$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore$ f(x) is neither even nor odd function.

(v) $\mathbf{f(x) = x^{2/3} + 6}$

$$f(-x) = (-x)^{2/3} + 6$$
$$= [(-x)^2]^{1/3} + 6$$
$$= (x^2)^{1/3} + 6$$
$$= x^{2/3} + 6$$
$$= f(x)$$

$\therefore$ f(x) is an even function.

(vi) $\mathbf{f(x) = \dfrac{x^3 - x}{x^2 + 1}}$

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1}$$
$$= \frac{-x^3 + x}{x^2 + 1}$$
$$= \frac{-(x^3 - x)}{x^2 + 1}$$
$$= -f(x)$$

$\therefore$ f(x) is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z. The composition of f and g is a function, denoted by gof, from X to Z and is defined by.

$$(gof)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y. The inverse function of f, denoted by f^{-1}, is a function from Y onto X and is defined by.

$$x = f^{-1}(y), \ \forall\ y \in Y \text{ if and only if } y = f(x), \forall\ x \in X$$

EXERCISE 1.2

Q.1 The real valued functions f and g are defined below. Find

(a) fog (x) (b) gof (x) (c) fof (x) (d) gog (x)

(i) $\mathbf{f(x) = 2x + 1 \ ; \ g(x) = \dfrac{3}{x-1} \ , \ x \neq 1}$

(ii) $f(x) = \sqrt{x+1}$; $g(x) = \frac{1}{x^2}$, $x \neq 0$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}$; $x \neq 1$; $g(x) = (x^2+1)^2$

(iv) $f(x) = 3x^4 - 2x^2$; $g(x) = \frac{2}{\sqrt{x}}$, $x \neq 0$

Solution:

(i) $f(x) = 2x + 1$; $g(x) = \frac{3}{x-1}$, $x \neq 1$

(a) $$\begin{aligned} fog\,(x) &= f(g(x)) \\ &= f\left(\frac{3}{x-1}\right) \\ &= 2\left(\frac{3}{x-1}\right) + 1 \\ &= \frac{6}{x-1} + 1 \\ &= \frac{6 + x - 1}{x-1} \\ &= \frac{x+5}{x-1} \quad \text{Ans.} \end{aligned}$$

(b) $$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(2x+1) \\ &= \frac{3}{2x+1-1} = \frac{3}{2x} \quad \text{Ans.} \end{aligned}$$

(c) $$\begin{aligned} fof(x) &= f(f(x)) \\ &= f(2x+1) \\ &= 2(2x+1) + 1 \\ &= 4x + 2 + 1 \\ &= 4x + 3 \quad \text{Ans.} \end{aligned}$$

(d) $$\begin{aligned} gog(x) &= g(g(x)) \\ &= g\left(\frac{3}{x-1}\right) \\ &= \frac{3}{\frac{3}{x-1} - 1} \end{aligned}$$

$$= \frac{3}{\frac{3-(x-1)}{x-1}}$$

$$= \frac{3(x-1)}{3-x+1}$$

$$= \frac{3(x-1)}{4-x} \quad \text{Ans.}$$

(ii) $\mathbf{f(x) = \sqrt{x+1} \quad ; \quad g(x) = \frac{1}{x^2} \quad , \quad x \neq 0}$

(a) $fog(x) = f(g(x))$

$$= f\left(\frac{1}{x^2}\right)$$

$$= \sqrt{\frac{1}{x^2}+1}$$

$$= \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{x} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\sqrt{x+1}\right)$$

$$= \frac{1}{\left(\sqrt{x+1}\right)^2} = \frac{1}{x+1} \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f\left(\sqrt{x+1}\right)$$

$$= \sqrt{\sqrt{x+1}+1} \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4 \quad \text{Ans.}$$

(iii) $\mathbf{f(x) = \frac{1}{\sqrt{x-1}} \quad ; \quad x \neq 1 \quad ; \quad g(x) = (x^2+1)^2}$

(a) $fog(x) = f(g(x))$

$$= f((x^2+1)^2)$$

$$= \frac{1}{\sqrt{(x^2+1)^2-1}}$$

$$= \frac{1}{\sqrt{x^4 + 1 + 2x^2 - 1}}$$

$$= \frac{1}{\sqrt{x^2(x^2+2)}} = \frac{1}{x\sqrt{x^2+2}} \quad \text{Ans.}$$

(b) $\text{gof}(x) = g(f(x))$

$$= g\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2$$

$$= \left(\frac{1}{x-1} + 1\right)^2 = \left(\frac{1 + x - 1}{x-1}\right)^2$$

$$= \left(\frac{x}{x-1}\right)^2 \quad \text{Ans.}$$

(c) $\text{fof}(x) = f(f(x))$

$$= f\left(\frac{1}{\sqrt{x-1}}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}}$$

$$= \frac{1}{\sqrt{\frac{1-\sqrt{x-1}}{\sqrt{x-1}}}} = \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x-1}}} \quad \text{Ans.}$$

(d) $\text{gog}(x) = g(g(x))$

$$= g((x^2+1)^2)$$

$$= [\{(x^2+1)^2\}^2 + 1]^2$$

$$= [(x^2+1)^4 + 1]^2 \quad \text{Ans.}$$

(iv) $\mathbf{f(x) = 3x^4 - 2x^2 \;;\; g(x) = \frac{2}{\sqrt{x}} \;,\; x \neq 0}$

(a) $\text{fog}(x) = f(g(x))$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2}$$

$$= \frac{8(6 - x)}{x^2} \qquad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g(3x^4 - 2x^2)$$

$$= \frac{2}{\sqrt{3x^4 - 2x^2}}$$

$$= \frac{2}{\sqrt{x^2(3x^2 - 2)}} \qquad = \frac{2}{x\sqrt{3x^2 - 2}} \qquad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f(3x^4 - 2x^2)$$

$$= 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)^2 \qquad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{2}{\sqrt{x}}\right)$$

$$= \frac{2}{\sqrt{2/\sqrt{x}}}$$

$$= 2\sqrt{\frac{\sqrt{x}}{2}}$$

$$= \sqrt{2} \times \sqrt{2}\ \frac{\sqrt{\sqrt{x}}}{\sqrt{2}}$$

$$= \sqrt{2\sqrt{x}} \qquad \text{Ans.}$$

Q.2 For the real valued function, f defined below, find:

(a) $f^{-1}(x)$

(b) $f^{-1}(-1)$ **and verify** $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

(i) $f(x) = -2x + 8$ ***(Lahore Board 2007,2009)*** **(ii)** $f(x) = 3x^3 + 7$

(iii) $f(x) = (-x + 9)^3$ **(iv)** $f(x) = \frac{2x + 1}{x - 1}$, $x > 1$

Solution:

(i) $\quad \mathbf{f(x) = -2x + 8}$

(a) Since $\quad y = f(x)$

$$\Rightarrow x = f^{-1}(y)$$

Now,

$$f(x) = -2x + 8$$
$$y = -2x + 8$$
$$2x = 8 - y$$
$$x = \frac{8 - y}{2}$$
$$f^{-1}(y) = \frac{8 - y}{2}$$

Replacing y by x

$$f^{-1}(x) = \frac{8 - x}{2}$$

Replacing y by x.

$$\boxed{f^{-1}(x) = \frac{8 - x}{2}}$$

(b) Put, $\quad x = -1$

$$f^{-1}(-1) = \frac{8 - (-1)}{2} = \frac{8 + 1}{2} = \frac{9}{2}$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{8 - x}{2}\right)$$
$$= -2\left(\frac{8 - x}{2}\right) + 8$$
$$= -8 + x + 8$$
$$= x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(-2x + 8)$$
$$= \frac{8 - (-2x + 8)}{2}$$
$$= \frac{8 + 2x - 8}{2}$$
$$= \frac{2x}{2} \quad = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ $\qquad$ Hence proved.

(ii) f(x) = $3x^3 + 7$

(a) Since $y = f(x)$

$\Rightarrow x = f^{-1}(y)$

Now

$$f(x) = 3x^3 + 7$$
$$y = 3x^3 + 7$$
$$3x^3 = y - 7$$
$$x^3 = \frac{y-7}{3}$$
$$x = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$
$$f^{-1}(y) = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{\frac{1}{3}}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{\frac{1}{3}}$$
$$= \left(\frac{-8}{3}\right)^{\frac{1}{3}}$$

$$f\left(f^{-1}(x)\right) = f\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]$$
$$= 3\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]^3 + 7$$
$$= 3\left(\frac{x-7}{3}\right) + 7$$
$$= x - 7 + 7 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(3x^3 + 7)$$
$$= \left(\frac{3x^3 + 7 - 7}{3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{3x^3}{3}\right)^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iii) $f(x) = (-x+9)^3$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$f(x) = (-x+9)^3$

$y = (-x+9)^3$

$y^{\frac{1}{3}} = -x+9$

$x = 9 - y^{\frac{1}{3}}$

Replacing y by x

$f^{-1}(x) = 9 - x^{\frac{1}{3}}$

(b) Put $x = -1$

$f^{-1}(-1) = 9 - (-1)^{\frac{1}{3}}$

$$f\left(f^{-1}(x)\right) = f\left(9 - x^{\frac{1}{3}}\right)$$

$$= \left[-\left(9 - x^{\frac{1}{3}}\right) + 9\right]^3$$

$$= \left(-9 + x^{\frac{1}{3}} + 9\right)^3$$

$$= \left(x^{\frac{1}{3}}\right)^3 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left((-x+9)^3\right)$$

$$= 9 - [(-x+9)^3]^{\frac{1}{3}}$$

$$= 9 - (-x+9)$$

$$= 9 + x - 9$$

$$= x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iv) $\mathbf{f(x) = \dfrac{2x + 1}{x - 1}}$, $\mathbf{x > 1}$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$$f(x) = \frac{2x + 1}{x - 1}$$

$$y = \frac{2x + 1}{x - 1}$$

$$y(x - 1) = 2x + 1$$

$$yx - y = 2x + 1$$

$$yx - 2x = 1 + y$$

$$x(y - 2) = y + 1$$

$$x = \frac{y + 1}{y - 2}$$

$$f^{-1}(y) = \frac{y + 1}{y - 2}$$

Replacing y by x

$$f^{-1}(x) = \frac{x + 1}{x - 2}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \frac{-1 + 1}{-1 - 2}$$

$$= \frac{0}{-3} = 0$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{x + 1}{x - 2}\right)$$

$$= \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1}$$

$$= \frac{\frac{2(x + 1) + (x - 2)}{x - 2}}{\frac{x + 1 - (x - 2)}{x - 2}}$$

$$= \frac{2x + 2 + x - 2}{x + 1 - x + 2}$$

$$= \frac{3x}{3} = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{2x+1}{x-1}\right)$$

$$= \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2}$$

$$= \frac{\frac{2x + 1 + x - 1}{x - 1}}{\frac{2x + 1 - 2(x - 1)}{x - 1}}$$

$$= \frac{3x}{2x + 1 - 2x + 2}$$

$$= \frac{3x}{3} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

Q.3 Without finding the inverse, state the domain and range of f^{-1}.

(i) $f(x) = \sqrt{x+2}$ **(ii)** $f(x) = \frac{x-1}{x-4}, x \neq 4$

(iii) $f(x) = \frac{1}{x+3}, x \neq -3$ **(iv)** $f(x) = (x-5)^2, x \geq 5$

Solution:

(i) $f(x) = \sqrt{x+2}$

Domain of $f(x)$ = $[-2, +\infty)$

Range of $f(x)$ = $[0, +\infty)$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $[0, +\infty)$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $[-2, +\infty)$

(ii) $f(x) = \frac{x-1}{x-4}, x \neq 4$

Domain of $f(x)$ = $R - \{4\}$

Range of $f(x)$ = $R - \{1\}$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $R - \{1\}$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $R - \{4\}$

(iii) $\mathbf{f(x) = \dfrac{1}{x+3}, x \neq -3}$

Domain of $f(x)$ = $R - \{-3\}$

Range of $f(x)$ = $R - \{0\}$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $R - \{0\}$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $R - \{-3\}$

(iv) $\mathbf{f(x) = (x-5)^2, x \geq 5}$ ***(Gujranwala Board 2007)***

Domain of $f(x)$ = $[5, +\infty)$

Range of $f(x)$ = $[0, +\infty)$

Domain of $f^{-1}(x)$ = Range of $f(x)$ = $[0, +\infty)$

Range of $f^{-1}(x)$ = Domain of $f(x)$ = $[5, +\infty)$

Limit of a Function:

Let a function f(x) be defined in an open interval near the number 'a' (need not at a) if, as x approaches 'a' from both left and right side of 'a', f(x) approaches a specific number 'L' then 'L', is called the limit of f(x) as x approaches a symbolically it is written as.

$$\lim_{x \to a} f(x) = L$$ read as "Limit of f(x) as $x \to a$, is L"

Theorems on Limits of Functions:

Let f and g be two functions, for which $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

Theorem 1: The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

Theorem 2: The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L - M$$

Theorem 3: If K is any real numbers, then.

$$\lim_{x \to a} [kf(x)] = K \lim_{x \to a} f(x) = kL$$

Theorem 4: The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x \to a} [f(x) \,.\, g(x)] = [\lim_{x \to a} f(x)]\, [\lim_{x \to a} g(x)] = LM$$

Theorem 5: The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad , \quad g(x) \neq 0,\ M \neq 0$$

Theorem 6: Limit of $[f(x)]^n$, where n is an integer.

$$\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$$

The Sandwich Theorem:

Let f, g and h be functions such that $f(x) \leq g(x) \leq h(x)$ for all number x in some open interval containing "C", except possibly at C itself.

If, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$

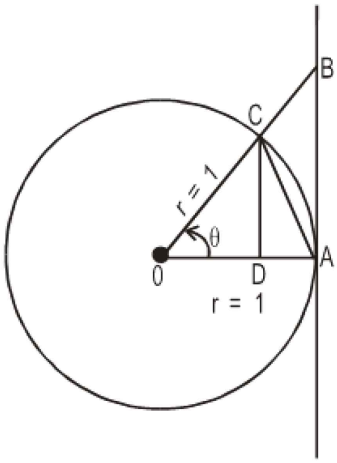
Prove that

If θ is measured in radian, then

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Proof:

Take θ a positive acute central angle of a circle with radius r = 1. OAB represents the sector of the circle.



$|OA| = |OC| = 1$ (radii of unit circle)

From right angle ΔODC

$$\text{Sin}\theta = \frac{|DC|}{|OC|} = |DC| \quad (\because\ |OC| = 1)$$

From right angle ΔOAB

$$\text{Tan}\theta = \frac{|AB|}{|OA|} = AB \quad (\because\ |OA| = 1)$$

In terms of θ, the areas are expressed as

$$\text{Area of } \Delta OAC = \frac{1}{2}\,|OA|\,|CD| = \frac{1}{2}(1)\sin\theta = \frac{1}{2}\sin\theta$$

Area of sector OAC $= \frac{1}{2} r^2\theta = \frac{1}{2} (1)(\theta) = \frac{1}{2}\theta$

Area of ΔOAB $= \frac{1}{2} |OA|\,|AB| = \frac{1}{2} (1) \tan\theta = \frac{1}{2}\tan\theta$

From figure

Area of ΔOAB > Area of sector OAC > Area of ΔOAC

$$\frac{1}{2}\tan\theta > \frac{1}{2}\theta > \frac{1}{2}\sin\theta$$

$$\frac{1}{2}\frac{\sin\theta}{\cos\theta} > \frac{\theta}{2} > \frac{\sin\theta}{2}$$

As $\sin\theta$ is positive, so on division by $\frac{1}{2}\sin\theta$, we get.

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1 \qquad (0 < \theta < \pi/2)$$

i.e.

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

When, $\theta \to 0$, $\cos\theta \to 1$

Since $\frac{\sin\theta}{\theta}$ is sandwitched between 1 and a quantity approaching 1 itself.

So by the sandwitch theorem it must also approach 1.

i.e.

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

Theorem: Prove that

$$\lim_{n\to +\infty} \left(1+\frac{1}{n}\right)^n = e$$

Proof:

Taking

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots\ldots\ldots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots\ldots$$

Taking $\lim_{n\to +\infty}$ on both sides.

$$\lim_{n\to +\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\ldots\ldots$$

$$= 1 + 1 + 0.5 + 0.166667 + 0.0416667 + \ldots\ldots\ldots$$

$$= 2.718281 \ldots\ldots\ldots$$

As approximate value of e is $= 2.718281$

$$\therefore \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^2 = e$$

Deduction:

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

We know that.

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Put $x = \frac{1}{n}$ then $\frac{1}{x} = n$

As $n \to +\infty$, $x \to 0$

$$\therefore \lim_{n \to +\infty} (1 + x)^{1/x} = e$$

Theorem:

Prove that:

$$\lim_{x \to a} \frac{a^x - 1}{x} = \log_e a$$

Proof:

Taking,

$$\lim_{x \to a} \frac{a^x - 1}{x}$$

Let $a^x - 1 = y$

$a^x = 1 + y$

$x = \log_a (1 + y)$

As, $x \to a$, $y \to 0$

$$\lim_{x \to a} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a(1 + y)}$$

$$= \lim_{y \to 0} \frac{1}{\frac{1}{y} \log_a(1 + y)} = \lim_{y \to 0} \frac{1}{\log_a(1 + y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a e} \qquad \because \lim_{y \to 0} (1 + y)^{1/y} = e$$

$$= \log_e a$$