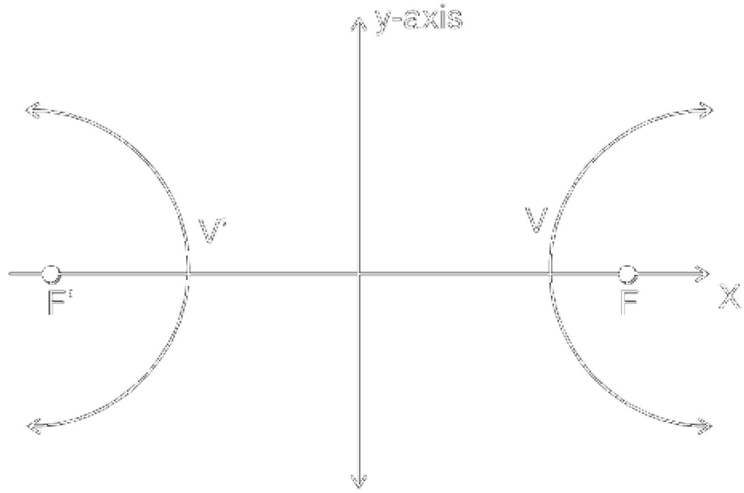
(5) Center (0, 0)

(6) Vertices $(0, \pm a)$

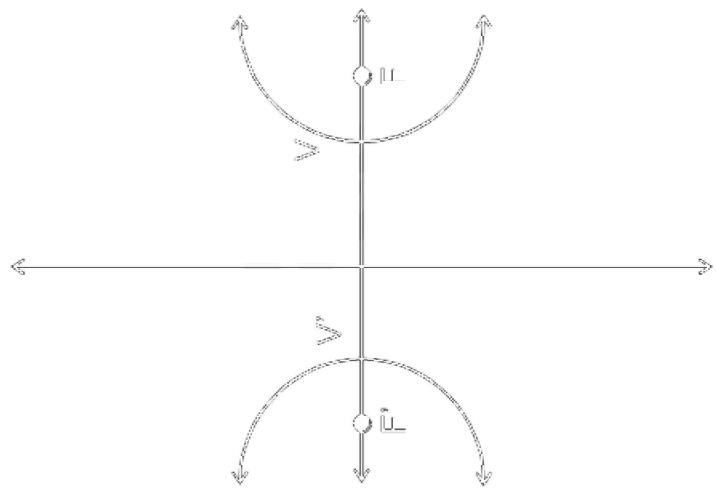
(7) Covertices $(\pm b, 0)$

Length of latus rectum is $\frac{2b^2}{a}$.

If center is not (0, 0) then equations become



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \& \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



EXERCISE 6.6

Q.1: Find an equation of the hyperbola with given data. Sketch the graph of each.

(i) Center (0, 0) Focus (6, 0) , Vertex (4, 0)

Solution:

Given $(ae, 0) = (6, 0)$, $(a, 0) = (4, 0)$

$(c, 0) = (6, 0)$ $a = 4$

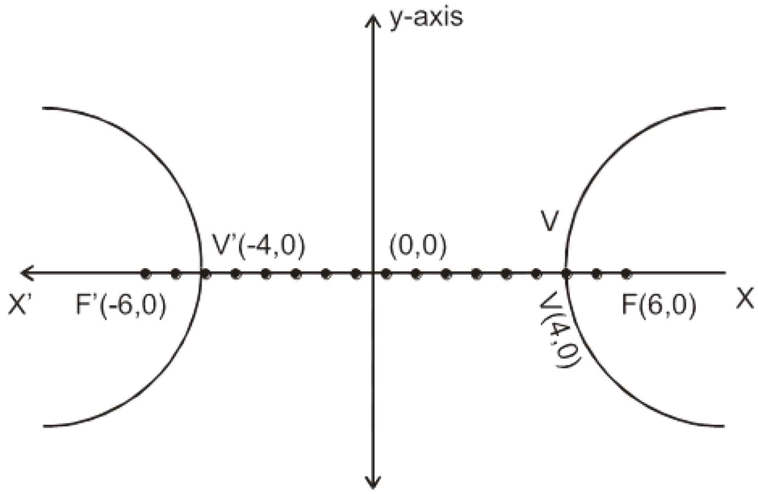
$c = 6$

For hyperbola

$$c^2 = a^2 + b^2$$
$$(6)^2 = (4)^2 + b^2$$
$$36 - 16 = b^2$$
$$20 = b^2$$

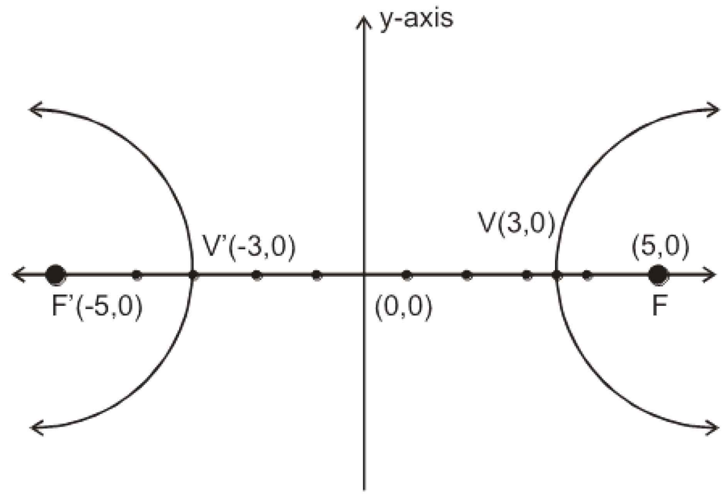
Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$



(ii) Foci (± 5, 0), vertex (3, 0)

Solution:



Foci (± 5, 0) Vertex (3, 0)

$(\pm c, 0) = (\pm 5, 0)$ $(a, 0) = (3, 0)$

$\boxed{c = 5}$ $\boxed{a = 3}$

For hyperbola

$c^2 = a^2 + b^2$

$(5)^2 = (3)^2 + b^2$

$25 = 9 + b^2$

$16 = 9 + b^2$

$16 = b^2$

Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{Ans.}$$

(iii) Foci $(2 \pm 5\sqrt{2}, -7)$, Length of the transverse axis 10.

Solution:

$(\pm ac, 0) = (2 \pm 5\sqrt{2}, -7)$

Foci are $(2 + 5\sqrt{2}, -7)$ & F' $(2 - 5\sqrt{2}, -7)$

Also given $2a = 10$

$a = 5$

Center = mid point of foci

$$= \left(\frac{2 + 5\sqrt{2} + 2 - 5\sqrt{2}}{2}, \quad \frac{-7-7}{2}\right)$$

Center $= (2, -7)$

We know that

c = distance between center and focus

$$= \sqrt{(2 - 2 - 5\sqrt{2})^2 + (-7+7)^2} \quad = \sqrt{50}$$

For hyperbola

$c^2 = a^2 + b^2$

$(\sqrt{50})^2 = (5)^2 + b^2$

$50 - 25 = b^2$

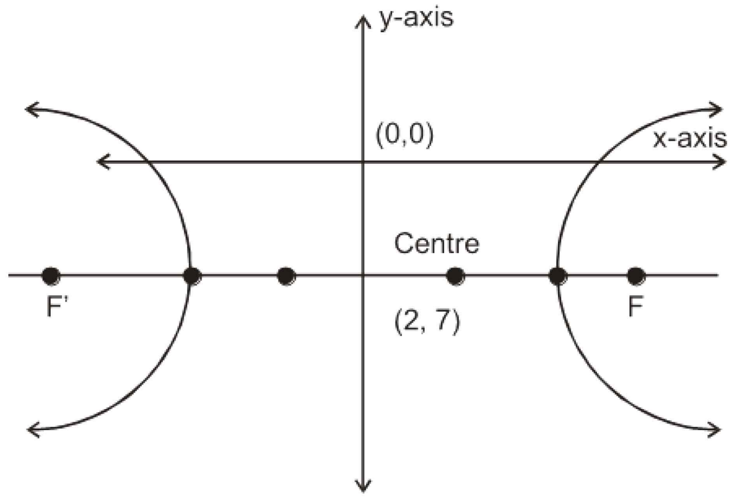
$25 = b^2$

$b^2 = 25$

with center $(2, -7)$ equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{25^2} - \frac{(y+7)^2}{25} = 1 \quad \text{Ans}$$



(iv) Foci (0, ± 6) , e = 2 (Lahore Board 2011)

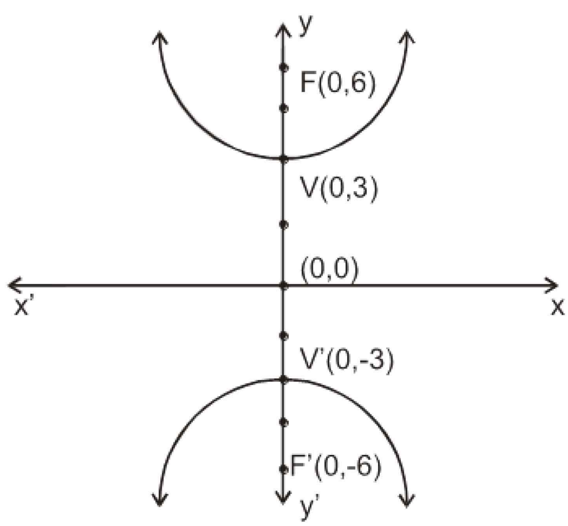
Solution:

Foci $(0, \pm c) = (0, \pm 6)$

$c = 6 \quad , \quad e = 2$

$ae = c$

$a = \dfrac{c}{e} = \dfrac{6}{2} = \boxed{3 = a}$



For hyperbola

$c^2 = a^2 + b^2$

$(6)^2 = (3)^2 + b^2$

$36 - 9 = b^2$

$b^2 = 27$

Equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{x^2}{27} = 1$$

(v) Foci (0, ± 9) directrix y = ± 4

Solution:

Foci $(0, \pm ae) = (0, \pm 9)$, $y = \pm\frac{a}{e}$

$ae = 9$, $\pm 4 = \pm\frac{a}{e}$

=> $c = 9$, $e = \frac{a}{4}$

=> $a\left(\frac{a}{4}\right) = 9$

$a^2 = 36$

For hyperbola

$c^2 = a^2 + b^2$

$(9)^2 = 36 + b^2$

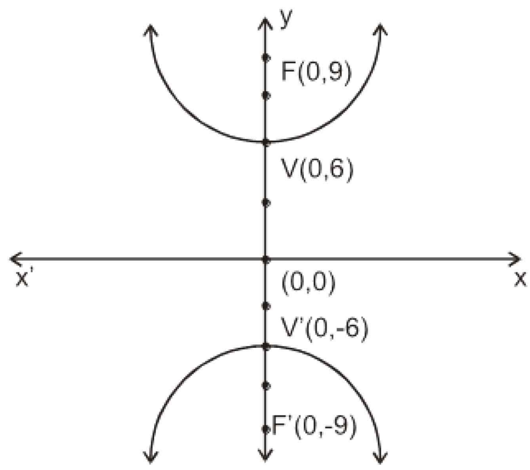
$81 - 36 = b^2$

$b^2 = 45$

Required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{36} - \frac{x^2}{45} = 1$$



(vi) Centre (2, 2) horizontal transverse axis of length 6 & eccentricity e = 2.

Solution:

Centre (2, 2) & $2a = 6 \Rightarrow$ $\boxed{a = 3}$ & $e = 2$

$c = ae = 3(2) = 6$

For hyperbola

$c^2 = a^2 + b^2$

$(6)^2 = (3)^2 + b^2$

$36 - 9 = b^2$

$b^2 = 27$

with centre (2, 2) required equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$$

foci are $(\pm ae, 0) = (\pm 6, 0)$

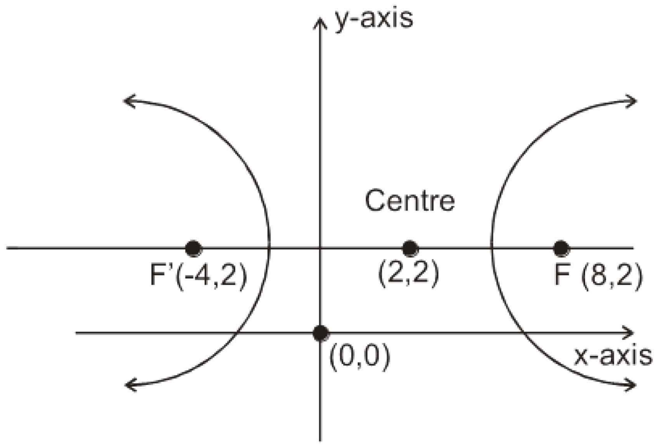
$(x-2, y-2) = (\pm 6, 0)$

$x - 2 = \pm 6 \qquad y - 2 = 0$

$x = \pm 6 + 2 \qquad y = 2$

$x = 8, -4 \qquad y = 2$

$\therefore$ F(8, 2) & (– 4, 2)



(vii) Vertices (2, ± 3) and (0, 5) lies on the curve.

Solution:

Center of the hyperbola = mid point of vertices

$$= \left(\frac{2+2}{2}, \frac{3-3}{2}\right)$$

Center $= (2, 0)$

Since V $=(2, 3)$ & V′ $=(2, -3)$

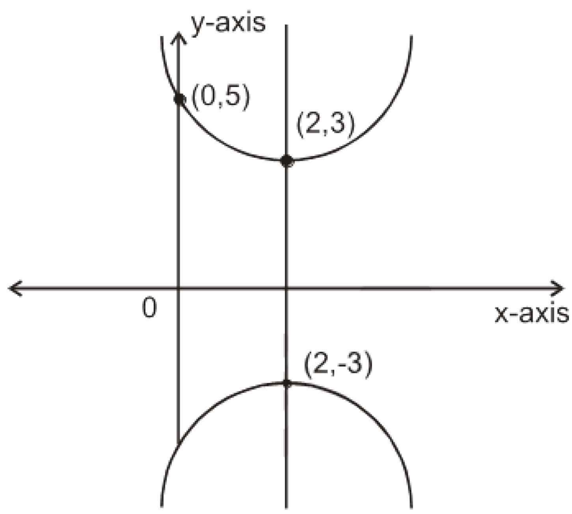
$$2a = |VV'| = \sqrt{(2-2)^2+(3+3)^2} = 6$$

$$\boxed{a = 3}$$

With center (2, 0) equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-0)^2}{9} - \frac{(x-2)^2}{b^2} = 1 \qquad (1)$$



Since (0, 5) lies on (1)

$$\frac{25}{9} - \frac{4}{b^2} = 1$$

$$\frac{25}{9} - 1 = \frac{4}{b^2}$$

$$\frac{16}{9} = \frac{4}{b^2}$$

$$b^2 = \frac{9}{4} \qquad (1) \text{ becomes } \frac{y^2}{9} - \frac{(x-2)^2}{\frac{9}{4}} = 1 \qquad \text{Ans.}$$

(viii) Foci (5, – 2) (5, 4) & One vertex (5, 3)

Solution:

Center = mid point of foci

$$= \left(\frac{5+5}{2}, \quad \frac{-2+4}{2}\right)$$

Center = (5, 1)

$$c = \text{distance between center \& focus}$$
$$= \sqrt{(5-5)^2 + (1-4)^2}$$
$$= \sqrt{9}$$
$$= 3$$
$$\boxed{c = 3}$$

$$a = \text{distance between vertex \& center}$$
$$= \sqrt{(5-5)^2 + (3-1)^2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\boxed{a = 2}$$

For hyper bola

$$c^2 = a^2 + b^2$$
$$(3)^2 = (2)^2 + b^2$$
$$9 - 4 = b^2$$
$$\boxed{b^2 = 5}$$

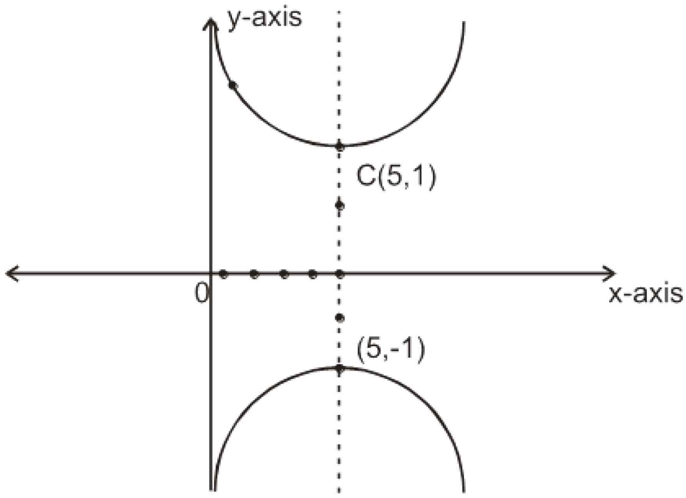
with center (5, 1) required equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$
$$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1$$

The second vertex is

$(5, 1 - 2)$

$(5, -1)$



Q.2: Find the center, foci, eccentricity, vertices and equations of directrices of each of the following. (i) $x^2 - y^2 = 9$

Solution:

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

$$\Rightarrow \quad a^2 = 9 \;, \quad b^2 = 9$$

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + 9 = 18$$

$$c^2 = 18 \Rightarrow \quad c = 3\sqrt{2}$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{18}{9} = 2$$

$$\boxed{e = \sqrt{2}}$$

foci $= (\pm c, 0) = (\pm 3\sqrt{2}\,, 0)$

Vertices $= (\pm a, 0) = (\pm 3, 0)$

Directrix $x = \pm\dfrac{a}{e}$

Directrix $x = \pm\dfrac{3}{\sqrt{2}}$

Center is $(0, 0)$

(ii) $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$a^2 = 4 \qquad b^2 = 9$$

$$\therefore \quad c^2 = a^2 + b^2 = 4 + 9 = 13 \quad \Rightarrow \quad c = \sqrt{13}$$

$$\therefore \quad ae = c \quad \Rightarrow \quad e = \frac{c}{a}$$

eccentricity $\boxed{e = \dfrac{\sqrt{13}}{2}}$

foci $= (\pm c, 0) = (\pm\sqrt{13}\,, 0)$

Vertices $= (\pm a, 0) = (\pm 2, 0)$

Directrix $x = \pm\dfrac{a}{e}$

$$x = \pm\frac{\frac{2}{\sqrt{13}}}{2}$$

$$x = \pm\frac{4}{\sqrt{13}}$$

Center is (0, 0)

(iii) $\mathbf{\frac{y^2}{16} - \frac{x^2}{9} = 1}$

Solution:

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$a^2 = 16$, $b^2 = 9$

$\therefore$ $c^2 = a^2 + b^2$

$= 16 + 9$

$c^2 = 25$

eccentricity $ac = c$ => $e = \frac{c}{a}$ => $\frac{5}{4}$

$$\boxed{e = \frac{5}{4}}$$

foci $= (0, \pm c)$

$= (0, \pm 5)$

Vertices $= (0, \pm a) = (0, \pm 4)$

Directrix $y = \pm\frac{a}{e}$

Directrix $y = \pm\frac{4}{\frac{5}{4}} = \pm\frac{16}{5}$

Center is (0, 0)

(iv) $\mathbf{\frac{y^2}{4} - x^2 = 1}$

Solution:

$$\frac{y^2}{4} - \frac{x^2}{1} = 1$$

Center = (0, 0)

$a^2 = 4$, $b^2 = 1$

$c^2 = a^2 + b^2$
$c^2 = 4 + 1$
$c^2 = 5$

eccentricity $ae = c \quad \Rightarrow \quad e = \dfrac{c}{a} = \boxed{\dfrac{\sqrt{5}}{2} = e}$

foci $= (0, \pm c) = (0, \pm\sqrt{5})$
Vertices $= (0, \pm a) = (0, \pm 2)$

Directrix $y = \pm\dfrac{a}{e}$

Directrix $y = \pm\dfrac{2}{\frac{\sqrt{5}}{2}}$

$= \pm\dfrac{4}{\sqrt{5}}$

(v) $\dfrac{(x-1)^2}{2} - \dfrac{(y-1)^2}{9} = 1$

Solution:

Let $x - 1 = X \quad , \quad y - 1 = Y$

$\dfrac{X^2}{2} - \dfrac{Y^2}{2} = 1$

$a^2 = 2 \quad , \quad b^2 = 9 \quad c^2 = a^2 + b^2 = 2 + 9 = 11$

$ae = c \quad , \quad e = \dfrac{c}{a} = \dfrac{\sqrt{11}}{\sqrt{2}}$

Eccentricity $e = \sqrt{\dfrac{11}{2}}$

Coordinates of foci $(\pm c, 0)$

$(X, Y) = (\pm\sqrt{11}, 0)$

$(x - 1 \ , \ y - 1) = (\pm\sqrt{11}, 0)$

$x - 1 = \pm\sqrt{11} \quad y - 1 = 0$

$x = 1 \pm\sqrt{11} \quad y = 1$

foci $(1 \pm \sqrt{11}, 1)$

For center Put $X = 0 \quad , \quad Y = 0$

$x - 1 = 0 \quad , \quad y - 1 = 0$

$\Rightarrow x = 1 \quad , \quad y = 1 \qquad \boxed{\text{cenere} = (1, 1)}$

Vertices $(X, Y) = (\pm\sqrt{2}, 0)$

$(x - 1, y - 1) = (\pm\sqrt{2}, 0)$

$x - 1 = \pm\sqrt{2} = \quad y - 1 = 0$

$x = 1 \pm \sqrt{2} \quad y = 1$

Vertices are $\boxed{(1 + \sqrt{2}, 1) \quad , \quad (1 - \sqrt{2}, 1)}$

Equation of directrix

$$X = \pm \frac{a}{e}$$

$$x - 1 = \pm \frac{\sqrt{2}}{\frac{\sqrt{11}}{\sqrt{2}}}$$

$$x - 1 = \pm \frac{2}{\sqrt{11}}$$

$$\boxed{x = 1 \pm \frac{2}{\sqrt{11}}}$$

(vi) $\mathbf{\dfrac{(y + 2)^2}{9} - \dfrac{(x - 2)^2}{16} = 1}$

Solution:

Let $y + 2 = Y \quad , \quad x - 2 = X$

Given equation becomes

$$\frac{Y^2}{9} - \frac{X^2}{16} = 1$$

$a^2 = 9 \quad , \quad b^2 = 16 \qquad c^2 = a^2 + b^2 = 9 + 16 = 25$

$\Rightarrow \quad c = \pm 5$

$ae = c$

Eccentricity $\Rightarrow \quad e = \dfrac{c}{a} = \dfrac{5}{3}$

Foci $(X, Y) = (0, \pm c)$

$(x - 2, y + 2) = (0, \pm 5)$

$x - 2 = 0 \quad , \quad y + 2 = \pm 5$

$x = 2 \quad , \quad y = 2 \pm 5$

$y = -7, 3$

Foci $(2, -7) \quad , \quad (2, 3)$

For the center put $\quad X = 0 \qquad Y = 0$

$$x - 2 = 0 \quad , \quad y + 2 = 0$$

$$x = 2 \quad , \quad y = -2$$

Center $(2, -2)$

Vertices $\quad (X, Y) = (0, \pm a)$

$$(x - 2, y + 2) = (0, \pm 3)$$

$$x - 2 = 0 \qquad y + 2 = \pm 3$$

$$x = 2 \qquad y = -2 \pm 3$$

$$y = -5, 1$$

Vertices

$$(2, -5) \ \& \ (2, 1)$$

Equations of directrices are

$$Y = \pm \frac{a}{e}$$

$$y + 2 = \pm \frac{3}{\frac{5}{3}}$$

$$y = -2 \pm \frac{9}{5} \qquad \text{Ans.}$$

(vii) $\quad \mathbf{9x^2 - 12x - y^2 - 2y + 2 = 0}$

Solution:

$$9x^2 - 12x - y^2 - 2y = -2$$

$$9(x^2 - \frac{12}{9}x) - 1\,(y^2 + 2y) = -2$$

$$9(x^2 - \frac{4}{3}x + \frac{4}{9}) - 1\,(y^2 + 2y + 1) = -2 + 4 - 1$$

$$9(x - \frac{2}{3})^2 - 1\,(y + 1)^2 = 1$$

$$\frac{\left(x - \frac{2}{3}\right)^2}{\frac{1}{9}} - \frac{(y + 1)^2}{1} = 1 \qquad (1)$$

Let $\quad x - \frac{2}{3} = X \quad , \quad y + 1 = Y \quad$ Equation (1) becomes

$$\frac{X^2}{\frac{1}{9}} - \frac{Y^2}{1} = 1 \qquad \text{which represents hyperbola}$$

$a^2 = \frac{1}{9}$, $b^2 = 1$

$e^2 = \frac{a^2 + b^2}{a^2} = \frac{\frac{1}{9} + 1}{\frac{1}{9}} = \frac{\frac{10}{9}}{\frac{1}{9}} = 10$

$e = \sqrt{10}$ $c^2 = \frac{10}{9}$ => $c = \frac{\sqrt{10}}{3}$

Coordinates of foci $(X, Y) = (\pm C = 0)$

$(x - \frac{2}{3}, y + 1) = (\pm\frac{\sqrt{10}}{3}, 0)$

$x - \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$ $y + 1 = 0$

$x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ $y = -1$

$x = \frac{2 \pm \sqrt{10}}{3}$, $y = -1$

Foci $(\frac{2 \pm \sqrt{10}}{3}, -1)$

For center put $X = 0$, $Y = 0$

$x - \frac{2}{3} = 0$, $y + 1 = 0$

$x = \frac{2}{3}$ $y = -1$ Center $(\frac{2}{3}, -1)$

Vertices $(\pm a, 0)$

$(X + Y) = (\pm\frac{1}{3}, 0)$

$(x - \frac{2}{3}, y + 1) = (\pm\frac{1}{3}, 0)$

$x - \frac{2}{3} = \pm\frac{1}{3}$, $y + 1 = 0$

$x = \frac{2}{3} \pm \frac{1}{3}$ $y = -1$

Vertices $(1, -1)$, $(\frac{1}{3}, -1)$

Equation of directrices are

$X = \pm\frac{a}{e}$

$x - \frac{2}{3} = \pm\frac{1}{3} \div \sqrt{10}$

$x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$

(viii) $\mathbf{4y^2 + 12y - x^2 + 4x + 1 = 0}$

Solution:

$$4(y^2 + 3y) - 1\,(x^2 - 4x) = -1$$

$$4(y^2 + 3y + \frac{9}{4}) - 1\,(x^2 - 4x + 4 = -1 + 9 - 4$$

$$4(y + \frac{3}{2})^2 - 1\,(x - 2)^2 = 4$$

$$\frac{\left(y + \frac{3}{2}\right)^2}{1} - \frac{(x - 2)^2}{4} = 1 \qquad (1)$$

Let $y + \frac{3}{2} = Y$, $x - 2 = X$

Equation (1) becomes

$$\frac{Y^2}{1} - \frac{X^2}{4} = 1 \qquad \text{(Represents a hyperbola)}$$

$a^2 = 1$, $b^2 = 4$

$\therefore$ $c^2 = a^2 + b^2 = 1 + 4 = 5 = c = \sqrt{5}$

Eccentricity $ae = c \Rightarrow e = \frac{c}{a} = \frac{\sqrt{5}}{1} = \boxed{\sqrt{5} = e}$

Foci $(X, Y) = (0, \pm c)$

$$(x - 2,\ y + \frac{3}{2}) = (0, \pm\sqrt{5})$$

$x - 2 = 0$ $\qquad y + \frac{3}{2} = \pm\sqrt{5}$

$x = 2$ $\qquad y = \frac{-3}{2} \pm \sqrt{5}$

Foci $(2, \frac{-3}{2} \pm \sqrt{5})$

For center Put $X = 0,\ Y = 0$

$x - 2 = 0$, $y + \frac{3}{2} = 0$

$x = 2$ $\qquad y = \frac{-3}{2}$

Center $(2, \frac{-3}{2})$

Vertices $(X, Y) = (0, \pm a)$

$$(x-2,\quad y+\frac{3}{2}) = (0, \pm 1)$$

$$x-2 = 0 \quad , \quad y + \frac{3}{2} = \pm 1$$

$$x = 2 \quad , \quad y = \frac{-3}{2} \pm 1$$

$$y = \frac{-1}{2}, \frac{-5}{2}$$

Vertices are $(2, \frac{-1}{2})$, $(2, \frac{-5}{2})$

Equation of directrices are

$$Y = \pm \frac{a}{e}$$

$$y + \frac{3}{2} = \pm \frac{1}{\sqrt{5}}$$

$$y = \frac{-3}{2} \pm \frac{1}{\sqrt{5}} \qquad \text{Ans}$$

(ix) $\mathbf{x^2 - y^2 + 8x - 2y - 10 = 0}$

Solution:

$$x^2 + 8x - y^2 - 2y = 10$$

$$(x^2 + 8x + 16) - (y^2 + 2y + 1) = 10 + 16 - 1$$

$$(x+4)^2 - (y+1)^2 = 25$$

$$\frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1 \qquad \text{(i)}$$

Let $x + 4 = X$, $y + 1 = Y$

(i) Becomes $\frac{X^2}{25} - \frac{Y^2}{25} = 1$ (which represents hyperbola)

$a^2 = 25$, $b^2 = 25$ $\quad c^2 = a^2 + b^2 = 50 \Rightarrow c = \sqrt{50}$

$ae = c$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{50}}{5} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

$$\text{Foci} = (\pm c,\ 0)$$

$$(X, Y) = (\pm 5\sqrt{2}, 0)$$

$$(x+4, y+1) = (\pm 5\sqrt{2}, 0)$$

$$x + 4 = \pm 5\sqrt{2} \quad , \quad y + 1 = 0$$

$$x = -4 \pm 5\sqrt{2} \qquad y = -1 \qquad \text{foci } (-4 \pm 5\sqrt{2}, -1)$$

For center put $X = 0$, $Y = 0$

$x + 4 = 0$, $y + 1 = 0$ => $x = -4$, $y = -1$

Center $(-4, -1)$

Vertices $(X, Y) = (\pm a, 0)$

$(x + 4, \; y + 1) = (\pm 5, 0)$

$x + 4 = \pm 5$ $\quad y + 1 = 0$

$x = -4 \pm 5$ $\quad y = -1$

$= -9, 1$

Vertices $(-9, -1)$ & $(1, -1)$

Equations of directrices

$$X = \pm \frac{a}{e}$$

$$x + 4 = \pm \frac{5}{\sqrt{2}}$$

$$x = -4 \pm \frac{5}{\sqrt{2}} \quad \text{Ans.}$$

(x) $\mathbf{9x^2 - y^2 - 36x - 6y + 18 = 0}$

Solution:

$9x^2 - 36x - y^2 - 6y = -18$

$9(x^2 - 4x) - 1(y^2 + 6y) = -18$

$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) = -18 + 36 - 9$

$9(x - 2)^2 - 1(y + 3)^2 = 9$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{1} = 1 \qquad \text{(i)}$$

Let $x - 2 = X$, $y + 3 = Y$ Equation (i) becomes

$$\frac{X^2}{1} - \frac{Y^2}{9} = 1$$

$a^2 = 1, \quad b^2 = 9$

$\therefore \quad c^2 = a^2 + b^2$ => $c^2 = 1 + 9 = 10$ $\quad \boxed{c^2 = 10}$

$\therefore \quad ae = c$

Eccentricity $\therefore \quad e = \frac{c}{a} = \frac{\sqrt{10}}{1}$ $\quad \boxed{e = \sqrt{10}}$

Foci $(\pm c, 0) = (\pm \sqrt{10}, 0)$

$(X, Y) = \pm \sqrt{10}, 0$

$(x - 2, \; y + 3) = (\pm\sqrt{10}, 0)$

$$x - 2 = \pm\sqrt{10} \quad , \quad y + 3 = 0$$
$$x = 2\pm\sqrt{10} \quad , \quad y = -3$$

Foci $\quad (2 \pm \sqrt{10}\,, \; -3)$

For center $\quad$ Put $\quad X = 0 \quad , \quad Y = 0$

$$x - 2 = 0 \qquad y + 3 = 0$$
$$x = 2 \quad , \quad y = -3 \qquad \boxed{\text{center } (2, -3)}$$

Vertices $(X, Y) = (\pm a, \; 0)$

$$(x - 2 \;, \; y + 3) = (\pm 1, 0)$$
$$x - 2 = \pm 1 \qquad y + 3 = 0$$
$$x = 2 \pm 1 \qquad y = -3$$
$$x = 3, 1 \quad , \quad y = -3 \qquad \boxed{\text{Vertices } (3, -3), (1, -3)}$$

Equations of directrices

$$X = \pm \frac{a}{e}$$
$$x - 2 = \pm \frac{1}{\sqrt{10}}$$
$$x = 2 \pm \frac{1}{\sqrt{10}} \qquad \text{Ans}$$

Q.3: Let $0<a<c$ and $F(-c, 0)$, $F'(c, 0)$ be two fixed points. Show that the set of points $P(x, y)$ such that $|PF| - |PF'| = \pm 2a$ is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Solution:

Given $F(-c, 0)$, $F'(c, 0)$ and $p(x, y)$

Also $|PF| - |PF'| = 2a$

$$\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$
$$\sqrt{(x + c)^2 + y^2} = 2a + \sqrt{(x - c)^2 + y^2}$$

Squaring on both sides

$$(\sqrt{(x + c)^2 + y^2}\,)^2 = (2a + \sqrt{(x - c)^2 + y^2}\,)^2$$
$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 + 4a\sqrt{(x - c)^2 + y^2}$$
$$4cx - 4a^2 = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$
$$4(cx - a^2) = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$

Again Squaring

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 - a^2x^2 - a^2c^2 + 2a^2cx - a^2y^2 = 0$$
$$x^2(c^2 - a^2) - a^2y^2 = a^2 (c^2 - a^2)$$
Dividing throughout by $c^2 - a^2$
$$\frac{x^2(c^2 - a^2)}{c^2 - a^2} - \frac{a^2 y^2}{c^2 - a^2} = \frac{a^2(c^2 - a^2)}{c^2 - a^2}$$
$$x^2 - \frac{a^2 y^2}{c^2 - a^2} = a^2$$
Dividing throughout by a^2 $\quad \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$ Hence proved.

Q.4: Find an equation of the hyperbola with foci $(-5, -5)$ & $(5, 5)$ vertices $(-3\sqrt{2}, -3\sqrt{2})$ and $(3\sqrt{2}, 3\sqrt{2})$.

Solution:

Since $2a =$ distance between two vertices
$$2a = \sqrt{(3\sqrt{2} + 3\sqrt{2})^2 + (3\sqrt{2} + 3\sqrt{2})^2}$$
$$2a = \sqrt{(6\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{72 + 72} = \sqrt{144} = 12$$
$$2a = 12 \quad \boxed{a = 6}$$

Also given $F(-5, -5)$ & $F'(5, 5)$. Let $P(x, y)$ be any point on the hyperbola such that $|PF| - |PF'| = 2a$
$$\sqrt{(x + 5)^2 + (y + 5)^2} - \sqrt{(x - 5)^2 + (y - 5)^2} = 12$$
Squaring $\sqrt{(x + 5)^2 + (y + 5)^2} = 12 + \sqrt{(x - 5)^2 + (y - 5)^2}$
$$x^2 + 25 + 10x + y^2 + 25 + 10y = x^2 + 25 - 10x + y^2 + 25 - 10y + 144 + 24\sqrt{(x - 5)^2 + (y - 5)^2}$$
$$20x + 20y - 144 = 24\sqrt{(x - 5)^2 + (y - 5)^2}$$
$$4(5x + 5y - 36) = 4[6\sqrt{(x - 5)^2 + (y - 5)^2}]$$
Again squaring on both sides
$$(5x + 5y - 36)^2 = 36(x^2 + 25 - 10x + y^2 + 25 - 10y)$$
$$25x^2 + 5y^2 + 1296 + 50xy - 360y - 360x = 36x^2 + 900 - 360x + 36y^2 + 900 - 360y$$
$$25x^2 + 25y^2 + 1296 + 50xy - 360y - 360x - 36x^2 - 900 + 360x - 36y^2 - 900 + 360y = 0$$
$$11x^2 - 50xy - 11y^2 + 504 = 0 \quad \text{Ans}$$

Q.5: For any point on hyperbola the difference of its distances from the points (2, 2) & (10, 2) is 6. Find an equation of hyperbola.

Solution:

Given $2a = 6$ $\quad F = (2, 2)$ & $F' = (10, 2)$

Let $P(x, y)$ be any point on the hyperbola, such that
$$|PF| - |PF'| = 2a$$

$$\sqrt{(x-2)^2+(y-2)^2} - \sqrt{(x-10)^2+(y-2)^2} = 6$$
$$\sqrt{(x-2)^2+(y-2)^2} = 6 + \sqrt{(x-10)^2+(y-2)^2}$$

Squaring

$$x^2+4-4x+y^2+4-4y = 36 + x^2 + 100 - 20x + y^2 + 4 - 4y + 12\sqrt{(x-10)^2+(y-2)^2}$$
$$16x - 132 = 12\sqrt{(x-10)^2+(y-2)^2}$$
$$4(4x-33) = 12\sqrt{(x-10)^2+(y-2)^2}$$
$$4x - 33 = 3\sqrt{(x-10)^2+(y-2)^2}$$

Again squaring

$$16x^2 + 1089 - 264 = 9(x^2 + 100 - 20x + y^2 + 4 - 4y)$$
$$16x^2 + 1089 - 264 - 9x^2 + 900 + 180x - 9y^2 - 36 + 36y = 0$$
$$7x^2 - 9y^2 - 84x + 36y + 153 = 0 \qquad \text{Ans}$$

Q.6: Two listening posts hear the sound of an enemy gun. The difference in time is one second. If the listening posts are 1400 feet apart, write an equation of the hyperbola passing though the position of the enemy gun.

Solution:

Since the difference in time is one second, so let the listening posts F&F′ near the second of the enemy gm after t, t–1 seconds respectively. If P is position of enemy then

$$|PF| - |PF'| = 2a \qquad \ldots\ldots \quad (1)$$

Since posts are 1400 feet apart, so

$$2c = 1400 \quad \Rightarrow \quad c = 700$$

since distance = (Velocity) (time)

So $|PF| = (1080)\,t$ & $|PF'| = (1080)(t-1)$

Putting these values in (1)

$$(1080)t - (1080)(t-1) = 2a$$
$$1080t - 1080\,t + 1080 = 2a$$
$$a = \frac{1080}{2} = 540$$

For hyperbola, we know that $c^2 = a^2 + b^2$

$$(700)^2 = (540)^2 + b^2$$
$$b^2 = 490000 - 291600$$
$$b^2 = 198400$$

Required equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x^2}{291600} - \frac{y^2}{198400} = 1 \qquad \text{Ans}$$

Tangents to Conics

(I) Parabola

Equations of Tangent in different forms

(i) Point form:

The equation of the tangent to the parabola

$y^2 = 4ax$ at the point (x, y) is $yy_1 = 2a(x + x_1)$

(ii) Slope form:

The equation of the tangent to the parabola

$y^2 = 4ax$ in the term of slope 'm' is

$$y = mx + \frac{a}{m} \qquad (\therefore \quad c = \frac{a}{m})$$

Note:

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{1}{2}(x + x_1)$, y by $\frac{1}{2}(y + y_1)$ and xy by $\frac{xy_1 + yx_1}{2}$

(II) Ellipse

Equations of tangents in different forms.

(i) Point form:

Equations of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope m is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \qquad (\because \quad c = \pm\sqrt{a^2m^2 + b^2})$$

(III) Hyperbola

Equations of tangent in different forms

(i) Point form:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of tangent to hyperbola in terms of 'm' is

$$y = mx \pm \sqrt{a^2m^2 - b^2} \qquad (c = \pm\sqrt{a^2m^2 - b^2})$$

(IV) Circle

Equations of tangent in different forms

(i) Point form:

Equation of tangent to the circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) Slope form:

Equation of tangent is terms of slope 'm' is $y = mx \pm a\sqrt{1 + m^2}$

$(\because \quad c^2 = a^2 (1 + m^2))$

Equations of Normal

(i) Parobola $y^2 = 4ax$ is at (x_1, y_1)

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

(ii) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(iii) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

EXERCISE 6.7

Q.1: Find equations of tangent and normal to each of the following at the indicated point.

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Solution:

Equation of tangent at $(at^2, 2at)$ is

$$\begin{aligned}
yy_1 &= 2a(x + x_1) \\
y(2at) &= 2a(x + at^2) \\
2ayt &= 2ax + 2a^2t^2 \\
2ayt &= 2a(x + at^2) \\
yt &= x + at^2
\end{aligned}$$

And equation of normal at $(at^2, 2at)$ is

$$\begin{aligned}
y - y_1 &= \frac{-y_1}{2a}(x - x_1) \\
y - 2at &= \frac{-2at}{2a}(x - at^2)
\end{aligned}$$

$$y - 2at = -tx + at^3$$
$$tx + y - 2at - at^3 = 0$$

(ii) $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ **at $(a\cos\theta, b\sin\theta)$**

Solution:

Equation of tangent $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$$\frac{x(a\cos\theta)}{a^2} + \frac{y(b\sin\theta)}{b^2} = 1$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

and equation of normal

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2 - b^2$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$$

(iii) $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ **at $(a\sec\theta, b\tan\theta)$**

Solution:

Equation of tangent

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\frac{x\,a\sec\theta}{a^2} - \frac{y\,b\tan\theta}{b^2} = 1$$

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

And equation of normal

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

$$\frac{xa^2}{a\sec\theta} + \frac{yb^2}{b\tan\theta} = a^2 + b^2$$

$$\frac{xa}{\sec\theta} + \frac{yb}{\tan\theta} = a^2 + b^2$$

OR $\quad x\,a\cos\theta + y\,b\cot\theta = a^2 + b^2$

Q.2: Write equation of the tangent to the given conic at the indicated point

(i) $3x^2 = -16y$ at the points whose ordinate is -3

Solution:

$3x^2 = -16y$ .... (1)

Put $y = -3$ in

$3x^2 = -16(-3)$

$3x^2 = 48 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Hence points are

$(4, -3)$ & $(-4, -3)$

Now diff. (1) w.r.t 'x'

$$6x = -16\frac{dy}{dx}$$

$$\frac{6x}{-16} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-3}{8}x$$

$$m = \text{Slope} = \frac{dy}{dx}\Big|_{(4,-3)} = \frac{-3}{8}(4) = \frac{-3}{2}$$

$$\text{Also} \quad m = \frac{dy}{dx}\Big|_{(-4,-3)} = \frac{-3}{8}(4) = \frac{3}{2}$$

$\therefore$ Equation of tangent at $(4, -3)$ is

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{-3}{2}(x - 4)$

$2y + 6 = -3x + 12$

$3x + 2y = 6$

$3x + 2y - 6 = 0$

Equation of tangent at $(-4, -3)$ is

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{+3}{2}(x + 4)$

$2y + 6 = 3x + 12$

$3x - 2y = -6$

$3x - 2y + 6 = 0$

(ii) $3x^2 - 7y^2 = 20$ at points where $y = -1$.

Solution:

$3x^2 - 7y^2 = 20$ .... (1)

Put $y = -1$ in (1)

$3x^2 - 7(-1)^2 = 20$

$3x^2 = 20 + 7$

$3x^2 = 27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

Thus the required points on the conic are $(3, -1)$ & $(-3, -1)$

Now diff (1) w.r.t. 'x' we have

$$6x - 14y\frac{dy}{dx} = 0$$

$$14\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{14y} = \frac{3x}{7y}$$

Now $m = \text{Slope} = \frac{dy}{dx}\Big|_{(3,-1)} = \frac{9}{-7}$ Also $m = \frac{dy}{dx}\Big|_{(-3,-1)} = \frac{9}{7}$

Therefore equation of tangent at (3, – 1) is

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{-9}{7}(x - 3)$$
$$7y + 7 = -9x + 27$$
$$9x + 7y = 20$$
$$9x + 7y - 20 = 0 \quad \text{Ans}$$

Equation of tangent at (– 3, –1)

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{9}{7}(x + 3)$$
$$7y + 7 = 9x + 27$$
$$9x - 7y = -20$$
$$9x - 7y + 20 = 0 \quad \text{Ans}$$

(iii) $3x^2 - 7y^2 + 2x - y - 48 = 0$, at point where $x = 4$

Solution:

$$3x^2 - 7y^2 + 2x - y - 48 = 0 \qquad ..... (1)$$

Put $x = 4$ in (1)

$$3(4)^2 - 7y^2 + 2(4) - y - 48 = 0$$
$$48 - 7y^2 + 8 - y - 48 = 0$$
$$\Rightarrow \quad -7y^2 - y + 8 = 0 \quad \Rightarrow \quad 7y^2 + y - 8 = 0$$
$$7y^2 + 8y - 7y - 8 = 0$$
$$y(7y + 8) - 1(7y + 8) = 0$$
$$(7y + 8)(y - 1) = 0$$

Either

$$7y + 8 = 0 \quad , \quad y - 1 = 0$$
$$y = \frac{-8}{7} \quad , \quad y = 1$$

Therefore, required points on the conic are $(4, \frac{-8}{7})$ & $(4, 1)$

Now diff. (1) w.r.t. 'x' $\quad 6x - 14y\frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$

$$(-14y - 1)\frac{dy}{dx} = -6x - 2$$

$$\frac{dy}{dx} = \frac{6x+2}{14y+1}$$

$$m = \frac{dy}{dx}\Big|_{(4,\,1)} = \frac{6(4)+2}{14(1)+1} = \frac{26}{15} \quad \text{Also } m = \frac{dy}{dx}\Big|_{(4,\,-\frac{8}{7})} = \frac{6(4)+2}{14\left(\frac{-8}{7}\right)+1} = \frac{26}{-15}$$

Equation of tangent at (4, 1) is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{26}{15}(x - 4)$$

$$15y - 15 = 26x - 104$$

$$26x - 15y - 89 = 0 \qquad \text{Ans}$$

Equation of tangent at $(4, \frac{-8}{7})$ is

$$y - y_1 = m(x - x_1)$$

$$y + \frac{8}{7} = \frac{-26}{15}(x - 4)$$

$$105y - 120 = -182x + 728$$

$$182x + 105y - 608 = 0 \qquad \text{Ans}$$

Q.3: Find equations of the tangents to each of the following through the given point

(i) $x^2 + y^2 = 25$, through (7, – 1)

Solution:

$$x^2 + y^2 = 25 \Rightarrow r = 5$$

We know that condition of tangency for the circle is

$$c^2 = r^2(1 + m^2)$$

$$c^2 = 25(1 + m^2)$$

$$\Rightarrow \quad c = \pm 5\sqrt{1 + m^2}$$

Let the required equation of tangent be

$y = mx + c$ .... (1) Putting value of C in (1)

$y = mx \pm 5\sqrt{1 + m^2}$ .... (2)

Since tangent line passes through point (7, – 1), therefore

$$-1 = 7m \pm 5\sqrt{1 + m^2}$$

$\pm 5\sqrt{1 + m^2} = 7m + 1$ Squaring

$$25(1 + m^2) = (7m + 1)^2$$

$$25 + 25m^2 = 49m^2 + 1 + 14m$$

$$-24m^2 - 14m + 24 = 0$$

$$12m^2 + 7m - 12 = 0$$

$$12m^2 + 16m - 9m - 12 = 0$$

$$4m(3m + 4) - 3(3m + 4) = 0$$

$$(3m + 4)(4m - 3) = 0$$

$$m = \frac{-4}{3} \qquad m = \frac{3}{4}$$

with $m = \frac{-4}{3}$ ...... (2) becomes

$$y = -\frac{4}{3}x \pm 5\sqrt{1+\frac{16}{9}}$$

$$= -\frac{4}{3}x \pm 5\ \frac{5}{3}$$

$$3y = -4x \pm 25$$

$$4x + 3y \pm 25 = 0$$

with $m = \frac{3}{4}$ ...... (2) becomes

$$y = \frac{3x}{4} \pm 5\sqrt{1+\frac{9}{16}}$$

$$= \frac{3x}{4} \pm \frac{25}{4}$$

$$4y = 3x \pm 25$$

$$3x - 4y \pm 25 = 0$$

(ii) $y^2 = 12x$ through (1, 4)

Solution:

$y^2 = 12\,x$

As standard form is

$y^2 = 4ax$

$4a = 12 \quad \Rightarrow \quad a = 3$

Let $y = mx + c$ ..... (1) be the required equation of tangent. For Parabola we know that condition of tangency is $c = \frac{a}{m} = \frac{3}{m}$ put in (1)

$y = mx + \frac{3}{m}$ .... (2)

Since tangent line passes through point (1, 4)

So (2) becomes

$4 = m + \frac{3}{m} \quad \Rightarrow \quad 4m = m^2 + 3$

$m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0$

$m = 1 \quad , \quad m = 3 \qquad$ Put in (1)

$y = x + 3 \qquad \& \qquad y = 3x + \frac{3}{m}$

$x - y + 3 = 0 \qquad\qquad y = 3x + \frac{3}{3}$

$y = 3x + 1$

$3x - y + 1 = 0 \qquad$ Ans

(iii) $x^2 - 2y^2 = 2$ through (1, – 2)

Solution:

$x^2 - 2y^2 = 2$

$\frac{x^2}{2} - \frac{y^2}{1} = 1$

$\Rightarrow \quad a^2 = 2 \quad , \quad b^2 = 1$

For hyperbola, we know that condition of tangent is

$$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad c^2 = 2m^2 - 1 \qquad \Rightarrow \qquad c = \pm\sqrt{2m^2 - 1}$$

Let $y = mx + c$ be tangent to the given hyperbola then $y = mx \pm \sqrt{2m^2 - 1}$ (1)

Since (1) passes through (1, – 2) (1) becomes

$$-2 = m \pm \sqrt{2m^2 - 1}$$

$$-2 - m = \pm\sqrt{2m^2 - 1} \qquad \text{Squaring}$$

$$4 + m^2 + 4m = 2m^2 - 1$$

$$2m^2 - 1 - m^2 - 4m - 4 = 0$$

$$m^2 - 4m - 5 = 0$$

$$\Rightarrow \quad (m - 5)(m + 1) = 0$$

$$\Rightarrow \quad m = 5, \quad m = -1$$

Putting values of m in (1) we get

$$y = 5x \pm \sqrt{2(25) - 1} \quad , \qquad y = -x \pm \sqrt{2 - 1}$$

$$y = 5x \pm \sqrt{49} \quad , \qquad = -x \pm 1$$

$$y = 5x \pm 7 \quad , \qquad y + x \pm 1 = 0$$

$$5x - y \pm 7 = 0 \qquad \text{Ans}$$

Q.4: Find equations of normal to the Parabola $y^2 = 8x$, which are parallel to the line $2x + 3y = 10$.

Solution:

$$y^2 = 8x \quad \text{...... (1)} \qquad\qquad 2x + 3y = 10 \quad \text{........ (2)}$$

Diff. (1) w.r.t. ‘x’ $\qquad\qquad m_2 = \text{Slope of line}$

$$2y\frac{dy}{dx} = 8 \qquad\qquad = \frac{-\text{coeff of } x}{\text{coeff of } y}$$

$$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y} \qquad\qquad = -\frac{2}{3}$$

$$m_1 = \frac{dy}{dx} = \frac{4}{y}$$

$$m_1 = \text{Slope of normal} = \frac{-y}{4}$$

Since normal and given line are Parallel

$$m_1 = m_2$$

$$\frac{-y}{4} = \frac{-2}{3} \qquad \Rightarrow \qquad y = \frac{8}{3} \qquad \text{Put in (1)}$$

$$\left(\frac{8}{3}\right)^2 = 8x$$

$$\frac{64}{9\times 8} = x \qquad \Rightarrow \qquad x = \frac{8}{9}$$

Required point $\left(\frac{8}{9}, \frac{8}{3}\right)$

with $y = \frac{8}{3}$, m_1 become

$$m_1 = -\frac{8}{3}\times\frac{1}{4} = \frac{-2}{3}$$

Required equation of normal at $\left(\frac{8}{9}, \frac{8}{3}\right)$ is

$$y - y_1 = m(x - x_1)$$
$$y - \frac{8}{3} = \frac{-2}{3}\left(x - \frac{8}{9}\right)$$
$$3y - 8 = -2\left(\frac{9x-8}{9}\right)$$
$$27y - 72 = -18x + 16$$
$$18x + 27y - 88 = 0$$

Q.5: Find equations of tangents to the ellipse $\frac{x^2}{4} + y^2 = 1$, which are parallel to the line $2x - 4y + 5 = 0$.

Solution:

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \qquad\qquad 2x - 4y + 5 = 0$$

$$\Rightarrow \quad a^2 = 4 \;,\quad b^2 = 1 \qquad\qquad m = \frac{-\text{coeff of } x}{\text{coeff of } y} = \frac{-2}{-4} = \frac{1}{2}$$

We know that condition of tangent for ellipse is

$$c^2 = a^2m^2 + b^2$$
$$c^2 = 4m^2 + 1$$
$$c = \pm\sqrt{4m^2+1}$$

Since tangent is parallel to line $2x - 4y + 5 = 0$

$\therefore$ Slope is also $m = \frac{1}{2}$

$$c = \pm\sqrt{4\frac{1}{4}+1} = \pm\sqrt{2}$$

Let the equation of required tangent by

$$y = mx + c$$

$$y = \frac{1}{2}x \pm \sqrt{2}$$

$$2y = x \pm 2\sqrt{2}$$

$$x - 2y \pm 2\sqrt{2} = 0 \quad \text{Ans}$$

Q.6: Find equations of the tangents to the conics $9x^2 - 4y^2 = 36$ Parallel to $5x - 2y + 7 = 0$.

Solution:

$$9x^2 - 4y^2 = 36$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \qquad \text{(Dividing by 36)}$$

$$\Rightarrow \quad a^2 = 4 \quad , \quad b^2 = 9$$

$$5x - 2y + 7 = 0$$

$$m = \text{slope of line} = \frac{5}{2}$$

For hyperbola, we know that

$$c^2 = a^2m^2 - b^2$$

$$c^2 = 4m^2 - 9$$

Since tangent and given line are parallel so their slopes are same. Thus $m = \frac{5}{2}$

$$c^2 = 4\left(\frac{25}{4}\right) - 9 \qquad c^2 = 16 \qquad \Rightarrow \qquad c = \pm 4$$

Let $y = mx + c$ be the required equation of the tangent then $y = \frac{5}{2}x \pm 4$

$$2y = 5x \pm 8$$

$$5x - 2y \pm 8 = 0 \qquad \text{Ans.}$$

Q.7: Find equations of common tangents to the given conics.

(i) $x^2 = 80y$ & $x^2 + y^2 = 81$

Solution:

$$x^2 = 80y \quad \text{.... (1)} \qquad x^2 + y^2 = 81 \quad \text{...... (2)}$$

Let $y = mx + c$ (3) be the required common tangent. Let a be radius of circle then (2) becomes $a^2 = 81$ Put in (1)

$$x^2 = 80 \quad (mx + c)$$

$$x^2 - 80\,mx - 80c = 0$$

For equal roots, we know that Disc = 0

$$b^2 - 4ac = 0$$

$$(-80\,m)^2 - 4(1)(-80\,c) = 0$$

$$80(80\,m^2 + 4c) = 0$$

$80\,m^2 + 4c = 0 \qquad c = -20m^2$

Condition of tangency for circle is $c^2 = a^2(1 + m^2)$ (4)

$(-20m^2)^2 = 81(1 + m^2)$

$400\,m^4 = 81 + 81m^2$

$400\,m^4 - 81m^2 - 81 = 0$

By Quadratic Formula

$$m^2 = \frac{-(-81) \pm \sqrt{(-81)^2 - 4(400)(-81)}}{2(400)}$$

$$= \frac{81 \pm \sqrt{136161}}{800} = \frac{9}{16}$$

$$m = \pm\frac{3}{4}$$

$$\therefore \quad c = -20\left(\frac{9}{16}\right) = \frac{-45}{4}$$

Putting values of m & c in y = mx + c

$$y = \pm\frac{3}{4}x - \frac{45}{4}$$

$$4y = \pm 3x - 45$$

$\pm 3x - 4y - 45 = 0$ Ans.

(ii) $\mathbf{y^2 = 16x}$ **&** $\mathbf{x^2 = 2y}$

Solution:

$y^2 = 16x$ .... (1) $\qquad x^2 = 2y$ .... (2)

$y^2 = 4ax$

$4a = 16$

$\boxed{a = 4}$

We know that condition of tangency for Parabola is $c = \dfrac{a}{m}$

$$c = \frac{4}{m}$$

Let $y = mx + c$ ...... (3) be required tangent

then $y = mx + \dfrac{4}{m}$ Putting value of y in (2)

$x^2 = 2(mx + \dfrac{4}{m}) \quad \Rightarrow \quad mx^2 = 2m^2x + 8$

$mx^2 - 2m^2x - 8 = 0$

For equal roots, we know that Disc = 0

i.e; $b^2 - 4ac = 0$

$$(-2m^2)^2 - 4(m)(-8) = 0$$
$$4m^4 + 32m = 0$$
$$4m(m^3 + 8) = 0$$
$$m = 0 \quad , \quad m^3 = -8 \quad , \quad m = -2$$

Equation of tangent is

$$y = mx + c$$
$$y = -2x + \frac{4}{-2}$$
$$y = -2x - 2$$
$$\boxed{2x + y + 2 = 0} \qquad \text{Ans.}$$

Q.8: Find the points of intersection of the given conics.

(i) $\mathbf{\dfrac{x^2}{18} + \dfrac{y^2}{8} = 1 \quad \& \quad \dfrac{x^2}{3} - \dfrac{y^2}{3} = 1}$

Solution:

$$\frac{x^2}{18} + \frac{y^2}{8} = 1 \qquad \& \qquad \frac{x^2}{3} - \frac{y^2}{3} = 1$$
$$8x^2 + 18y^2 = 144 \qquad x^2 - y^2 = 3 \;\; \text{........ (2)}$$
$$4x^2 + 9y^2 = 72 \text{ .... (1)} \qquad \text{(Dividing by 2)}$$

Multiplying Eq. (2) by 9 & add in (1)

$$9x^2 - 9y^2 = 27$$
$$\underline{4x^2 + 9y^2 = 72}$$
$$13x^2 = 99$$
$$x^2 = \frac{99}{13} \quad \Rightarrow \quad x = \pm\sqrt{\frac{99}{13}}$$

Put in (2)

$$\frac{99}{13} - y^2 = 3$$
$$\frac{99}{13} - 3 = y^2$$
$$\frac{99 - 39}{13} = y^2$$
$$y^2 = \frac{60}{13} \quad \Rightarrow \quad y = \pm\sqrt{\frac{60}{13}}$$

Points of intersection are $\left(\pm\sqrt{\frac{99}{13}}\ ,\ \pm\sqrt{\frac{60}{13}}\right)$ Ans.

(ii) $x^2 + y^2 = 8$ & $x^2 - y^2 = 1$

Solution:

$x^2 + y^2 = 8$ .... (1) $\qquad x^2 - y^2 = 1$ .... (2)

Adding (1) & (2)

$$\begin{array}{l} x^2 + y^2 = 8 \\ \underline{x^2 - y^2 = 1} \\ 2x^2 \quad = 9 \end{array}$$

$2x^2 = 9 \quad => \quad x^2 = \frac{9}{2} \quad => \quad x = \pm\frac{3}{\sqrt{2}}$

Put in (1) $\quad \frac{9}{2} + y^2 = 8$

$y^2 = 8 - \frac{9}{2}$

$y^2 = \frac{16-9}{2} = \frac{7}{2}$

$y = \pm\sqrt{\frac{7}{2}}$

Hence points of intersection are $\left(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{\frac{7}{2}}\right)$ Ans

(iii) $3x^2 - 4y^2 = 12$ & $3y^2 - 2x^2 = 7$

Solution:

$3x^2 - 4y^2 = 12$ .... (1)

$3y^2 - 2x^2 = 7$ .... (2)

Multiplying equation (1) by (2) & (2) by 3 and adding

$$\begin{array}{l} 6x^2 - 8y^2 = 24 \\ \underline{-6x^2 + 9y^2 = 21} \\ \qquad\quad y^2 = 45 \end{array}$$

$y^2 = 45 \quad => \quad y = \pm\sqrt{45}$

Put in (2)

$-2x^2 + 3(45) = 7$

$-2x^2 + 135 = 7$

$135 - 7 = 2x^2$

$128 = 2x^2$

$x^2 = 64 \quad => \quad x = \pm 8$

Hence points of intersection are

$(\pm 8\ ,\ \pm\sqrt{45})$ Ans.

(iv) $3x^2 + 5y^2 = 60$ and $9x^2 + y^2 = 124$

Solution:

$3x^2 + 5y^2 = 60$ .... (1) $\quad 9x^2 + y^2 = 124$ .... (2)

Multiplying (1) by (3) & Subtracting from (2)

$$\begin{array}{rcl} 9x^2 + \quad y^2 & = & 124 \\ -9x^2 \pm 15y^2 & = & -180 \\ \hline -14y^2 & = & -56 \end{array}$$

$y^2 = 4 \quad \Rightarrow \quad y = \pm 2$

Put in (1)

$9x^2 + 4 = 124$

$9x^2 = 120$

$x^2 = \dfrac{120}{9} = \dfrac{40}{3} \qquad x = \pm\sqrt{\dfrac{40}{3}}$

Hence points of intersection are $\left(\pm\sqrt{\dfrac{40}{3}} \; \pm 2 \right)$

EXERCISE 6.8

1. **Find an equation of each of the following with respect to new parallel axes obtained by shifting the origin to the indicated points.**

(i) $x^2 + 16y - 16 = 0, O'(0, 1)$

Solution. $x^2 + 16y - 16 = 0 \quad \ldots (1), \quad O'(0, 1) = (h, k)$

Equations of transformation are

$$x = X + h = X + 0 = X \quad , \quad y = Y + k = Y + 1$$

Substituting these values of x, y into (1), we have

$$X^2 + 16(Y + 1) - 16 = 0$$

or $\quad X^2 + 16Y + 16 - 16 = 0 \quad \Rightarrow \quad X^2 + 16Y = 0$

is the required transformed equation.

(ii) $4x^2 + y^2 + 16x - 10y + 37 = 0, O'(-2, 5)$

Solution. $4x^2 + y^2 + 16x - 10y + 37 = 0 \quad \ldots (1), \quad O'(-2, 5) = (h, k)$

Equations of transformation are

$$x = X + h = X - 2 \quad , \quad y = Y + k = Y + 5$$

Substituting these values of x, y into (1), we have

$$4(X-2)^2+(Y+5)^2+16(X-2)-10(Y+5)+37 = 0$$

or $\quad 4(X^2-4X+4)+(Y^2+10Y+25)+16(X-2)-10(Y+5)+37 = 0$

or $\quad 4X^2-16X+16+Y^2+10Y+25+16X-32-10Y-50+37 = 0$

or $\quad 4X^2+Y^2-4 = 0$ is the required transformed equation.

(iii) $\quad 9x^2+4y^2+18x-16y-11 = 0, \ O'(-1,2)$

Solution. $9x^2+4y^2+18x-16y-11 = 0 \quad \ldots (1), \qquad O'(-1,2)=(h,k)$

Equations of transformation are

$$x = X+h = X-1, \quad y = Y+k = Y+2$$

Putting these values of x and y into (1), we have

$$9(X-1)^2+4(Y+2)^2+18(X-1)-16(Y+2)-11 = 0$$

or $\quad 9(X^2-2X+1)+4(Y^2+4Y+4)+18(X-1)-16(Y+2)-11=0$

or $\quad 9X^2-18X+9+4Y^2+16Y+16+18X-18-16Y-32-11 = 0$

or $\quad 9X^2+4Y^2-36 = 0$ is the required transformed equation.

(iv) $\quad x^2-y^2+4x+8y-11 = 0, \ O'(-2,4)$

Solution. $x^2-y^2+4x+8y-11 = 0 \quad \ldots (1), \qquad O'(-2,4)=(h,k)$

Equations of transformation are

$$x = X+h = X-2, \quad y = Y+k = Y+4$$

Substituting these values of x and y into (1), we have

$$(X-2)^2-(Y+4)^2+4(X-2)+8(Y+4)-11 = 0$$

or $\quad X^2-4X+4-(Y^2+8Y+16)+4X-8+8Y+32-11 = 0$

or $\quad X^2-4X+4-Y^2-8Y-16+4X-8+8Y+32-11 = 0$

or $\quad X^2-Y^2+1 = 0$ is the required transformed equation.

(v) $\quad 9x^2-4y^2+36x+8y-4 = 0, \ O'(-2,1)$

Solution. $9x^2-4y^2+36x+8y-4 = 0 \quad \ldots (1) \quad O'(-2,1)=(h,k)$

Equations of transformation are

$$x = X+h = X-2, \quad y = Y+k = Y+1$$

Substituting these values of x and y into (1), we have

$$9(X-2)^2-4(Y+1)^2+36(X-2)+8(Y+1)-4 = 0$$

or $\quad 9(X^2-4X+4)-4(Y^2+2Y+1)+36(X-2)+8(Y+1)-4 = 0$

or $\quad 9X^2-36X+36-4Y^2-8Y-4+36X-72+8Y+8-4 = 0$

or $9X^2 - 4Y^2 - 36 = 0$ is the required transformed equation.

2. Find coordinates of the new origin (axes remaining parallel) so that first degree terms are removed from the transformed equation of each of the following. Also find the transformed equation.

(i) $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

Solution. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$ ... (1)

Let the coordinates of the new origin be $O'(h, k)$.

Equations of transformation are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$3(X+h)^2 - 2(Y+k)^2 + 24(X+h) + 12(Y+k) + 24 = 0$$

or $3(X^2 + 2Xh + h^2) - 2(Y^2 + 2kY + k^2) + 24(X+h) + 12(Y+k) + 24 = 0$

or $3X^2 + 6hX + 3h^2 - 2Y^2 - 4kY - 2k^2 + 24X + 24h + 12Y + 12k + 24 = 0$

or $3X^2 - 2Y^2 + X(6h + 24) + Y(-4k + 12) + 3h^2 + 24h - 2k^2 + 12k + 24 = 0$ ...(2)

To remove first degree terms, we have

$$6h + 24 = 0 \Rightarrow h = -4$$

and $-4k + 12 = 0 \Rightarrow k = 3$

New origin is $O'(-4, 3)$. Putting $h = -4$ and $k = 3$ into (2), the transformed equation is

$$3X^2 - 2Y^2 + X[6(-4) + 24] + Y[-4(3) + 12] + 3(-4)^2 + 24(-4) - 2(3)^2 + 12(3) + 24 = 0$$

or $3X^2 - 2Y^2 + X(-24 + 24) + Y(-12 + 12) + 3(16) - 96 - 2(9) + 36 + 24 = 0$

or $3X^2 - 2Y^2 + X(0) + Y(0) + 48 - 96 - 18 + 36 + 24 = 0$

or $3X^2 - 2Y^2 - 6 = 0$

which is the required transformed equation with new origin at $(-4, 3)$

(ii) $25x^2 + 9y^2 + 50x - 36y - 164 = 0$

Solution. $25x^2 + 9y^2 + 50x - 36y - 164 = 0$ ... (1)

Let the coordinates of the new origin be $O'(h, k)$. Equations of transformed are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$25(X+h)^2+9(Y+k)^2+50(X+h)-36(Y+k)-164 = 0$$

or $\quad 25(X^2+2Xh+h^2)+9(Y^2+2Yk+k^2)+50X+50h-36Y-36k-164=0$

or $\quad 25X^2+50Xh+25h^2+9Y^2+18Yk+9k^2+50X+50h-36Y-36k-164=0$

or $25X^2+9Y^2+(50h+50)X+(18k-36)Y+25h^2+9k^2+50h-36k-164=0 \quad ...(2)$

To reove the first degree terms, we put

$$50h+50 = 0 \Rightarrow 50h = -50 \quad \Rightarrow \quad h = -1$$

and $\quad 18k-36 = 0 \Rightarrow 18k = 36 \quad \Rightarrow \quad k = 2$

$\therefore$ New origin is $O'(-1, 2)$.

Putting $h = -1$, $k = 2$ into (2), the transformed equation is

$$25X^2+9Y^2+(-50+50)X+(36-36)Y+25(-1)^2+9(2)^2+50(-1)$$
$$-36(2)-164 = 0$$

or $\quad 25X^2+9Y^2+(0)X+(0)Y+25(1)+9(4)-50-72-164 = 0$

or $\quad 25X^2+9Y^2+25+36-50-72-164 = 0$

or $\quad 25X^2+9Y^2-225 = 0$

which is the required equation with new origin at $(-1, 2)$.

(iii) $\quad x^2-y^2-6x+2y+7 = 0$

Solution. $x^2-y^2-6x+2y+7 = 0 \quad ... (1)$

Let the coordinates of the new origin be $O'(h,k)$. Equations of transformation are

$$x = X+h \quad , \quad y = Y+k$$

Substituting these values of x, y into (1), we get

$$(X+h)^2-(Y+k)^2-6(X+h)+2(Y+k)+7 = 0$$

or $\quad X^2+2Xh+h^2-(Y^2+2Yk+k^2)-6X-6h+2Y+2k+7 = 0$

or $\quad X^2-Y^2+X(2h-6)+Y(-2k+2)+h^2-k^2-6h+2k+7 = 0 \quad ... (2)$

To remoe the first gdegree terms, we have

$$2h-6 = 0 \quad \Rightarrow \quad h = 3$$

and $\quad -2k+2 = 0 \quad \Rightarrow \quad k = 1$

New origin is $O'(3, 1)$. Putting $h = 3, k = 1$ into (2) the transformed equation is

$$X^2-Y^2+X(0)+Y(0)+(3)^2-(1)^2-6(3)+2(1)+7 = 0$$

or $X^2 - Y^2 + 9 - 1 - 18 + 2 + 7 = 0$

or $X^2 - Y^2 - 1 = 0$

which is the required equation with new origin at (3, 1).

3. In each of the following, find an equation referred to the new axes obtained by rotation of axes about the origin through the given angle:

(i) $xy = 1, \ \theta = 45^\circ$

Solution. Here $\theta = 45^\circ$. Equations of transformation are

$$x = X\cos 45^\circ - Y\sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X\sin 45^\circ + Y\cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these values for x, y into the given equation, we have

$$\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) = 1$$

or $\dfrac{X^2 - Y^2}{2} = 1$ or $X^2 - Y^2 = 2$

is the required transformed equation

(ii) $7x^2 - 8xy + y^2 - 9 = 0, \ \theta = \tan^{-1} 2$

Solution. Here $\theta = \tan^{-1} 2 \Rightarrow \tan\theta = 2 = \dfrac{2}{1}$

$$\Rightarrow \text{base} = 2, \perp = 1 \text{ and hypotenuse} = \sqrt{4+1} = \sqrt{5}$$

$\therefore \sin\theta = \dfrac{2}{\sqrt{5}}$ and $\cos\theta = \dfrac{1}{\sqrt{5}}$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Using the above values

$$x = X \cdot \frac{1}{\sqrt{5}} - Y \cdot \frac{2}{\sqrt{5}} = \frac{X - 2Y}{\sqrt{5}}$$

$$y = X \cdot \frac{2}{\sqrt{5}} + Y \cdot \frac{1}{\sqrt{5}} = \frac{2X + Y}{\sqrt{5}}$$

Substituting these values for x, y into the given equation, we have

$$7\left(\frac{X-2Y}{\sqrt{5}}\right)^2-8\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right)+\left(\frac{2X+Y}{\sqrt{5}}\right)^2-9=0$$

By multiplying by 5, we get

$$7\left(\frac{X^2-4XY+4Y^2}{5}\right)-8\left(\frac{2X^2-3XY \quad 2Y^2}{5}\right)+\left(\frac{4X^2+4XY+Y^2}{5}\right)-9=0$$

or $\quad 7X^2-28XY+28Y^2-16X^2+24XY+16Y^2+4X^2+4XY+Y^2-45=0$

or $\quad -5X^2+45Y^2-45=0$

or $\quad X^2-9Y^2+9=0$

which is the required transformed equation.

(iii) $\quad 9x^2+12xy+4y^2-x-y=0\ ,\ \theta=\tan^{-1}\frac{2}{3}$

Solution. Here $\theta=\tan^{-1}\frac{2}{3} \Rightarrow \tan\theta=\frac{2}{3}$

$\Rightarrow$ base = 3, $\perp = 2$ and hypotenuse = $\sqrt{9+4}=\sqrt{13}$

$\therefore \quad \sin\theta=\frac{2}{\sqrt{13}}$ and $\cos\theta=\frac{3}{\sqrt{13}}$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Putting values, we have

$$x = X\cdot\frac{3}{\sqrt{13}} - Y\cdot\frac{2}{\sqrt{13}} = \frac{3X-2Y}{\sqrt{13}}$$

$$y = X\cdot\frac{2}{\sqrt{13}} + Y\cdot\frac{3}{\sqrt{13}} = \frac{2X+3Y}{\sqrt{13}}$$

Substituting these values for x, y in to the given equations, we have

$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2+12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right)+4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2-\left(\frac{3X-2Y}{\sqrt{13}}\right)$$
$$-\left(\frac{3X+2Y}{\sqrt{13}}\right)=0$$

or $\quad 9\left(\frac{9X^2-12XY+4Y^2}{13}\right)+12\left(\frac{6X^2+5XY-6Y^2}{13}\right)$

$$+ 4\left(\frac{4X^2 + 12XY + 9Y^2}{13}\right) - \left(\frac{3X - 2Y}{\sqrt{13}}\right) - \left(\frac{2X + 3Y}{\sqrt{13}}\right) = 0$$

or $\quad 81X^2 - 108XY + 36Y^2 + 72X^2 + 60XY - 72Y^2 + 16X^2 + 48XY + 36Y^2$

$$- 3\sqrt{13}\,X + 2\sqrt{13}\,Y - 2\sqrt{13}\,X - 3\sqrt{13}\,Y = 0 .$$

or $\quad 169X^2 - 5\sqrt{13}\,X - \sqrt{13}\,Y = 0$

or $\quad 13\sqrt{13}\,X^2 - 5X - Y = 0$

is the required transformed equation

(iv) $\quad x^2 - xy + y^2 - 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0,\ \theta = 45^\circ$

Solution. Here $\theta = 45^\circ$

Equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}$$

Putting values, ewe have

$$x = X\cos 45^\circ - Y\sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X\sin 45^\circ + Y\cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these expressions for x, y into the given equation, we have

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 2\sqrt{2}\left(\frac{X - Y}{\sqrt{2}}\right)$$

$$- 2\sqrt{2}\left(\frac{X + Y}{\sqrt{2}}\right) + 2 = 0$$

or $$\left(\frac{X^2 - 2XY + Y^2}{2}\right) - 2\left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right)$$

$$- 2(X - Y) - 2(X + Y) + 2 = 0$$

(by $\times 2$): $X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 4X + 4Y - 4X - 4Y + 4 = 0$

or $\quad 4Y^2 - 8X + 4 = 0 \quad$ or $\quad Y^2 - 2X + 1 = 0$

which is the required transformed equation.

4. Find measure of the angle through which the axes be rotated so that the product term XY is removed from the transformed equation. Also find the transformed equation.

(i) $2x^2 + 6xy + 10y^2 - 11 = 0$

Solution. Let the axes be rotated through an angle θ.

Equations of transformation are

$$x = X\cos\theta - Y\sin\theta \quad , \quad y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation, we get

$$2(X\cos\theta - Y\sin\theta)^2 + 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ 10(X\sin\theta + Y\cos\theta)^2 - 11 = 0$$

or $$2(X^2\cos^2\theta - 2XY\sin\theta\cos\theta + Y^2\sin^2\theta)$$
$$+ 6(X^2\sin\theta\cos\theta - Y^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta)$$
$$+ 10(X^2\sin^2\theta + 2XY\sin\theta\cos\theta + Y^2\cos^2\theta) - 11 = 0$$

or $$X^2(2\cos^2\theta + 6\sin\theta\cos\theta + 10\sin^2\theta)$$
$$+ XY(-4\sin\theta\cos\theta + 6\cos^2\theta - 6\sin^2\theta + 20\sin\theta\cos\theta)$$
$$+ Y^2(2\sin^2\theta - 6\sin\theta\cos\theta + 10\cos^2\theta) - 11 = 0 \qquad \text{(i)}$$

Since this equation is to be free from the product term, i.e.;

$$-4\sin\theta\cos\theta + 6\cos^2\theta - 6\sin^2\theta + 20\sin\theta\cos\theta = 0$$

or $$-6\sin^2\theta + 16\sin\theta\cos\theta + 6\cos^2\theta = 0$$

or $$3\tan^2\theta - 8\tan\theta - 3 = 0 \qquad \{\text{by} \div \cos^2\theta\}$$

$$\therefore \quad \tan\theta = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-3)}}{2(3)} = \frac{8 \pm \sqrt{64+36}}{6}$$

$$= \frac{8 \pm \sqrt{100}}{6} = \frac{8 \pm 10}{6} = \frac{8+10}{6}, \frac{8-10}{6} = \frac{18}{6}, \frac{-2}{6}$$

$$= 3, \frac{-1}{3} \Rightarrow \tan\theta = 3 \text{ (as } \theta \text{ is in the first quadrant)}$$

$\Rightarrow$ base = 1, $\perp = 3$ and hypotenuse $= \sqrt{1+9} = \sqrt{10}$

$$\therefore \quad \sin\theta = \frac{3}{\sqrt{10}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{10}}$$

Substituting $\sin\theta = \dfrac{3}{\sqrt{10}}$ and $\cos\theta = \dfrac{1}{\sqrt{10}}$ in (i), then

$$X^2\left(\left(2\left(\frac{1}{\sqrt{10}}\right)\right)^2 + 6\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{3}{\sqrt{10}}\right)^2\right)$$
$$+ XY\left(-4\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 6\left(\frac{1}{\sqrt{10}}\right)^2 - 6\left(\frac{3}{\sqrt{10}}\right)^2 + 20\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right)\right)$$

$$+ Y^2\left(2\left(\frac{3}{\sqrt{10}}\right)^2 - 6\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{1}{\sqrt{10}}\right)^2\right) - 11 = 0$$

or $$X^2\left(\frac{2}{10} + \frac{18}{10} + \frac{90}{10}\right) + XY\left(-\frac{12}{10} + \frac{6}{10} - \frac{54}{10} + \frac{60}{10}\right)$$

$$+ Y^2\left(\frac{18}{10} - \frac{18}{10} + \frac{10}{10}\right) - 1 = 0$$

or $\quad 11X^2 + Y^2 - 11 = 0$

(ii) $\quad xy + 4x - 3y - 10 = 0 \quad \ldots (1)$

Solution. Let the axes be rotated through an angle θ.

Equation of transformation are

$$x = X\cos\theta - Y\sin\theta \ ; \ y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation (1), we have

$$(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 4(X\cos\theta - Y\sin\theta)$$

$$+ XY\cos^2\theta - XY\sin^2\theta - 3(X\sin\theta + Y\cos\theta) - 10 = 0$$

$$X^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta + 4X\cos\theta$$

$$-4Y\sin\theta - 3X\sin\theta - 3Y\cos\theta - 10 = 0$$

or $$X^2\sin\theta\cos\theta + XY(\cos^2\theta - \sin^2\theta) - Y^2\sin\theta\cos\theta + X(4\cos\theta - 3\sin\theta)$$

$$-Y(4\sin\theta + 3\cos\theta) - 10 = 0 \quad \ldots (2)$$

Since this equation is to be free from the product term XY

i.e., $\quad \cos^2\theta - \sin^2\theta = 0$

or $\quad \cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°$

Thus axes be rotated through an angle of $45°$. So that XY term is removed from the transformed equation.

Setting $\theta = 45°$ in (i), the transformed equation is

$$X^2\sin 45°\cos 45° + XY(\cos^2 45° - \sin 2\,45°) - Y^2\sin 45°\cos 45°$$

$$+ X(4\cos 45° - 3\sin 45°) - Y(4\sin 45° + 3\cos 45°) - 10 = 0$$

$$X^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + XY(0) - Y^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + X\left(4 \cdot \frac{1}{\sqrt{2}} - 3 \cdot \frac{1}{\sqrt{2}}\right)$$

$$- Y\left(4 \cdot \frac{1}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{2}}\right) - 10 = 0$$

or $$X^2 \cdot \frac{1}{2} - Y^2 \cdot \frac{1}{2} + X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{7}{\sqrt{2}} - 10 = 0$$

or $\quad X^2 - Y^2 + \sqrt{2}\, X - 7\sqrt{2}\, Y - 20 = 0$

is the requied equation.

(iii) $\quad 5x^2 - 6xy + 5y^2 - 8 = 0 \quad \ldots (1)$

Solution. Let the axes be rotated through an angle θ.

Equation of transformation are

$$x = X\cos\theta - Y\sin\theta \; ; \; y = X\sin\theta + Y\cos\theta$$

Substituting into the given equation, we get

$$5(X\cos\theta - Y\sin\theta)^2 - 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 5(X\sin\theta + Y\cos\theta)^2 - 8 = 0$$

or $$5(X^2\cos^2\theta - 2XY\sin\theta\cos\theta + Y^2\sin^2\theta) - 6(X^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta) + 5(X^2\sin^2\theta + 2XY\sin\theta\cos\theta + Y^2\cos^2\theta) - 8 = 0$$

or $$X^2(5\cos^2\theta - 6\sin\theta\cos\theta + 5\sin^2\theta) + XY(-10\sin\theta\cos\theta - 6\cos^2\theta + 6\sin^2\theta + 10\sin\theta\cos\theta) + Y^2(5\sin^2\theta + 6\sin\theta\cos\theta + 5\cos^2\theta) - 8 = 0$$

Since this equation is to be free from the product term XY

i.e., $\quad -6\cos^2\theta - 6\sin^2\theta = 0$

or $\quad \sin^2\theta = \cos^2\theta \;\Rightarrow\; \tan^2\theta = 1 \;\Rightarrow\; \tan\theta = 1$

$\Rightarrow \quad \theta = 45^\circ$ (as θ is in the first quadrant)

Setting $\theta = 45^\circ$ in (1), the transformed equation is

$$X^2\left(5\cdot\left(\frac{1}{\sqrt{2}}\right)^2 - 6\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 5\cdot\left(\frac{1}{\sqrt{2}}\right)^2\right) + XY\left(-10\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - 6\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 6\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 10\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right) + Y^2\left(5\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + 6\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 5\cdot\left(\frac{1}{\sqrt{2}}\right)^2\right) - 8 = 0$$

or $$X^2\left(\frac{5}{2} - \frac{6}{2} + \frac{5}{2}\right) + XY(0) + Y^2\left(\frac{5}{2} + \frac{6}{2} + \frac{5}{2}\right) - 8 = 0$$

or $\quad 2X^2 + 8Y^2 - 8 = 0 \;\Rightarrow\; X^2 + 4Y^2 - 4 = 0$

which is the required equation.

EXERCISE 6.9

1. By a rotation of axes, eliminate the xy-term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2 - 4xy + y^2 - 6 = 0$

Solution. $4x^2 - 4xy + y^2 - 6 = 0$ ... (I)

Here $a = 4$, $b = 1$, $2h = -4$ the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-1} = -\frac{4}{3} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{4}{3}$$

$$6\tan\theta = 4\tan^2\theta - 4 \Rightarrow 4\tan^2\theta - 6\tan\theta - 4 = 0$$

$$2\tan^2\theta - 3\tan q - 2 = 0$$

$$\tan\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$= 2, -\frac{1}{2} \Rightarrow \tan\theta = 2 \text{ (as } \theta \text{ is in the first quadrant)}$$

Now $\tan\theta = 2 = \frac{2}{1} \Rightarrow$ base $= 1$, $\perp = 2$, hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore$ $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = \frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y \\ y &= X\sin\theta + Y\cos\theta = \frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y \end{aligned}\right\} \quad \dots (2)$$

Substituting these expressions for x and y into (1), we get

$$4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)^2 - 4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)\left(\frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y\right) + \left(\frac{2}{\sqrt{5}}X + \frac{2}{\sqrt{5}}Y\right)^2 - 6 = 0$$

$$4\left(\frac{1}{5}X^2 - \frac{4}{5}XY + \frac{4}{5}Y^2\right) - 4\left(\frac{2}{5}X^2 - \frac{3}{5}XY + \frac{2}{5}Y^2\right) + \left(\frac{4}{5}X^2 - \frac{4}{5}XY + \frac{1}{5}Y^2\right) - 6 = 0$$

$$\left(\frac{4}{5} - \frac{8}{5} + \frac{4}{5}\right)X^2 + \left(-\frac{16}{5} + \frac{12}{5} + \frac{4}{5}\right)XY + \left(\frac{16}{5} + \frac{8}{5} + \frac{1}{5}\right)Y^2 - 6 = 0$$

$$25Y^2 - 30 = 0 \Rightarrow Y^2 = \frac{6}{5} \Rightarrow Y = \pm\sqrt{\frac{6}{5}}$$

represents a pair of lines. To find their equations in xy - plane, we have

From (2), we have

$$X - 2Y = \sqrt{5}\,x \qquad (3)$$

$$2X + Y = \sqrt{5}\,y \qquad (4)$$

Multiplying (3) by 2, we get

$$2X - 4Y = 2\sqrt{5}\,x \qquad (5)$$

Subtracting equation (5) from (6), we get

$$5Y = \sqrt{5}\ y - 2\sqrt{5}\ x \quad \Rightarrow \quad Y = \frac{\sqrt{5}}{5}(y - 2x) = -\frac{1}{\sqrt{5}}(2x - y)$$

$$\pm\sqrt{\frac{6}{5}} = -\frac{1}{\sqrt{5}}(2x - y) \quad \Rightarrow \quad \pm\sqrt{6} = -(2x - y)$$

$$2x - y \pm \sqrt{6} = 0 \quad \Rightarrow \quad 2x - y + \sqrt{6} = 0\ ,\ 2x - y - \sqrt{6} = 0$$

(ii) **Identify :** $x^2 - 2xy + y^2 - 8x - 8y = 0$

Solution. $x^2 - \ xy + y^2 - 8x - 8y = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = -2$ the angle θ though which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-2}{1 - 1} = \infty \quad \Rightarrow \quad 2\theta = 90^\circ \quad \Rightarrow \quad \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\,.\,\frac{1}{\sqrt{2}} - Y\,.\,\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\,.\,\frac{1}{\sqrt{2}} + Y\,.\,\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{aligned}\right] \ldots \quad (2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 8\left(\frac{X - Y}{\sqrt{2}}\right) - 8\left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) - \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2X.Y + Y^2)$$

$$-\frac{8}{\sqrt{2}}(X - Y) - \frac{8}{\sqrt{2}}(X + Y) = 0$$

$$X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 8\sqrt{2}\ X$$

$$+ 8\sqrt{2}\ Y - 8\sqrt{2}\ X - 8\sqrt{2}\ Y = 0$$

$$4Y^2 - 16\sqrt{2}\ X = 0 \quad \Rightarrow \quad Y^2 = 4\sqrt{2}\ X \qquad (3)$$

which represents a parabola. In xy-plane, we have

From 2i), we have

$$X - Y = \sqrt{2}\ x \qquad (4)$$

and $\quad X + Y = \sqrt{2}\ y \qquad (5)$

Adding (3) and (4), we get

$$2X = \sqrt{2}\,(x+y) \quad \Rightarrow \quad X = \frac{1}{\sqrt{2}}(x+y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x+y) - Y = \sqrt{2}\,x \quad \Rightarrow \quad Y = -\sqrt{2}\,x + \frac{1}{\sqrt{2}}(x+y)$$

$$= \frac{-2x+x+y}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y-x)$$

Thus $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(y-x)$

Elements of the parabola are

Focus of (3) is $Y = 0$, $X = \sqrt{2}$

i.e., $\frac{1}{\sqrt{2}}(x+y) = \sqrt{2}$ and $\frac{1}{\sqrt{2}}(y-x) = 0$

$x + y = 2$ and $y - x = 0$

Adding: $x + y = 2$

$\underline{-x + y = 0}$

$2y = 2 \Rightarrow y = 1$

Put $y = 1$ in $x + y = 2$, we get

$x + 1 = 2 \Rightarrow x = 1$

$(1, 1)$ is the focus of (1)

Vertex of (3) is $X = 0$, $Y = 0$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0 \Rightarrow x + y = 0$

and $\frac{1}{\sqrt{2}}(y-x) = 0 \Rightarrow -x + y = 0$

Solving, we get $x = 0$, $y = 0$

Vertex: $(0, 0)$ is the vertex of (1).

Axis: $Y = 0$ i.e., $\frac{1}{\sqrt{2}}(y-x) = 0 \Rightarrow x - y = 0$

Equation of directrix of (3) is

$$X = -\sqrt{2} \quad \Rightarrow \quad \frac{x+y}{\sqrt{2}} = -\sqrt{2} \quad \Rightarrow \quad \frac{x+y}{\sqrt{2}} + \sqrt{2} = 0$$

$x + y + 2 = 0$ is the directrix in xy-coodinate system.

(iii) Identify: $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$

Solution. $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = 2$ the angle θ through which axes be rotated to given by

$$\tan 2q = \frac{2h}{a-b} = \frac{2}{1-1} = \frac{2}{0} \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\,.\,\frac{1}{\sqrt{2}} - Y\,.\,\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\,.\,\frac{1}{\sqrt{2}} + Y\,.\,\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right\} \quad ...(2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)$$

$$-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) + \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2XY + Y^2)$$

$$+ 2(X - Y) - 2(X + Y) + 2 = 0$$

$$X^2 - 2XY + Y^2 + 2X^2 - 2Y^2 + X^2 + 2XY + Y^2 + 4X - 4Y - 4X$$

$$- 4Y + 4 = 0$$

$$4X^2 - 8Y + 4 = 0 \quad \Rightarrow \quad X^2 - 2Y + 1 = 0$$

$$\therefore X^2 = 2\left(Y - \frac{1}{2}\right) \quad ... (3)$$

Which represents a parabola.

From (ii), we have

$$X - Y = \sqrt{2}\,x \quad ... (4)$$

and $$X + Y = \sqrt{2}\,y \quad ... (5)$$

Adding (4) and (5), we get

$$2X = \sqrt{2}\,x + \sqrt{2}\,y \Rightarrow 2X = \sqrt{2}\,(x + y) \Rightarrow X = \frac{1}{\sqrt{2}}(x + y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x+y) - Y = \sqrt{2}\,x \implies Y = \frac{1}{\sqrt{2}}(x+y) - \sqrt{2}\,x$$

$$= \frac{x+y-2x}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y-x)$$

Thus $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(y-x)$

Elements of parabola are

Focus of (3) is $X = 0,\ Y - \frac{1}{2} = \frac{1}{2} \implies Y = 1$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$ and $\frac{1}{\sqrt{2}}(y-x) = 1$

i.e., $x+y = 0$ and $y - x = \sqrt{2}$

Adding: $x + y = 0$

$$\underline{-x+y = \sqrt{2}}$$

$$2y = \sqrt{2} \implies y = \frac{1}{\sqrt{2}}$$

Put $y = \frac{1}{\sqrt{2}}$ in $x+y=0 \implies x = -y = -\frac{1}{\sqrt{2}}$

Focus: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the focus of (1)

Vertex of (3) is $X = 0,\ Y - \frac{1}{2} = 0 \implies Y = \frac{1}{2}$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0 \implies x+y = 0$

and $\frac{1}{\sqrt{2}}(y-x) = \frac{1}{2} \implies y - x = \frac{1}{\sqrt{2}}$

Solving, we get $x = -\frac{1}{2\sqrt{2}}$, $y = \frac{1}{2\sqrt{2}}$

Vertex $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ is the vertex of (1)

Axis $X = 0$ i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$

$\implies x + y = 0$

Equation of directrix of (3) is

$$Y - \frac{1}{2} = -\frac{1}{2} \Rightarrow \frac{y-x}{\sqrt{2}} = 0 \Rightarrow y - x = 0 \Rightarrow x - y = 0$$

is the directrix in xy–coordinate system.

(iv) $x^2 + xy + y^2 - 4 = 0$

Solution. $x^2 + xy + y^2 - 4 = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = 1$ the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{1}{1-1} = \frac{1}{0} = \infty \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right] \quad \text{(ii)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 4 = 0$$

$$\left(\frac{X^2 - 2XY + Y^2}{2}\right) + \left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right) - 4 = 0$$

$$X^2 - 2XY + Y^2 + X^2 - Y^2 + X^2 + 2XY + Y^2 - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{X^2}{8/3} + \frac{Y^2}{8} = 1 \quad \text{... (3)}$$

Which represents an ellipse.

From (2), we have

$$X - Y = \sqrt{2}\, x \quad \text{... (4)}$$

$$X + Y = \sqrt{2}\, y \quad \text{... (5)}$$

Adding (4) and (5) $2X = \sqrt{2}\, x + \sqrt{2}\, y \Rightarrow X = \frac{1}{\sqrt{2}}(x + y)$

Subtracting (iv) and (v):

$$-2Y = \sqrt{2}\, x - \sqrt{2}\, y \Rightarrow Y = \frac{1}{\sqrt{2}}(y - x)$$

Elements of ellipse are

Centre of (3), is $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x + y) = 0 \implies x + y = 0$$

and $\frac{1}{\sqrt{2}}(y - x) = 0 \implies -x + y = 0 \implies x = 0 , y = 0$

Hence $C(0, 0)$ is the centre of (1)

Vertices of (3) are: $X = 0$, $Y = \pm 2\sqrt{2}$

$$X = 0 \implies \frac{1}{\sqrt{2}}(x + y) = 0 \implies x + y = 0$$

and $Y = \pm\sqrt{2} \implies \frac{1}{\sqrt{2}}(y - x) = \pm 2\sqrt{2} \implies -x + y = \pm 4$

$\implies$ $x + y = 0$

$-x + y = 4$

Adding: $2y = 4 \implies y = 2$

$\implies x = -y = -2$

$(-2, 2)$

and

$x + y = 0$

$-x + y = -4$

Adding: $2y = -4 \implies y = -2$

$x = -y = -(-2) = 2$

$(2, -2)$

$(-2, 2), (2, -2)$, as vertices of (1)

Equation of major axis: $X = 0 \implies x + y = 0$

Equation of minor axis: $Y = 0 \implies x - y = 0$

Ecentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{8 - \frac{8}{3}}}{2\sqrt{2}} = \dfrac{4}{2\sqrt{6}} = \dfrac{2}{\sqrt{6}}$

Foci of (3) are $X = 0$, $Y = \pm\sqrt{8}\left(\dfrac{2}{\sqrt{6}}\right)$

i.e., $\dfrac{1}{\sqrt{2}}(x + y) = 0$, $-\dfrac{1}{\sqrt{2}}(x - y) = \pm\sqrt{8}\left(\dfrac{2}{\sqrt{6}}\right)$

$\implies$ $x + y = 0$,

$-x + y = \dfrac{2\sqrt{8}}{\sqrt{3}}$

Adding: $2y = \dfrac{2\sqrt{8}}{\sqrt{3}} \implies y = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$x + y = 0$

$-x + y = \dfrac{-2\sqrt{8}}{\sqrt{3}}$

Adding: $2y = -\dfrac{2\sqrt{8}}{3} \implies y = \dfrac{-2\sqrt{2}}{3}$

$x + y = 0 \Rightarrow x = -y = -\dfrac{2\sqrt{2}}{3}$

$x + y = 0$

$\Rightarrow \quad x = -y = -\left(\dfrac{-2\sqrt{2}}{3}\right) = \dfrac{2\sqrt{2}}{3}$

Hence $\left(\dfrac{-2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}\right)$ and $\left(\dfrac{2\sqrt{2}}{3}, \dfrac{-2\sqrt{2}}{3}\right)$ are the foci of (1).

(v) $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$

Solution. $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$ ... (1)

Here $a = 7$, $b = 13$, $2h = -6\sqrt{3}$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-6\sqrt{3}}{7-13} = \frac{-6\sqrt{3}}{-6} = \sqrt{3}$$

$\Rightarrow \quad 2\theta = 60^\circ \quad \Rightarrow \quad \theta = 30^\circ$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 30^\circ - Y\sin 30^\circ = X \cdot \frac{\sqrt{3}}{2} - Y \cdot \frac{1}{2} = \frac{\sqrt{3}\,x - Y}{2} \\ y &= X\sin 30^\circ + Y\cos 30^\circ = X \cdot \frac{1}{2} + Y\,\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}\,Y}{2} \end{aligned}\right] \quad \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$7\left(\frac{\sqrt{3}\,X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}\,X - Y}{2}\right)\left(\frac{X + \sqrt{3}\,Y}{2}\right) + 13\left(\frac{\sqrt{3}\,X + Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 - 2\sqrt{3}\,XY + Y^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}\,X^2 + 2XY - \sqrt{3}\,Y^2}{4}\right) + 13\left(\frac{X^2 + 2\sqrt{3}\,Y + 3Y^2}{4}\right) - 16 = 0$$

$$\frac{(21X^2 - 14\sqrt{3}\,XY + 7Y^2)}{4} - \frac{(18X^2 + 12\sqrt{3}\,XY - 18Y^2)}{4} + \frac{(13X^2 + 26\sqrt{3}\,XY + 39Y^2)}{4} - 16 = 0$$

$$21X^2 - 14\sqrt{3}\,XY + 7Y^2 - 18Y^2 - 12\sqrt{3}\,XY + 18Y^2 + 13X^2 + 26\sqrt{3}\,XY + 39Y^2 - 64 = 0$$

$$16X^2 + 64Y^2 = 64$$

$$\Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{1} = 1 \quad \dots (3)$$

Which represents an ellipse.

From (ii), we have

$$\sqrt{3}\, X - Y = 2x \quad \dots (4)$$

$$X + \sqrt{3}\, Y = 2y \quad \dots (5)$$

Multiplying (iv) by $\sqrt{3}$, we get

$$3X - \sqrt{3}\, Y = 2\sqrt{3}\, x \quad \dots (6)$$

Adding (5) and (6), we get

$$4X = 2y + 2\sqrt{3}\, x \quad \Rightarrow \quad X = \frac{1}{2}(\sqrt{3}\, x + y)$$

Multiplying (5) by $\sqrt{3}$, we get

$$\sqrt{3}\, X + 3Y = 2\sqrt{3}\, Y \quad \dots (7)$$

Subtracting (7) from (4), we get

$$-4Y = 2\sqrt{3}\, y - 2x \quad \Rightarrow \quad Y = \frac{1}{2}\left(x - \sqrt{3}\, y\right)$$

Thus $X = \frac{1}{2}(\sqrt{3}\, x + y)$ and $Y = \frac{1}{2}\left(x - \sqrt{3}\, y\right)$

Elements of ellipse are

Centre of (3) is $X = 0,\ Y = 0$

$$X = 0 \quad \left(\ \frac{1}{2}(\sqrt{3}\, x + y) = 0 \quad \left(\ \sqrt{3}\, x + y = 0\right.\right.$$

$$Y = 0 \quad \left(\ \frac{1}{2}(x - \sqrt{3}\, y) = 0 \quad \left(\ x - \sqrt{3}\, y = 0\right.\right.$$

Solving these equations, we get $x = 0,\ y = 0$

Hence, $C\ (0, 0)$ centre of (1).

Vertices of (3) are $X = \pm a = \pm 2$ and $Y = 0$

$$X = \pm 2 \quad \Rightarrow \quad \frac{1}{2}(\sqrt{3}\, x + y) = \pm 2 \quad \Rightarrow \quad \sqrt{3}\, x + y = \pm 4$$

$$Y = 0 \quad \Rightarrow \quad \frac{1}{2}(x - \sqrt{3}\, y) = 0 \quad \Rightarrow \quad x - \sqrt{3}\, y = 0$$

$$\sqrt{3}\, x + y = 4 \quad \dots (4)$$

$$x - \sqrt{3}\, y = 0 \quad \dots (5)$$

Multiplying (4) by $\sqrt{3}$ and adding these equations, we get

$$\sqrt{3}\, x + y = -4 \quad \dots (6)$$

$$x - \sqrt{3}\, y = 0 \quad \dots (7)$$

Multiplying (6) by $\sqrt{3}$ and adding these equations, we get

$4x = 4\sqrt{3} \Rightarrow x = \sqrt{3}$

(5) $\Rightarrow \sqrt{3}\,y = x = \sqrt{3} \Rightarrow y = 1$

$4x = -4\sqrt{3} \Rightarrow x = -\sqrt{3}$

(7) $\Rightarrow \sqrt{3}\,y = x = -\sqrt{3} \Rightarrow y = -1$

$(\sqrt{3}, 1), (-\sqrt{3}, -1)$, as vertices of (1)

Eccentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{4-1}}{2} = \dfrac{\sqrt{3}}{2}$

Foci of (3) are: $X = \pm\sqrt{3},\ Y = 0$

$$X = \pm\sqrt{3} \Rightarrow \frac{1}{2}(\sqrt{3}\,x + y) = \pm\sqrt{3} \Rightarrow \sqrt{3}\,x + y = \pm 2\sqrt{3}$$

$$Y = 0 \Rightarrow \frac{1}{2}(\sqrt{3}\,y - x) = 0 \Rightarrow -x + \sqrt{3}\,y = 0$$

$\sqrt{3}\,x + y = 2\sqrt{3}$ ... (8)

$-x + \sqrt{3}\,y = 0$ ... (9)

Multiplying (9) by $\sqrt{3}$ and adding these equations, we get

$4y = 2\sqrt{3} \Rightarrow y = \dfrac{\sqrt{3}}{2}$

(9) $\Rightarrow$ x \r(3) y = \r(3) . \f(\r(3),2) = \f(3,2)

$\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$

$\sqrt{3}\,x + y = -2\sqrt{3}$ ... (10)

$-x + \sqrt{3}\,y = 0$ ... (11)

Multiplying (11) by $\sqrt{3}$ and adding these equations, we get

$4y = -2\sqrt{3} \Rightarrow y = \dfrac{-\sqrt{3}}{2}$

(11) $\Rightarrow$ x = \r(3) y

$= \sqrt{3}\left(\dfrac{-\sqrt{3}}{2}\right) = \dfrac{-3}{2}$

$\left(\dfrac{-3}{2}, \dfrac{-\sqrt{3}}{2}\right)$

Hence $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(\dfrac{-3}{2}, \dfrac{-\sqrt{3}}{2}\right)$, as foci of (1).

Equation of major axis: $Y = 0 \Rightarrow \dfrac{1}{2}\left(\sqrt{3}\,y - x\right) = 0 \Rightarrow x - \sqrt{3}y = 0$

Equation of minor axis: $X = 0 \Rightarrow \dfrac{1}{2}\left(\sqrt{3}\,x + y\right) = 0 \Rightarrow \sqrt{3}x + y = 0.$

(vi) Identify. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$ ... (1)

Here $a = 4,\ b = 7,\ 2h = -4$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\frac{2\tan q}{1-\tan^2 q} = \frac{4}{3} \Rightarrow 6\tan\theta = 4-4\tan^2\theta$$

$$4\tan^2\theta + 6\tan\theta - 4 = 0 \Rightarrow 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} = \frac{-3\pm\sqrt{9+16}}{4}$$

$$= \frac{-3\pm\sqrt{25}}{4} = \frac{-3\pm 5}{4} = -2, \frac{1}{2} \Rightarrow \tan\theta = \frac{1}{2}$$

Now $\tan\theta = \frac{1}{2} \Rightarrow$ base $= 2$, $\perp = 1$, so hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore$ $\sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X\cdot\frac{2}{\sqrt{5}} - Y\cdot\frac{1}{\sqrt{5}} = \frac{2X-Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\sin\theta = X\cdot\frac{1}{\sqrt{5}} + Y\cdot\frac{2}{\sqrt{5}} = \frac{X+2Y}{\sqrt{5}} \end{aligned}\right\} \ldots (2)$$

Substituting these expressions for x, and y into (i), we get

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2 + 12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\left(\frac{4X^2-4XY+Y^2}{5}\right) - 4\left(\frac{2X^2+3XY-2Y^2}{5}\right) + 7\left(\frac{X^2+4XY+4Y^2}{5}\right) + \frac{24X-12Y}{\sqrt{5}} + \frac{6X+12Y}{\sqrt{5}} - 9 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2 + \sqrt{5}\,(24X-12Y) + \sqrt{5}\,(6X+12Y) - 45 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2 + 24\sqrt{5}\,X - 12\sqrt{5}\,Y + 6\sqrt{5}\,X + 12\sqrt{5}\,Y - 45 = 0$$

$$15X^2 + 40Y^2 + 30\sqrt{5}\,X - 45 = 0 \Rightarrow 3X^2 + 8Y^2 + 6\sqrt{5}\,X - 9 = 0$$

$$3(X^2 + 2\sqrt{5}\,X) + 8Y^2 = 9 \Rightarrow 3(X^2 + 2\sqrt{5}X + (\sqrt{5})^2) + 8Y^2 = 9 + 15$$

$$3(X+\sqrt{5})^2 + 8Y^2 = 24 \Rightarrow \frac{(X+\sqrt{5})^2}{8} + \frac{Y^2}{3} = 1 \qquad (3)$$

which represents an ellipse.

From (2), we have

$$2X - Y = \sqrt{5}\ x \quad \dots (4)$$

$$X + 2Y = \sqrt{5}\ y \quad \dots (5)$$

Multiplying (5) by 2 and subtracting from (4), we get

$$5Y = 2\sqrt{5}\ y - \sqrt{5}\ x \Rightarrow Y = \frac{1}{\sqrt{5}}(-x + 2Y)$$

Put $Y = \frac{1}{\sqrt{5}}(-x + 2y)$ in (5), we get

$$X + \frac{2}{\sqrt{5}}(-x + 2y) = \sqrt{5}\ y \Rightarrow X = \sqrt{5}\ y - \frac{2}{\sqrt{5}}(-x + 2y)$$

$$= \sqrt{5}\ y + \frac{2}{\sqrt{5}}\ x - \frac{4}{\sqrt{5}}\ y$$

$$= \frac{1}{\sqrt{5}}(2x + y)$$

Thus $X = \frac{1}{\sqrt{5}}(2x + y)$ and $Y = \frac{1}{\sqrt{5}}(-x + 2y)$

For centre of (3) $X + \sqrt{5} = 0, Y = 0 \Rightarrow X = -\sqrt{5}, Y = 0$

$$X = -\sqrt{5} \Rightarrow \frac{1}{\sqrt{5}}(2x + y) = -\sqrt{5} \Rightarrow 2x + y = -5 \quad (7)$$

$$Y = 0 \Rightarrow \frac{1}{\sqrt{5}}(-x + 2y) = 0 \Rightarrow -x + 2y = 0 \quad (8)$$

Multiplying equation (8) by 2, we get

$$-2x + 4y = 0 \quad (9)$$

Adding equation (7) and (9), we get

$$5y = -5 \Rightarrow y = -1$$

Equation. (8) $\Rightarrow x = 2y = 2(-1) = -2$

Hence $C(-2, -1)$ is the centre of (1)

Vertices of (3) are $X + \sqrt{5} = \pm\sqrt{8}$, $Y = 0$

$$X + \sqrt{5} = \sqrt{8}, Y = 0$$

$$X + \sqrt{5} = \sqrt{8} \Rightarrow \frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = \sqrt{8}$$

$2x + y + 5 = \sqrt{40}$

$2x + y = -5 + \sqrt{40}$ ... (10)

$Y = 0 \implies -x + 2y = 0$ ... (11)

Multiplying (11) by 2

$-2x + 4y = 0$ ... (12)

Adding equation (10) and equation (12), we get

$$5y = -5 + \sqrt{40} \implies y = -1 + \sqrt{\frac{8}{5}}$$

$$(12) \implies x = 2y = 2\left(-1 + \sqrt{\frac{8}{5}}\right) = - + \sqrt{\frac{32}{5}}$$

Similarly, solving $X + \sqrt{5} = -\sqrt{8}$, and $Y = 0$, we get

$$x = -2 - \sqrt{\frac{32}{5}}, \ y = -1 - \sqrt{\frac{8}{5}}$$

$$\therefore \left(-2 + \sqrt{\frac{32}{5}}, -1 + \sqrt{\frac{8}{5}}\right), \left(-2 - \sqrt{\frac{32}{2}}, -1 - \sqrt{\frac{8}{5}}\right)$$

are the vertices of (1)

Eccentricity:- $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{8 - 3}}{\sqrt{8}} = \dfrac{\sqrt{5}}{\sqrt{8}} = \sqrt{\dfrac{5}{8}}$

Foci of (3) are $X + \sqrt{5} = \pm\sqrt{5}$, $Y = 0$

$X + \sqrt{5} = \sqrt{5}$ and $Y = 0$

$X + \sqrt{5} = \sqrt{5} \implies X = 0$

$\dfrac{2x + y}{\sqrt{5}} = 0 \implies 2x + y = 0$ ... (13)

$Y = 0 \implies x + 2y = 0$ ... (14)

Multiplying equation (14) by 2, we get

$-2x + 4y = 0$ ... (15)

Adding (13) and (15), we get

$5y = 0 \implies y = 0$

Equation (14) $\implies x = 2y = 2(0) = 0$

Similarly, solving $X + \sqrt{5} = -\sqrt{5}$ and $Y = 0$.

We get $x = -4$, $y = -2$

Thus $(0, 0), (-4, -2)$ are the foci of (1).