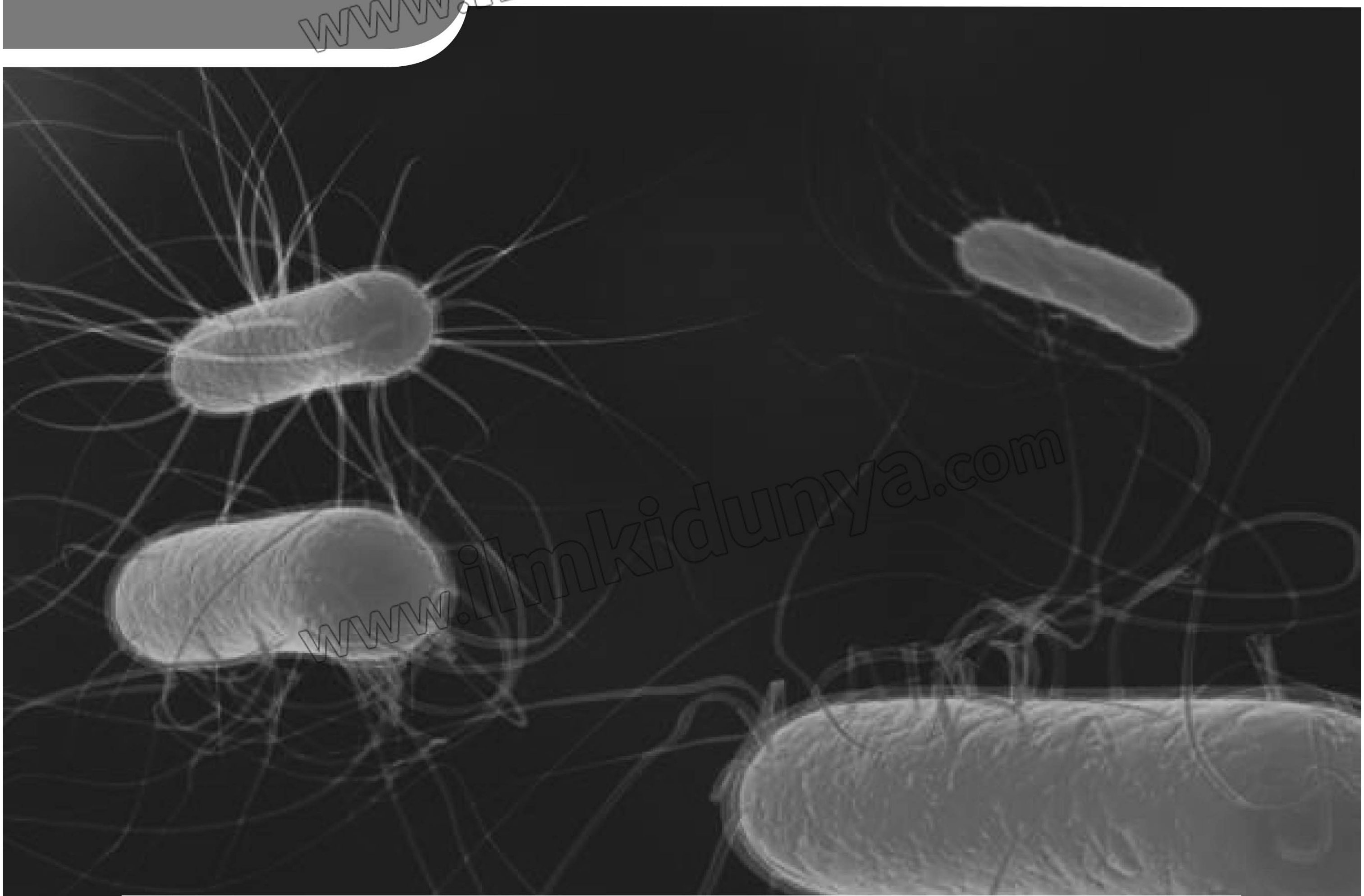
CHAPTER

2 Logarithm



Logarithms: The Secret Behind Predicting Growth in Science

Did you know that logarithms are super useful for figuring out how quickly things grow or shrink over time, like in science experiments with bacteria? Imagine you've got a tiny bunch of bacteria that doubles in size every hour. If you start with just a little bit, they can fill up a whole petri dish in no time! By using logarithms, scientists can work out how fast the bacteria will grow without having to watch them double and double again for hours on end. This is because logarithms help turn that doubling (which happens again and again) into a simple problem, kind of like solving a puzzle. So, if you're curious about how long it will take for those tiny bacteria to take over the dish, logarithms are your best friend. This makes it super easy for scientists to predict stuff, like how diseases spread or how fast a plant will grow, just by using this cool math trick!

Students' Learning Outcome

1. Express a number in scientific notations and vice versa.
2. Describe logarithm of a number.
3. Differentiate between common and natural logarithm.
4. Apply laws of logarithm to real life situations such as growth and decay, loudness of sound.

Pre & Post Requisite

Class 8
Chapter # 2
Estimation and Approximation

Class 9 (physics)
Chapter # 1
Scientific Notation

Class 9
Chapter # 1
Logarithm

Knowledge

① **Understanding Scientific Notation:**
- Comprehend the concept of expressing large numbers in scientific notation for simplification and clarity.
- Recognize how to convert numbers from scientific notation back to standard form.

② **Fundamentals of Logarithms:**
- Grasp the definition and the mathematical significance of the logarithm of a number.
- Identify the characteristics and differences between common logarithms (base 10) and natural logarithms (base *e*).

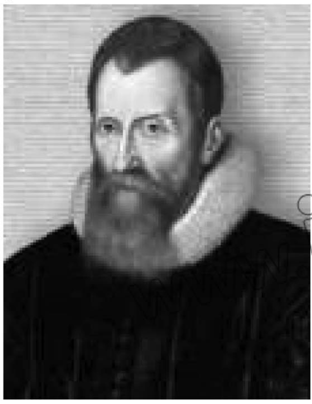
③ **Distinction Between Logarithm Types:**
- Distinguish between common logarithms and natural logarithms, noting their specific bases and typical applications.

④ **Applications of Logarithms:**
- Understand how logarithms are used in real-life situations, such as modeling exponential growth and decay in populations, finances or radioactive substances.
- Learn about the application of logarithms in measuring the loudness of sound in decibels and their role in acoustic engineering.

Skills

- Employ scientific notation in various contexts to simplify numerical expressions and calculations.
- Transition seamlessly between scientific notation and standard number form.
- Converting logarithms into exponential and vice versa.
- Applying definition of log to find the unknowns.
- Choose appropriately between common and natural logarithms based on the context of the problem.
- Utilizing the laws of logarithms to combine or separate logarithmic terms.
- Utilizing the laws of logarithms to evaluate logarithmic equations.
- Use logarithmic functions to analyze real-life situations such as population growth, radioactive decay, and sound intensity levels.

History



In 1614, Scottish mathematician John Napier introduced logarithms, a breakthrough simplifying multiplication and division, speeding up calculations before calculators existed. This invention transformed mathematics, aiding scientific discoveries and advancing navigation, marking a significant impact on the development of modern mathematics.

Student Learning Outcomes

- ✧ Express a number in scientific notations and vice versa.

Introduction

Logarithms have been instrumental in the evolution of mathematics and its myriad applications, notably through the enhancements by Henry Briggs. These refinements significantly broadened the utility of logarithms, making complex calculations more manageable across various fields such as science, engineering, and navigation. By simplifying the processes of multiplication and division into more straightforward operations, logarithms enabled significant advancements in technology and scientific understanding. The work of Briggs, in particular, made logarithmic methods more accessible, cementing their role as a cornerstone in the development of modern computational techniques and our exploration of the natural world.

Knowledge 2.1 Scientific Notation

Scientific notation is developed to efficiently represent large and small numbers. It simplifies handling large or small values with many zeros and facilitates calculations in scientific and technical fields.

Scientific notation is a concise mathematical representation that expresses numbers as a product of a significant digit and a power of 10 exponent.

For example, 45×10^{-3}, $0.0367\times10^{+4}$, 7.3×10^{-3}, $3.21\times10^{+4}$

2.1.1 Standard Form

Any number X can be written as power of 10 as $X = A\times10^{n}$ where n is the power of ten and A is non-zero digit ranging $1\leq A<10$.

Positive Power of Ten

To create a positive power of 10, move the decimal point to the left; the exponent is determined by the count of places the decimal has shifted. In the example below, shifting two digits left gives 10^{+2}.

$$475.31=4.7531\times10^{+2}$$

Similarly, $70100000 = 7.01\times10^{7}$, $34860 = 3.486\times10^{4}$, $54000 = 5.4\times10^{4}$

Negative Power of Ten

To form a negative power of 10, shift the decimal point to the right. The value of the exponent is equal to the total number of positions the decimal has been moved. In the example below, shifting the point three digits left gives 10^{-3}.

$$0.00325 = 3.25 \times 10^{-3}$$

Similarly, $0.0308 = 3.08 \times 10^{-2}$, $0.00097 = 9.7 \times 10^{-4}$, $0.0006741 = 6.741 \times 10^{-4}$

2.1.2 Ordinary Form

By ordinary form we mean to eliminate power of ten from scientific notation.

Eliminating Positive Powers

To create a positive power of ten, we move the decimal point to the left; to counteract a positive power, we shift the decimal point to the right. Let's consider an example where shifting the decimal point two digits to the right eliminates a +2 power.

$$4.7531 \times 10^{+2} = 475.31$$

Eliminating Negative Power

To create a negative power of ten, we shift the decimal point to the right; to eliminate a negative power, we move the decimal point to the left. Consider the following example, where shifting the decimal point three digits to the left eliminates a −3 power.

$$3.25 \times 10^{-3} = 0.00325$$

2.1.3 Arithmetic Operations on Scientific Notations

1. Multiplication Rule:

Expression: $(a \times 10^{m}) \times (b \times 10^{n}) = (a \times b) \times 10^{n+m}$

Explanation: To multiply numbers in scientific notation, multiply their significant figures and add their exponents of 10 to get the product's scientific notation form.

2. Division Rule:

Expression: $\dfrac{(a \times 10^{m})}{(b \times 10^{n})} = \left(\dfrac{a}{b}\right) \times 10^{m-n}$

Explanation: When dividing numbers in scientific notation, divide their significant figures and subtract the exponent of the divisor from the exponent of the dividend to obtain the quotient's scientific notation form.

Class Activity

This headline appeared in a local newspaper: "A significant portion of Pakistanis eat at biryani restaurants every day." Analyze the following information to estimate the percentage of Pakistanis engaging in this dining habit:

- There are about 1×10^{5} biryani Restaurants in Pakistan.
- Each restaurant serves about 2.5×10^{2} people every day.
- The population of Pakistan is about 2.4149×10^{8}

Example 2.1

Evaluate the result of multiplying $(6.8\times10^{5})\times(2.5\times10^{-3})$.

Solution:

Step 1: Multiply the significant figures: $6.8\times2.5=17$

Step 2: Add the exponents of 10: $5+(-3)=2$

The product 17×10^{2} is in scientific notation.

Example 2.2

Perform the division of $\frac{(7.2\times10^{4})}{3\times10^{2}}$.

Solution:

Step 1: Divide the significant figures: $\frac{7.2}{3}=2.4$

Step 2: Subtract the exponents of 10: $4-2=2$.

The quotient 2.4×10^{2} is in scientific notation.

3. Addition and Subtraction Rule:

Addition $(a\times10^{n})+(b\times10^{n})=(a+b)\times10^{n}$

Subtraction $(a\times10^{n})-(b\times10^{n})=(a-b)\times10^{n}$

Explanation: For addition or subtraction, align the exponents of 10 for all numbers, adjust the significands accordingly and perform the operation while maintaining the common exponent. Alternatively, convert them to ordinary form for regular addition or subtraction and later switch back to scientific notation.

Example 2.3

Perform the subtraction of $(5.3\times10^{-3})-(4.9\times10^{-4})$.

Solution:

First adjust the power of 1st value to -4 or adjust the power of 2nd value to -3.

Step 1: We are adjusting the 1st value, $5.3\times10^{-3}=53\times10^{-4}$

Step 2: Take 10^{-4} common, $10^{-4}(53-4.9)$

Step 3: Perform the subtraction: $53-4.9=48.1$

The resultant 48.1×10^{-4} is in scientific notation.

Skill 2.1

- Employ scientific notation in various contexts to simplify numerical expressions and calculations.
- Transition seamlessly between scientific notation and standard number form.

Exercise 2.1

1. Write the following numbers in standard form.

   i. 723000 ii. 53400 iii. 6934390 iv. 412300000

   v. 37.42 million vi. 0.000082 vii. 0.0060 viii. 0.00000000056

2. Write the following in ordinary form.

   i. 6.9×10^{6} ii. 6.07×10^{-4} iii. 4.73×10^{4} iv. 2.79×10^{7}

   v. 4.83×10^{-5} vi. 2.61×10^{-6} vii. 3.69×10^{3} viii. 6.07×10^{-4}

3. Write the following in standard form.

   i. 68×10^{-5} ii. 720×10^{6} iii. 8×10^{5} iv. 0.75×10^{7}

   v. 0.4×10^{-10} vi. 50×10^{-6}

4. Write these numbers in order of magnitude starting with the largest.

   $3.2\times 10^{-4}, 6.8\times 10^{-5}, 5.57\times 10^{-9}, 5.8\times 10^{-7}, 6.741\times 10^{-4}, 8.414\times 10^{2}$

5. Deduce the value of n in each of the following cases.

   i. $0.00025 = 2.5\times 10^{n}$ ii. $0.00357 = 3.57\times 10^{n}$ iii. $0.0000006 = 6\times 10^{n}$

   iv. $0.004^{2} = 1.6\times 10^{n}$ v. $0.00065^{2} = 4.225\times 10^{n}$ vi. $0.0002^{n} = 8\times 10^{-12}$

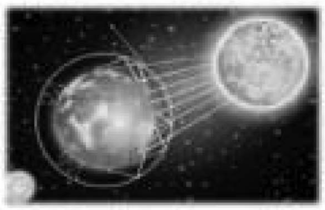
6. Find the value of the following and write your answer in scientific notation.

   i. $(9.6\times 10^{11}) \div (2.4\times 10^{5})$ ii. $3.15\times 10^{-9} \div 7.0\times 10^{6}$ iii. $8.5\times 10^{-6}\times 6\times 10^{15}$

   iv. $3.8\times 10^{5}\times 4.6\times 10^{4}$ v. $6.6\times 10^{7} - 4.9\times 10^{6}$ vi. $4.07\times 10^{7} - 5.1\times 10^{6}$

7. Population of Pakistan according to latest digital census is 241.49 million, convert this in standard form.

8. Light from the sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^{8} m/s. Calculate to 3 significant figures the distance from sun to Earth using $S = vt$.



9. A computer chip has 2.1×10^{9} transistors. If a new version of the chip has 4.5×10^{9} transistors, how many more transistors does the new version have?

10. A certain species of bacteria doubles in population every 2 hours. If there are initially 1.5×10^{3} bacteria, how many bacteria will there be after 100 hours?

Knowledge 2.2 Logarithm

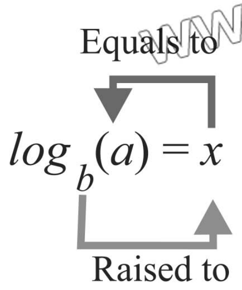
Logarithm, in mathematics, is another word for the exponentiation. It determines the power to which a number must be raised to obtain another number.

Student Learning Outcomes

✧ Describe logarithm of a number.

The logarithm of a positive real number a with respect to a base b (which is also a positive real number different from 1), is defined as the exponent x to which b must be raised to obtain a.

$$b^x = a \Leftrightarrow log_b\, a = x, \text{ where } b > 0\,, b \neq 1 \text{ and } a > 0.$$



This relationship indicates that logarithms are the inverse operations of exponentiation, Here "a" is called the argument which is inside the log and "b" is called the base which is at the bottom of the log.

The logarithm answers the question: How many times must one number be multiplied to obtain another?

For example, how many 2's are multiplied to get the answer 128?

If we multiply 2 for 7 times, we get the answer 128.

Therefore, the logarithm of 128 with respect to base 2 is 7.

The logarithm form can be written as

$$log_2 128 = 7 \cdots\cdots\cdots (i)$$

The above logarithm form can also be written as:

$$2^7 = 128 \cdots\cdots\cdots (ii)$$

Here, the equations (i) and (ii) both represent the same meaning.

Examples of conversion	
Exponents	Logarithms
$6^2 = 36$	$\log_6 36 = 2$
$10^2 = 100$	$\log_{10} 100 = 2$
$3^3 = 27$	$\log_3 27 = 3$

2.2.1 Understanding the same concept with number line representation

Take a geometric sequence with common ratio "3" on number line 1 line and a general arithmetic sequence with common difference as "1" on number line 2. The exponents in the geometric sequence terms with respect to base 3 are basically the logs of the geometric sequence which are plotted on 2nd number line (see figure 2.1).

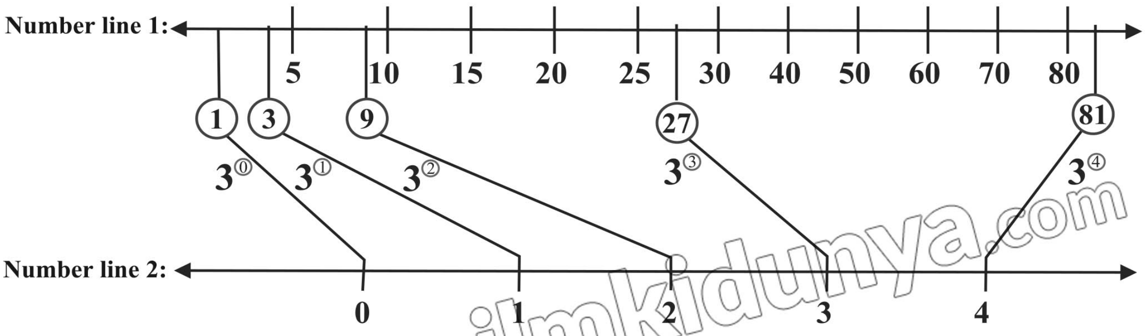


Figure 2.1

Number line 1: Displays a number sequence within a blue circle, where each term increases by a factor of three

Number line 2: Displays the exponents for the numbers highlighted in the first line, where the logarithm of a number encircled in blue is the exponent(encircled red) to which 3 must be raised to yield that numbered circle)

2.2.2 Important Logarithmic Understandings

1. Logarithm of a Positive Number can be Negative

A negative answer in log means that we will use a negative exponent (power) when we convert from log form to exponential form. In exponential form, a negative exponent implies a division.

For instance, a^{-n} is equivalent to $\frac{1}{a^n}$.

Negative logarithms typically arise when the base is positive and the input falls between 0 and 1.

Test Yourself

Identify as true or false

1. $log\,(1000)$ in any base is always 3.

In the expression

2. $log_2 4$ and $log_4 2$ are reciprocals of each other.

Example 2.4

Find x if $log_5 \frac{1}{25} = x$.

Solution:

We can convert from log form to exponential form to get

$$5^x = \frac{1}{25} \Rightarrow 5^x = 5^{-2}.$$

$$x = -2$$

Equal bases imply equal exponents, resulting in a negative logarithm in this case.

2. Logarithm of a Negative Number

You cannot take the log of a negative number (unless you want to deal with complex numbers). Once again, converting from log form to exponential form will help us to see why this is the case.

For example, let's say you wanted to take the log of –100, using a base of 10. So, we want to solve this log form equation for x:

$$log_{10}(-100) = x$$

Converting from log form to exponential form, we get

$$10^x = -100$$

The base is 10 (which is positive), while the argument is −100 (which is negative). This means that there is no real number we can substitute for x to get a result of −100 (any real number we put in for x will give us a positive value).

3. Concept of Negative Base

A logarithm cannot have a negative base (unless we are dealing with complex numbers).

4. Logarithm of "1"

A logarithm can be zero if the argument to the log function is 1 and this is true for any valid base. We can see this easily by converting the log form to exponential form.

$$log_b\, a = 0$$

$$b^0 = a$$

Since $b^0 = 1$, so

$$a = 1$$

5. Undefined Logarithm

The log function is undefined for zero or negative arguments. While negative arguments can be used with complex numbers, an argument of zero is always illogical in a log function. Once again, we'll convert from log form to exponential form to see why,

$$log_2\, 0 = x$$

Converting to exponential form, we get,

$$2^x = 0$$

There is no number we can substitute for x to make this equation true. Any real value of x we choose will give us a positive number.

Let us the consider the following examples to comprehend the concept of logarithms.

Note

- Log of zero and a negative number is undefined.
- Log of 1 is always zero with respect to any base.
- Base of logarithm can be any non-negative real number except zero and 1.
- Argument of log is always a positive real number.
- Argument of the log can never be negative and zero because the result is undefined.
- For argument between 0 and 1 logarithm is always negative.

Note

Complex numbers are a type of number that include the square root of negative one, represented as i, allowing us to solve equations with negative square roots and work with numbers beyond the usual real number system

Example 2.5

Convert from exponential to logarithmic and vice versa.

i. $2^4 = x$ ii. $log_7 49 = 2$ iii. $log_4 \frac{1}{16} = y$

Solution:

i. $log_2\, x = 4$ ii. $7^2 = 49$ iii. $4^y = \frac{1}{16}$

Example 2.6 Find the unknown variable.

i. $2^{3x} = 8$ ii. $log\, x^2 = 4$ iii. $log_9 \frac{1}{81} = y$

Solution:

i. $2^{3x} = 8$
$2^{3x} = 2^3$
Comparing,
$3x = 3$
$x = 1$

ii. $log\, x^2 = 4$
Common log has a base 10 so,
$10^4 = x^2$
$\sqrt{10^4} = \sqrt{x^2}$
$10^2 = x$
$x = 100$

iii. $log_9 \frac{1}{81} = y$
$9^y = \frac{1}{81}$
$9^y = 9^{-2}$
$9^y = \frac{1}{9^2}$
$y = -2$

How can one manually determine the integers between which a logarithm lies?

Example 2.7 Write the two consecutive integers between which $log_4 100$ lies.

Solution:

Since, $4^3 = 64$ and $4^4 = 128$. 100 lies somewhere between 64 and 128. So, by understanding of exponentiation nature of logarithm we can say that $log_4 100$ will lie somewhere between 3 and 4 (greater than 3 and less than 4).

Knowledge 2.3 Natural Logarithm

Student Learning Outcomes

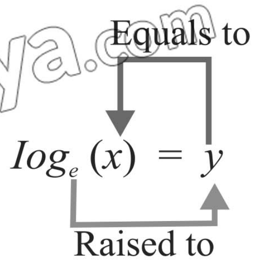
- Differentiate between common and natural logarithm of a number.

A natural logarithm is a logarithm that uses the mathematical constant *"e"* (which is approximately equal to 2.71828…) as its base. It is written as *"ln"* and is used to determine how many times we need to multiply

$$e^y = x \Leftrightarrow ln(x) = y$$

For example, the natural logarithm of 7.389 is about 2, because $2.71828^2 \approx 7.389$.

2.3.1 Euler's number "*e*"



The natural logarithm base, denoted by "*e*," not only features in compound interest but naturally emerges in various fields like finance, calculus and number theory. It's a go-to for modeling exponential growth and decay, playing a crucial role in advanced mathematical studies compared to the common logarithm.

Class Activity

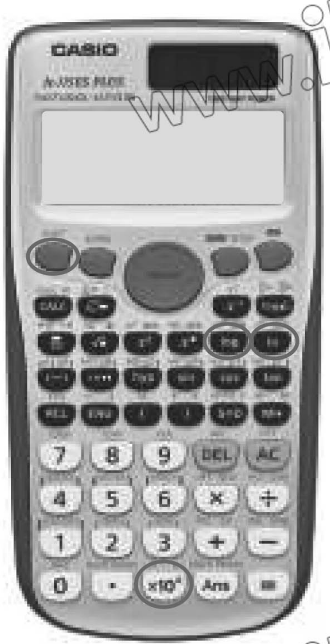
The expression below has three digits. Swap the position of two of the digits to make an expression with:

$$log_8 \frac{2}{4}$$

- The highest value.
- The lowest value.

Note

Remember, while the bases of these logarithms are different, the fundamental properties and laws of logarithms (like the product rule, quotient rule and power rule) apply to both types. The primary distinction is the base and the contexts in which they are typically used.



Relation between common log and natural log by using change of base formula, which is,

$$log_{10} x = \frac{ln\,x}{ln10},$$

where:

- $log_{10} x$ is the common logarithm of x.
- $ln\,x$ is the natural logarithm of
- $ln10$ is the natural logarithm of 10, which is approximately equal to 2.30259.

Therefore, $ln\,x = 2.3025 \times log_{10} x.$

Knowledge 2.4 Antilogarithm

The antilogarithm is about "undoing" the logarithm. If you take the antilog of a log, you get back to your original number.

Relationship with Exponentiation

It is inherently an operation of exponentiation. For base 10 logarithms, finding the antilog of a number is equivalent to raising 10 to the power of that number.

The antilogarithm (often called the antilog) is the inverse function of the logarithm function. This means if $y = \log_b x$, then $x = b^y$ or $x = \text{antilog}_b(y)$.

For common logarithms the antilog is the same as raising 10 to the power of y, $10^y = x$.

For natural logarithms the antilog is the same as raising e to the power of y, $e^y = x$.

How to calculate Log & Antilog on a calculator?

Common Log: To find the common logarithm, Press the **"log"** button and type [Your Number].

Natural Log: To find the natural logarithm, Press the **"ln"** button and type [Your Number].

For Common Antilog: Press **"Shift"** then **"log"** then [Your Number].

For Natural Antilog: Press **"Shift"** then **"ln"** then [Your Number].

If the *ln* and *log* functions are not available, you can use the highlighted buttons in the last row:

- Press 10^x for the common log.
- Press "e^x" for the natural log.

Skill 2.2

- Converting logarithms into exponential and vice versa.
- Applying definition of log to find the unknowns.
- Choose appropriately between common and natural logarithms based on the context of the problem.

Exercise 2.2

1. Convert the following from exponential form to logarithmic form.

i. $9^{-2} = \frac{1}{81}$ ii. $10^{-3} = 0.001$ iii. $4^{-2} = \frac{1}{16}$

iv. $3^{-4} = \frac{1}{81}$ v. $\left(\frac{1}{5}\right)^{2} = \frac{1}{25}$ vi. $\left(\frac{1}{3}\right)^{-3} = 27$

2. Convert the following from logarithmic form to exponential form.

i. $log_u v = -16$ ii. $log_7 7 = 1$ iii. $log_{\frac{1}{2}} \frac{1}{8} = 3$ iv. $log_3 1 = 0$

v. $log_e \frac{1}{64} = -x$ vi. $log_6 \frac{1}{36} = -2 log_{\frac{7}{4}} x$ vii. $log_5 25 = y$ viii. $ln\, x = -8$

3. Find the value of unknown variable

i. $log_{16} 4 = y$ ii. $log_2 8 = y$ iii. $log_7 \frac{1}{7} = y$

iv. $log_y 32 = 5$ v. $log_5 n = 2$ vi. $log_{64} 8 = \frac{x}{2}$

4. Find the value of unknown variable.

i. $e^x = 4$ ii. $lnx = 6$ iii. $ln(2x-1) = 1$ iv. $e^{3x+5} = 10$

v. $ln\left(e^{3-x}\right) = 8$ vi. $e^{e^x} = 5$ vii. $lnx = \frac{1}{2}$ viii. $e^x = 7$

5. Evaluate

i. $log_{2\sqrt{2}} 512$ ii. $log_3 \frac{1}{243}$ iii. $log_3 1 = y$ iv. $log_6 \frac{1}{216}$

v. log (0.01) vi. $log_{343} 7 log_{\frac{1}{6}} 216$ vii. $log_{\frac{2}{3}} \frac{8}{27}$ viii. $log_5 \sqrt[3]{5}$

6. Find between which two consecutive integers the logarithm lies.

i. $log_2 30$ ii. $log_7 9$ iii. $log_3 75$ iv. $log_{10} 7500$

Knowledge 2.5 Laws and Properties of Logarithm

The laws and properties of logarithms are essential tools in mathematics, transforming complex multiplicative relationships into simpler additive forms. They are pivotal for solving equations, understanding exponential growth and decay and facilitating advanced mathematical analysis.

(a) Power Rule

The logarithm of a number raised to an exponent equals the exponent multiplied by the logarithm of the base number. This law enables us to extract the exponent from the logarithm and use it as a multiplier.

Expression: $log_b m^n = n \times log_b m$

Let $log_b m^n = x$ ·········(i) and $log_b m = y$ ·········(ii)

Exponential Form: $b^x = m^n$ and $b^y = m$

$b^x = m^n$

$b^x = (b^y)^n$ ∵ Replaced with b^y

$b^x = b^{yn} \Rightarrow x = yn$ ·········(iii) ∵ By Comparing

We have,

$log_b m^n = x$ and $x = yn$ ∵ From (i) and (iii)

$log_b m^n = ny$

$log_b m^n = nlog_b m$ ∵ Replaced y with $log_b m$

Hence proved.

Let us consider the following example.

Example 2.8

Solve $log_4(4^5)$.

Solution:

$log_4(4^5) = 5log_4(4) = 5$ ∵ $log_b b = 1$

(b) Product Rule

The logarithm of a product equals the sum of the logarithms of its factors. This law enables us to break down the logarithm of a multiplication into the sum of two separate logarithms.

Expression: $\log_b(m \times n) = \log_b(m) + \log_b(n)$

Let $\log_b m = x$ and $\log_b n = y$

Exponential form: $b^x = m$·········(i) and $b^y = n$·········(ii)

Multiplying (i) and (ii)

$$mn = b^x \times b^y \Rightarrow mn = b^{x+y}$$

Taking log with base b on both sides,

$\log_b(m \times n) = \log_b(b^{x+y})$

$\log_b(m \times n) = (x + y)(\log_b b)$ ∵ By power law

$\log_b(m \times n) = x + y$ ∵ By identity law

$\log_b(m \times n) = \log_b m + \log_b n$ ∵ $\log_b m = x$ and $\log_b n = y$

Let us consider the following example.

Example 2.9

Solve $log_5(5^2 \times 5^3)$.

Solution:

$$log_5(5^2 \times 5^3) = log_5 5^2 + log_5 5^3 = 2log_5 5 + 3log_5 5$$

$$= 2(1) + 3(1) = 5 \qquad \because log_b b = 1$$

(c) Quotient Rule

The logarithm of a quotient is the difference between the logarithms of the numerator and the denominator. This law allows us to transform the logarithm of a division into the subtraction of two logarithms.

Expression: $log_b\left(\frac{m}{n}\right) = log_b(m) - log_b(n)$

Let $log_b m = x$ and $log_b n = y$

Exponential form: $b^x = m \cdots\cdots (i)$ and $b^y = n \cdots\cdots (ii)$

Dividing (i) by (ii),

$$\frac{m}{n} = \frac{b^x}{b^y} \Rightarrow \frac{m}{n} = b^{x-y}$$

Taking log with base b on both sides ,

$$log_b\left(\frac{m}{n}\right) = log_b\left(b^{x-y}\right)$$

$$log_b\left(\frac{m}{n}\right) = (x-y)(log_b b) \quad \because \text{By power Law: } log_b m^n = n\, log_b m$$

$$log_b\left(\frac{m}{n}\right) = x - y \quad \because \text{By identity law: } log_b b = 1$$

$$log_b\left(\frac{m}{n}\right) = log_b m - log_b n \quad \because \text{By substitution: } log_b m = x \text{ and } log_b n = y$$

Test Yourself

Express

$$log\left(\frac{75}{16}\right) - 2log\left(\frac{5}{9}\right) + log\left(\frac{32}{243}\right)$$

in terms of log 2 and log 3.

Let us consider the following example.

Example 2.10

Solve $log_3\left(\frac{27}{3}\right)$

Solution:

$$log_3\left(\frac{27}{3}\right) = log_3(27) - log_3 3 = log_3 3^3 - log_3 3$$

$$3log_3 3 - log_3 = 3(1) - 1 = 2$$

(d) Identity Law

The logarithm of a number to its own base is always 1. This reflects the idea that any number raised to the power of 1 is itself.

Expression: $log_b b = 1$

Let $log_b b = x$ ....(i)

Exponential form: $b^x = b \quad \Rightarrow \quad b^x = b^1 \quad \Rightarrow \quad x = 1$

From (i) we have,

$log_b b = x = 1$

$log_b b = 1$

For example, $log_2 2 = 1$

(e) Change of Base Formula

This formula offers a method to express a logarithm in one base in terms of logarithms in another base, proving especially useful for computations involving specific bases.

Expression: $log_b a = log_b c \times log_c a = \dfrac{log_c a}{log_c b}$

From R.H.S, let $log_c a = x$ ....(i)

Exponential Form: $c^x = a$

Taking log with base b on both sides,

$log_b a = log_b c^x$

$log_b a = x log_b c$ $\quad$ ∵ By power law: $log_b m^n = n log_b m$

$log_b a = log_c a . log_b c$ ....(ii) $\quad$ ∵ Replaced x with $log_c a$

Now replace a with b,

$log_b b = log_c b . log_b c$

$1 = log_c b . log_b c$

$\dfrac{1}{log_c b} = log_b c$ ....(iii)

Replace $log_b c$ with $\dfrac{1}{log_c b}$ in (ii), we get

$log_b a = \dfrac{log_c a}{log_c b}$ or we can also say that

$log_c a = log_b a \times log_c b$

Test Yourself

Prove that
$log_{12} 1728 \times log_9 6561 = 7$

Teacher's Guidelines

Use measuring cups to teach logarithms, highlighting how doubling or halving recipes reflect logarithmic scaling. This visual method, showing exponential growth or division, makes the abstract concept of logarithms more tangible and relatable by linking it to everyday objects.

Example 2.11 Express $log_5 130$ in base 10.

Solution:

Using change of base formula,

$$log_5 130 = \frac{log_{10} 130}{log_{10} 5}$$

(f) Inverse Property

Exponentiation and logarithms are inverse operations. Raising a base to its logarithm (with the same base) yields the original number, reinforcing the concept that logarithms and exponentiation are operations that undo each other.

Expression: $b^{log_b x} = x$ and $log_b b^x = x$

For example, $10^{log_{10} 100} = 100$

Skill 2.3
- Utilizing the laws of logarithms to combine or separate logarithmic terms.
- Utilizing the laws of logarithms to evaluate logarithmic equations.

Exercise 2.3

1. Expand the following using laws and properties of logarithm.

   i. $log(A \times B \times C)$ ii. $log\frac{15.2 \times 30.5}{81.8}$ iii. $log\sqrt[3]{\frac{7}{15}}$ iv. $log_a \frac{y}{\sqrt[3]{x}}$

   v. $log_5 (\frac{x\sqrt{x}}{125})$ vi. $log(x^2 y^3)$ vii. $log_3 \sqrt[4]{m^5 n^2}$ viii. $log_3 \frac{xy^3}{a^3 b^2 c}$

2. Write the following in single simplified form.

   i. $log_2 7 + log_2 4$ ii. $log 25 + log 4$ iii. $log_4 2x + log_4 4x^2$

   iv. $log_3 24 - log_3 8$ v. $log_4 x^9 - log_4 x^2$ vi. $log_3 4 + log_3 y + \frac{1}{2} log_3 49$

   vii. $\frac{1}{3}(log_5 8 + log_5 27) - log 3$ viii. $2 log 6 - \frac{1}{4} log 16 + log 3$

3. Find x if

   i. $log 2x + 1 = 2$ ii. $2 log_b 4 + log_b 5 - log_b 10 = log_b x$ iii. $log_b 30 - log_b 5^2 = log_b x$

   iv. $log_b 8 - log_b x^2 = log_b x$ v. $log_b (x+2) - log_b 4 = log_b 3x$ vi. $log_b (x-1) + log_b 3 = log_b x$

   vii. $log_x 81 = 4$

4. If $log 8 = x$ and $log 3 = y$, express the following in terms of x and y.

   i. $log 24$ ii. $log\frac{9}{8}$ iii. $log 720$

5. Evaluate each of the following logarithms.

   i. $log_{25} 5$ ii. $log_{64} 2$ iii. $log_{\frac{1}{4}} 8$

6. Express each of the following as simply as possible in terms of logarithms to base b for the given value of b.

i. $log_5 7,\ b=2$ ii. $log_{\frac{1}{3}} 8,\ b=10$ iii. $\frac{1}{log_6 31},\ b=5$

7. If $log\,2=0.3010$, $log\,3=0.4771$ and $log\,5=0.6990$ then find the values of the following

a. $log\,30$ b. $log\sqrt{4\frac{4}{5}}$ c. $log\frac{20}{3}$ d. $log(5^{2.5}\times\sqrt[3]{3})$

Knowledge 2.6 Real World Applications of Logarithm

Student Learning Outcomes

✧ Apply laws of logarithm to real life situations such as growth and decay, loudness of sound.

Teaching Guidelines

Use measuring cups to teach logarithms, highlighting how doubling or halving recipes reflect logarithmic scaling. This visual method, showing exponential growth or division, makes the abstract concept of logarithms more tangible and relatable by linking it to everyday objects.

Logarithms are used for extracting powers or roots and are, in fact, corollaries to the theory of indices. This is why they can transform very complex problems into simpler ones. They help us manage and visualize both extremely large and tiny numbers, ensuring that the results remain consistent.

Understanding Scales: Linear vs. Logarithmic

Linear Scaling: Features equal intervals, such as 1, 2, 3,… or 10, 20, 30,… resembling an arithmetic sequence. Each step adds a constant value.

Logarithmic Scaling: Involves multiplicative steps, for example, 1, 10, 100, 1000,… following a geometric progression. This scaling is exemplified by the number line discussed previously in section 2.2, where each increase is by a factor, not a sum.

The Usability of Logarithmic Scales

Linear scales can make it challenging to discern the growth of human population from thousands to billions, as they compress the vast differences into uniform intervals. In contrast, logarithmic scales adeptly illustrate and allow for easy comparison of both early and recent population growth within a single graph, as demonstrated below.

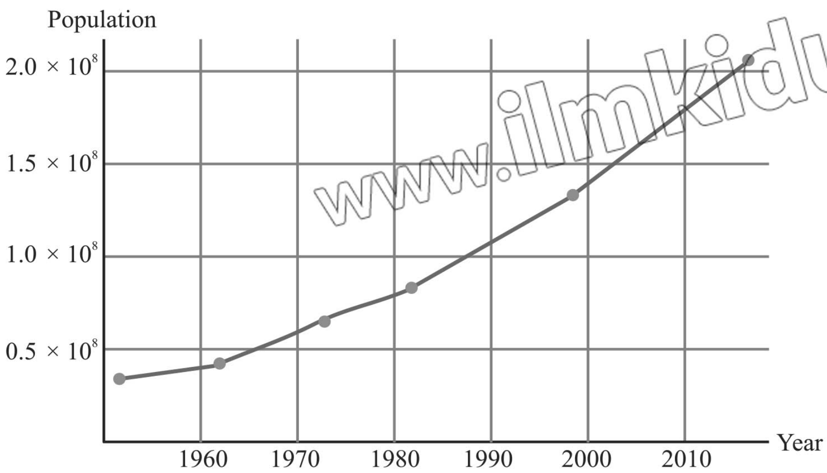
In order to have a better understanding of logarithmic scale,w plot the data mentioned in the table 2.1 on a cartesian plane.

Table 2.1

X[Years]	Y [Population]
1951	33,740,167
1961	42,880,378
1972	65,309,340
1981	84,254,644
1998	132,352,279
2017	207,774,520

Pakistan Population Census Trends: 1961-2017

The y-axis values double proportionally, while the x-axis uses a linear scale incrementing by 10.



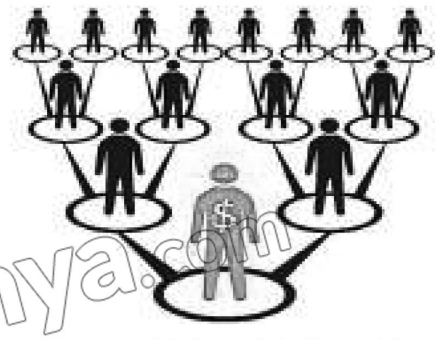
Interesting Information

Logarithmic scales make it easier to see big differences in data and are used in measuring sound (decibels), earthquake strength (Richter scale), acidity (pH scale) and light intensity (stellar magnitudes).

Exponential Growth and Decay

Exponential growth occurs when something multiplies at a rate that depends on how much of it is already there. This leads to rapid increases, like when one flower produces three, then nine flowers.

On the other hand, exponential decay is when something decreases faster when there's more of it, gradually slowing down as it gets smaller. It's a bit like eating a chocolate bar and halving what's left each day.



Exponential Growth in Pyramid Schemes: The Recruitment Model

Logarithm in Complex Numerical Calculations

Logarithms are indispensable in mathematics and practical applications, aiding in complex calculations and understanding growth, decay and sound intensity in fields like finance, biology and physics.

Riddle

You'll find me between an ant's weight and a mountain's height. In linear terms, I'm vast, but in my terms, I'm just right. What am I?

Example 2.12 Solve $\sqrt[3]{\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}}$

Solution:

Let $y=\sqrt[3]{\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}} \Rightarrow y=\left(\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}\right)^{\frac{1}{3}}$

Taking log on both sides

$$\log y=\log\left(\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}\right)^{\frac{1}{3}}$$

$$\log y = \frac{1}{3}\log\frac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}$$

$$\log y = \frac{1}{3}\left[\log 0.821 + \log(45.23)^4 - \log(5.79)^3 - \log 0.942\right]$$

$$\log y = \frac{1}{3}\left[\log 0.821 + 4\log 45.23 - 3\log 5.79 - \log 0.942\right]$$

$$\log y = \frac{1}{3}\left[-0.0856 + 4(1.6554) - 3(0.7626) - (-0.0259)\right]$$

$$\log y = \frac{1}{3}\left[4 - 0.0856 + 6.6216 - 2.2878 + 0.0259\right]$$

$$\log y = \frac{1}{3}\left[4.2741\right] \Rightarrow \log y = 1.4274$$

To eliminate log, take antilog on both sides

$$\text{antilog}(\log y) = \text{antilog}(1.4274) = 26.5888$$

Therefore, $y = 26.5888$

Example 2.13

The intensity I_1 of a whisper is 1×10^{-12} watts per square meter (W/m^2). The intensity I_2 of normal conversation is 1×10^{-6}. Calculate the difference in sound intensity levels (in decibels) between a whisper and normal conversation.

Solution:

The formula to calculate the sound intensity level L in decibels (dB) is given by:

$$L = 10\times\log\left(\frac{I}{I_o}\right),$$

where,

- I is the intensity of the sound.
- I_o is the reference intensity, which is 1×10^{-12} W/m^2 (it is always constant).

Sound intensity level for whispers:

$$L_1 = 10\times\log\left(\frac{1\times10^{-12}}{1\times10^{-12}}\right) \Rightarrow L_1 = 10\times\log 10^0$$

$L_2 = 10 \times 0 \Rightarrow L_1 = 0$ dB

Sound intensity level for normal conversation:

$$L_2 = 10 \times \log\left(\frac{1\times10^{-6}}{1\times10^{-12}}\right) \Rightarrow L_2 = 10 \times \log 10^6$$

$$L_2 = 10 \times 6 \Rightarrow L_2 = 60 \text{ dB}$$

$$L = L_2 - L_1 = (60 - 0) = 60 \text{ dB}$$

Example 2.14

Starting with 500 users, a startup experiences continuous growth at a rate of 8% per year. To determine the number of years it takes for the company to reach 3000 users, use the formula $N(t) = N_0 \times e^{rt}$ where $N(t)$ is the number of users after time t, N_0 is the initial number of users, r is the growth rate and t is the time in years

Solution:

Given that, N_0 = 500 users, r = 0.08 (8% expressed as a decimal), $N(t)$ =3000, t=?

Now insert the values into the formula,

$$N(t) = N_0 \times e^{rt}$$

$$3000 = 500e^{0.08\times t}$$

To isolate "t" in the equation, we apply the natural logarithm (ln) to both sides, utilizing the power rule of ln, which allows us to move the exponent to the coefficient position for easier manipulation.

$$ln\,3000 = ln(500e^{0.08t})$$

$$ln\,3000 = ln\,500 + 0.08t(ln\,e)$$

$$8.0063 = 6.2146 + 0.08t(1) \Rightarrow 8.0063 - 6.2146 = 0.08t$$

$$1.7917 = 0.08t$$

$$\frac{1.7917}{0.08} = t \Rightarrow t = 22.39 \approx 22.4 \text{ years}$$

The company is projected to reach 3,000 members from an initial count of 500 in approximately 22 years and 3 months.

Skill 2.4

✧ Use logarithmic functions to analyze real-life situations such as population growth, radioactive decay, and sound intensity levels.

Exercise 2.4

1. Use logarithm to evaluate the following expressions.

i. $\sqrt[4]{2.145}\times\dfrac{\sqrt{15.236}}{6000}$ ii. $\dfrac{183\times\sqrt[3]{2}}{0.2356\times\sqrt[5]{2578}}$ iii. $\dfrac{(438)^{3}\times\sqrt{0.056}}{(388)^{4}}$

2. If an amount of PKR 50,000 is invested in a bank account that offers compound interest at a rate of 4% annually, compounded quarterly, how many years will it take for the amount to grow to PKR 70,000? (Hint: $A=P\left(1+\dfrac{r}{n}\right)^{nt}$, where A is the future value, P is the principal amount, r is the annual interest rate, n is the number of times interest is compounded per year and t is the time in years.)

3. The value, PKR V, of an investment after t years is given by the formula $V = Ae^{0.03t}$, where PKR A is the initial investment.

i. How much will an investment of PKR 4000 be worth after 3 years?

ii. To the nearest year, how long will you need to keep an investment for it to double in value?

4. The value of a new car depreciates at a rate of 20% per year. If the car is initially worth PKR 25,000,00 how many years will it take for its value to drop below PKR 10,000,00?

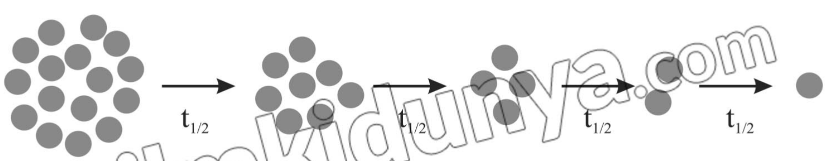
(Hint: $V(t)=V_0\times(1-r)^t$, where $V(t)$ is the value after time t, V_0 is the initial value and r is the annual depreciation rate.)

5. A town has a population of 50,000 people. The population is increasing at an annual rate of 6%. If this growth rate continues, in how many years will the town's population reaches 100,000?

(Hint: $P(t)=P_0\times e^{rt}$, where $P(t)$ is the population after time t, P_0 is the initial population, r is the growth rate, and e is the base of the natural logarithm.)

6. A radioactive element has a half-life (time in which half of the radioactive sample is left) of 5,000 years. If a sample initially contains 80 grams of the element, how long will it take for only 10 grams to remain?

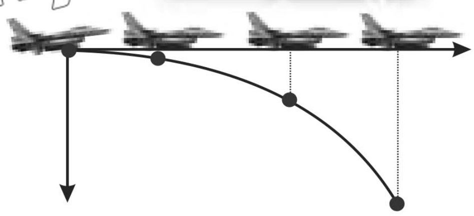
(Hint: $A(t)=A_0\times\left(\dfrac{1}{2}\right)^{\frac{T}{t}}$, where $A(t)$ is the amount after time t, A_0 is the initial amount and T is the half-life.)



7. Considering that the intensity of a quiet room (I_1) is 1×10^{-1} watts per square meter $(\mathrm{W/m^2})$ and the intensity of busy street traffic (I_2) is $1\times10^{-5}\ W/m^2$, can you determine the difference in sound levels in decibels (dB) between these two environments using logarithmic calculations?

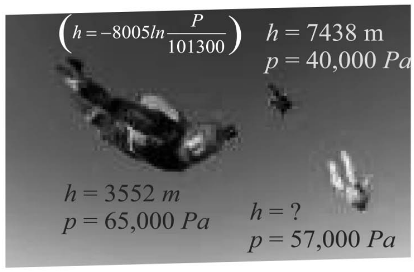
8. Considering that prolonged exposure to sounds above 85 decibels can cause hearing damage or loss and considering that a gunshot from a .22 rimfire rifle has an intensity of about $I = (2.5 \times 10^{13})$, should you follow the rules and wear ear protection when practicing at the rifle range?

9. The path of a projectile launched from an aircraft is given by the equation $h = 5000 - e^{0.2t}$, where h is the height in meters and t is the time in seconds.



- **i.** From what height was the projectile launched?
- **ii.** The projectile is aimed at a target at ground level. How long does it take to reach the target?

10. Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude h (in meters) above sea level is related to the air pressure P (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals? $\left(h = -8005 ln \dfrac{P}{101300}\right)$



11. The energy magnitude M of an earthquake can be modeled by $M = \dfrac{2}{3} log\, E - 99$, where E is the amount of energy released. What is the magnitude of an earthquake with an energy release of 7.079×10^{26}? Round your answer to the nearest whole number.

Review Exercise 2

1. Identify True or False

- **i.** Converting a number from standard form to scientific notation always results in a smaller numerical value.
- **ii.** The expression 3.6×10^4 can be written as 36×10^3 in scientific notation.
- **iii.** The logarithm of a negative number is always undefined.
- **iv.** Logarithms are only defined for positive base values.
- **v.** The natural logarithm (ln) is always larger than the common logarithm (log_{10}) for the same input value.
- **vi.** The natural logarithm (ln) has special significance in calculus due to its role in solving exponential growth and decay problems.
- **vii.** $log\left(\dfrac{x}{y^3}\right) = logx - 3logy$
- **viii.** $log(a-b) = loga - logb$
- **ix.** $-ln\left(\dfrac{1}{x}\right) = lnx$
- **x.** $ln_{\sqrt{x}} x^k = 2k$

2. Four every question, there are four options, choose the right one.

i. If $a^x = n$, then

(a) $a = log_x n$ (b) $x = log_n$ a (c) $x = log_a n$ (d) $a = log_n x$

ii. $log_e 10 =$

(a) 2.3026 (b) 0.4343 (c) e^{10} (d) 10

iii. If $log_2 x = 5$ then x is:

(a) 25 (b) 32 (c) 10 (d) 2^{5x}

iv. If $log\,27 = 1.431$, then the value of $log\,9$ is:

(a) 0.934 (b) 0.945 (c) 0.954 (d) 0.958

v. If $log\frac{a}{b} + log\frac{b}{a} = log(a+b)$, then:

(a) $a+b=1$ (b) $a-b=1$ (c) $a=b$ (d) $a^2-b^2=1$

vi. If $log_{10}70 = a$, then $log_{10}\left(\frac{1}{70}\right)$ is equal to

(a) $-(1+a)$ (b) $(1+a)^{-1}$ (c) $\frac{a}{10}$ (d) $-a$

vii. $log\,4 + log\,25 = ?$

(a) 2 (b) 3 (c) 4 (d) 5

viii. $log_2 7$ is

(a) An integer (b) A rational number (c) An irrational number (d) A prime number

ix. 0.36^2 in standard form is

(a) 0.01296×10^1 (b) 1.296×10^{-1} (c) 0.1296×10^0 (d) 12.96×10^{-2}

x. What is the value of x in the exponential equation $9 + e^{2x-4} = 1$?

(a) 2 (b) 3 (c) 4 (d) 5

xi. $log_3 \sqrt[9]{81} =$

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$

xii. $log_b a \times log_c b$ can be written as

(a) $log_c a$ (b) $log_a c$ (c) $log_a b$ (d) $log_b c$

xiii. $log_y x$ will be equal to

(a) $\frac{log_z x}{log_y z}$ (b) $\frac{log_x z}{log_y z}$ (c) $\frac{log_z x}{log_z y}$ (d) $\frac{log_z y}{log_z x}$

xiv. Simplify $10^{12} \div 10^5 \div 10^3$

(a) 10^{20} (b) 10^{12} (c) 10^{10} (d) 10^4

xv. If $log_b x = 4$ and $log_b y = 2$, what is $log_b\left(x \times y^3\right)$?

(a) 5 (b) 7 (c) 9 (d) 10

3. Write the following in standard form.

i. 0.000094 ii. 865000000 iii. 729.89×10^3

4. Write the following in ordinary form.

i. 1.6×10^{-3} ii. 4.8×10^{-2} iii. 5.12×10^{-3}

5. Solve for x.

i. $log_{625} 5 = \frac{1}{4}x$ ii. $log_{64} x = -\frac{2}{3}$ iii. $log_{36} x = -\frac{1}{2}$ iv. $log_x 16 = 2$

6. Express in terms of x, y and z if $log_7 2 = x$, $log_7 3 = y$ and $log_7 5 = z$.

i. $log_7 60$ ii. $log_7 \frac{50}{27}$ iii. $log_7 \frac{15}{2}$ iv. $log 10.5$

7. Simplify

i. $log_b x^2 + log_b x^3 - log_b x^4$ ii. $log_k \frac{a}{b} + log_k \frac{b}{a}$ iii. $log_b(x^2 - a^2) - log_b(x - a)$ if $x > a$

8. Prove the following statements.

i. $log_{\sqrt{b}} x = 2 log_b x$ ii. $log_{\frac{1}{\sqrt{b}}} \sqrt{x} = -log_b x$

9. Solve the following logarithmic equations.

i. $log x + log(x-3) = 1$ ii. $log(x-2) + log(x+1) = 2$

iii. $2 log x = log 2 + log(3x-4)$ iv. $log x + log(x-1) = log 4x$

10. A radioactive substance is decaying such that its mass, m grams, at a time t years after initial observation is given by $m = 240e^{kt}$ where k is a constant. Given that when $t = 180$ and $m = 160$,

i. find the value of k. ii. calculate the time it takes for the mass of the substance to be halved.

11. A quantity N is increasing such that at time t, $N = 20e^{0.04t}$.

i. Find the value of N when $t = 15$. ii. Find, in terms of the constant k, expressions for the value of t when $N = k$.

12. A quantity N is decreasing such that at time t, $N = N_0 e^{kt}$. Given that at time $t = 10$, $N = 300$ and that at time $t = 20$, $N = 225$, find

i. the values of the constants N_0 and k. ii. the value of t when $N = 150$.

Review Check List

1. ☐ **Scientific Notation**
A method to represent numbers as a significant digit multiplied by 10^n. Useful for managing very large or small numbers.

a. ☐ **Standard Form**
Describes how any number, X, can be expressed as $A\times10^n$. Details conversion based on shifting the decimal.

b. ☐ **Ordinary Form**
A method to revert from scientific notation by removing the power of ten.

c. ☐ **Arithmetic Operations on Scientific Notations**
Rules for multiplication, division, addition, and subtraction using numbers in scientific notation.

2. ☐ **Logarithm**
"The logarithm of a positive real number "a" with respect to base "b", a positive real number not equal to 1 is the exponent by which b must be raised to yield "a". For instance, $\log_{10}100=2$.
Relationship between exponents and logarithms.
$b^x = a \Leftrightarrow \log_b a = x$, where $b > 0$, $b \neq 1$ & $a > 0$.

a. ☐ **Understanding via number line and graph**

b. ☐ **Points to Remember**
Logarithm of a Positive Number is Negative
Negative results in logarithms lead to expressions like $a^{-n} = \frac{1}{a^n}$.
Logarithm of a Negative Number
Logarithm of a negative number isn't possible in real numbers.
Concept of Negative Base
Logarithms can't have negative bases in real numbers context.
Logarithm of One
Logarithms with an argument of 1 are always 0 regardless of the base.
Undefined Logarithm
Logarithms are undefined for arguments of zero or negative numbers in real numbers context.

3. ☐ **Natural Logarithm**
Logarithm with base 'e', where 'e' is Euler's number (~2.71828).

a. ☐ **Euler's Number "e"**
Explains 'e' as a constant arising in various mathematical contexts, particularly in growth and decay models.

4. ☐ **Antilogarithm**
if $y = \log_b(x)$, then $x = b^y$ or $x = \text{antilog}_b(y)$

5. ☐ **Laws and Properties of Logarithm**

a. **Product Law:** $\log_b(m \times n) = \log_b m + \log_b n$

b. **Quotient Law:** $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$

c. **Power Law:** $\log_b m^n = n\log_b m$

d. **Change of Base Formula:** $\log_b a = \log_c b \times \log_c a$ for any base c.

e. **Identity Property:** $\log_b b = 1$

f. **Inverse Property**: $b^{\log_b x} = x$ & $\log_b b^x = x$

6. ☐ **Real World Applications & Uses of Logarithm**
Logarithms, acting as the inverse of exponentials, simplify complex calculations: they turn multiplication into addition, division into subtraction, and exponentiation into multiplication, making large and small number operations manageable

a. **Logarithmic Scaling**
Linear Scales: Uniform steps, e.g., 1, 2, 3 or 10, 20, 30, comparable to an arithmetic sequence.
Logarithmic Scales: Multiplicative progression (e.g., 1, 10, 100) and geometric in nature.

b. **Clarity in Visualization:** Log scales depict datasets with vast variations without losing details.

c. **Mathematical Ease:** Makes operations simpler

d. **In Sound Engineering:** Decibel (dB) scale helps in representing sound intensities.

Mathematics

9

"One Curriculum, One Nation"

قومی ترانہ

پاک سرزمین شاد باد　　کشورِ حسین شاد باد
تو نشانِ عزمِ عالی شان　　ارضِ پاکستان
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