

## Deduction

$$\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) = \log_e e = 1$$

We know that

$$\lim_{x \rightarrow 0} \left( \frac{a^x - 1}{x} \right) = \log_a a$$

Put  $a = e$

$$\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) = \log_e e = 1$$

## Important results to remember

$$(i) \quad \lim_{x \rightarrow +\infty} (e^x) = \infty \quad (ii) \quad \lim_{x \rightarrow -\infty} (e^x) = \lim_{x \rightarrow -\infty} \left( \frac{1}{e^{-x}} \right) = 0$$

$$(iii) \quad \lim_{x \rightarrow \pm\infty} \left( \frac{a}{x} \right) = 0, \text{ where } a \text{ is any real number.}$$

## EXERCISE 1 . 3

**Q.1 Evaluate each limit by using theorems of limits.**

$$(i) \quad \lim_{x \rightarrow 3} (2x + 4)$$

$$(ii) \quad \lim_{x \rightarrow 1} (3x^2 - 2x + 4)$$

$$(iii) \quad \lim_{x \rightarrow 3} \sqrt{x^2 + x + 4}$$

$$(iv) \quad \lim_{x \rightarrow 2} x \sqrt{x^2 - 4}$$

$$(v) \quad \lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5}) \quad (vi) \quad \lim_{x \rightarrow 2} \frac{2x^3 + 5x}{3x - 2}$$

**Solution:**

$$(i) \quad \lim_{x \rightarrow 3} (2x + 4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4) \\ = 2 \lim_{x \rightarrow 3} x + 4 \\ = 2(3) + 4 = 6 + 4 = 10 \quad \text{Ans.}$$

$$(ii) \quad \lim_{x \rightarrow 1} (3x^2 - 2x + 4) = \lim_{x \rightarrow 1} (3x^2) - \lim_{x \rightarrow 1} (2x) + \lim_{x \rightarrow 1} (4) \\ = 3 \lim_{x \rightarrow 1} x^2 - 2 \lim_{x \rightarrow 1} x + 4 \\ = 3(1)^2 - 2(1) + 4 \\ = 3 - 2 + 4 \\ = 5 \quad \text{Ans.}$$

$$(iii) \quad \lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = [\lim_{x \rightarrow 3} (x^2 + x + 4)]^{1/2}$$

**Important Limits**

I.  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ , where  $n$  is integer and  $a > 0$ .

II.  $\lim_{x \rightarrow 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$ .

III.  $\lim_{n \rightarrow 0} \left(1 + \frac{1}{n}\right)^n = e$ .

IV.  $\lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}} = e$ .

V.  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$ , where  $a > 0$ .

VI.  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1$ .

VII. If  $\theta$  is measured in radian, then  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ .

**Question # 1**

(i)  $\lim_{x \rightarrow 3} (2x+4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4) = 2\lim_{x \rightarrow 3} (x) + 4 = 2(3) + 4 = 10$ .

(ii)  $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = 3(1)^2 - 2(1) + 4 = 3 - 2 + 4 = 5$ .

(iii)  $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = \sqrt{(3)^2 + (3) + 4} = \sqrt{9 + 3 + 4} = \sqrt{16} = 4$ .

(iv)  $\lim_{x \rightarrow 2} x\sqrt{x^2 - 4} = 2\sqrt{2^2 - 4} = 0$ .

(v)  $\lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5}) = \lim_{x \rightarrow 2} (\sqrt{x^3 + 1}) - \lim_{x \rightarrow 2} (\sqrt{x^2 + 5})$   
 $= (\sqrt{(2)^3 + 1}) - (\sqrt{(2)^2 + 5})$   
 $= \sqrt{8 + 1} - \sqrt{4 + 5} = \sqrt{9} - \sqrt{9} = 0$ .

(vi)  $\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2} = \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} = \frac{-16 - 10}{-6 - 2} = \frac{-26}{-8} = \frac{13}{4}$ .

**Question # 2**

(i)  $\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} = \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{x + 1} = \lim_{x \rightarrow -1} \frac{x(x+1)(x-1)}{x+1}$   
 $= \lim_{x \rightarrow -1} x(x-1) = (-1)(-1-1) = 2$

(ii)  $\lim_{x \rightarrow 0} \left( \frac{3x^3 + 4x}{x^2 + x} \right) = \lim_{x \rightarrow 0} \frac{x(3x^2 + 4)}{x(x+1)} = \lim_{x \rightarrow 0} \frac{3x^2 + 4}{x+1} = \frac{3(0) + 4}{0+1} = 4$ .

$$\begin{aligned}
 \text{(iii)} \quad & \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6} \\
 &= \lim_{x \rightarrow 2} \frac{x^3 - (2)^3}{x^2 + 3x - 2x - 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{x(x+3) - 2(x+3)} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x+3)(x-2)} = \lim_{x \rightarrow 2} \frac{(x^2 + 2x + 4)}{(x+3)} \\
 &= \frac{(2)^2 + 2(2) + 4}{(2+3)} = \frac{12}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x^2 - 1)} = \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x-1)(x+1)} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x+1)} = \lim_{x \rightarrow 1} \frac{(1-1)^2}{(1)(1+1)} = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & \lim_{x \rightarrow -1} \left( \frac{x^3 + x^2}{x^2 - 1} \right) = \lim_{x \rightarrow -1} \frac{x^2(x+1)}{(x+1)(x-1)} = \lim_{x \rightarrow -1} \frac{x^2}{(x-1)} \\
 &= \frac{(-1)^2}{(-1-1)} = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2} = \lim_{x \rightarrow 4} \frac{2(x^2 - 16)}{x^2(x-4)} \\
 &= \lim_{x \rightarrow 4} \frac{2(x+4)(x-4)}{x^2(x-4)} = \lim_{x \rightarrow 4} \frac{2(x+4)}{x^2} \\
 &= \frac{2(4+4)}{4^2} = \frac{16}{16} = 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad & \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x-2} = \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x-2} \left( \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} \right) \\
 &= \lim_{x \rightarrow 2} \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x-2)(\sqrt{x} + \sqrt{2})} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(\sqrt{x} + \sqrt{2})}
 \end{aligned}$$

$$= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\begin{aligned}
 \text{(viii)} \quad & \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
 &= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

(ix)  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m}$

$$\begin{aligned}
 &= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x-a)(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
 &= \lim_{x \rightarrow a} \frac{(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
 &= \frac{a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1}}{a^{m-1} + a^{m-2}a + a^{m-3}a^2 + \dots + a^{m-1}} \\
 &= \frac{a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1}}{a^{m-1} + a^{m-1} + a^{m-1} + \dots + a^{m-1}} \quad (n \text{ terms}) \\
 &= \frac{n a^{n-1}}{m a^{m-1}} = \frac{n}{m} a^{n-1-m+1} = \frac{n}{m} a^{n-m}
 \end{aligned}$$

### Law of Sine

If  $\theta$  is measured in radian, then  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

*See proof on book at page 25*

### Question # 3

(i)  $\lim_{x \rightarrow 0} \frac{\sin 7x}{x}$

Put  $t = 7x \Rightarrow \frac{t}{7} = x$

When  $x \rightarrow 0$  then  $t \rightarrow 0$ , so

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin 7x}{x} &= \lim_{t \rightarrow 0} \frac{\sin t}{t} \\
 &= 7 \lim_{t \rightarrow 0} \frac{\sin t}{t} = 7(1) = 7 \quad \text{By law of sine.}
 \end{aligned}$$

(ii)  $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$

Since  $180^\circ = \pi$  rad  $\Rightarrow 1^\circ = \frac{\pi}{180}$  rad  $\Rightarrow x^\circ = \frac{x\pi}{180}$  rad

So  $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x}$

Now put  $\frac{\pi x}{180} = t$  i.e.  $x = \frac{180t}{\pi}$

When  $x \rightarrow 0$  then  $t \rightarrow 0$ , so

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin \pi x / 180}{x} &= \lim_{x \rightarrow 0} \frac{\sin t}{180t / \pi} \\ &= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin t}{t} = \frac{\pi}{180} (1) = \frac{\pi}{180} \quad \text{by law of sine}\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(1 + \cos \theta)} = \frac{\sin(0)}{1 + \cos(0)} = \frac{0}{1+1} = 0\end{aligned}$$

$$\text{(iv)} \quad \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

Put  $t = \pi - x \Rightarrow x = \pi - t$

When  $x \rightarrow \pi$  then  $t \rightarrow 0$ , so

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} &= \lim_{t \rightarrow 0} \frac{\sin(\pi - t)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\sin t}{t} \quad \because \sin(\pi - t) = \sin\left(2 \cdot \frac{\pi}{2} - t\right) = \sin t \\ &= 1 \quad \text{By law of sine.}\end{aligned}$$

$$\begin{aligned}\text{(v)} \quad \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \sin ax \cdot \frac{1}{\sin bx} \\ &= \lim_{x \rightarrow 0} \sin ax \cdot \left(\frac{ax}{ax}\right) \frac{1}{\sin bx \cdot \left(\frac{bx}{bx}\right)} = \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot ax \frac{1}{\frac{\sin bx}{bx} \cdot bx} \\ &= \frac{a}{b} \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot \frac{1}{\lim_{x \rightarrow 0} \frac{\sin bx}{bx}} = \frac{a}{b} \cdot (1) \cdot \frac{1}{(1)} = \frac{a}{b} \quad \text{by law of sine}\end{aligned}$$

$$\begin{aligned}\text{(vi)} \quad \lim_{x \rightarrow 0} \frac{x}{\tan x} &= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \cos x \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} \cdot \cos x = \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \cdot \lim_{x \rightarrow 0} \cos x = \frac{1}{1} \cdot 1 = 1\end{aligned}$$

$$\begin{aligned}\text{(vii)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} &\quad \because \sin^2 x = \frac{1 - \cos 2x}{2} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \quad \therefore 2 \sin^2 x = 1 - \cos 2x\end{aligned}$$

$$= 2 \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^2 = 2 \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 = 2(1)^2 = 2$$

(vii) *Do yourself by rationalizing*

$$\begin{aligned} \text{(viii)} \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \sin \theta \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \sin \theta = (1) \cdot (0) = 0 \end{aligned}$$

$$\begin{aligned} \text{(x)} \quad \lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} - \cos x}{x} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos^2 x}{\cos x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = (1) \frac{\sin(0)}{\cos(0)} = (1) \cdot \frac{0}{1} = 0 \end{aligned}$$

$$\begin{aligned} \text{(xi)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{p\theta}{2}}{2 \sin^2 \frac{q\theta}{2}} \quad \because \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \\ &= \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2}} = \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2} \cdot \frac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}} \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2} \cdot \left(\frac{p\theta}{2}\right)^2 \cdot \frac{1}{\sin^2 \frac{q\theta}{2} \cdot \left(\frac{q\theta}{2}\right)^2} = \lim_{x \rightarrow 0} \left( \frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left( \frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} \cdot \frac{p^2 \theta^2 / 4}{q^2 \theta^2 / 4} \end{aligned}$$

$$\begin{aligned} &= \frac{p^2}{q^2} \left( \lim_{x \rightarrow 0} \frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left( \lim_{x \rightarrow 0} \frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} = \frac{p^2}{q^2} (1)^2 \cdot \frac{1}{(1)^2} = \frac{p^2}{q^2} \end{aligned}$$

$$\begin{aligned}
 \text{(xii)} \quad & \lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta} - \sin \theta}{\sin^3 \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta - \sin \theta \cos \theta}{\cos \theta}}{\sin^3 \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\sin \theta - \sin \theta \cos \theta}{\sin^3 \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta(1 - \cos \theta)}{\sin^3 \theta \cos \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} \\
 &= \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} = \lim_{x \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} \\
 &= \frac{1}{\cos(1)(1 + \cos(1))} = \frac{1}{1 \cdot (1+1)} = \frac{1}{2}
 \end{aligned}$$

**Note:**

a)  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

b)  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$  where  $e = 2.718281\dots$

*See proof of (a) and (b) on book at page 23*

c)  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$  or  $\ln a$

**Proof:**

Put  $y = a^x - 1 \dots \dots (i)$

When  $x \rightarrow 0$  then  $y \rightarrow 0$

Also from (i)  $1 + y = a^x$

Taking log on both sides

$$\begin{aligned}
 \ln(1+y) &= \ln a^x \Rightarrow \ln(1+y) = x \ln a & \because \ln x^m = m \ln x \\
 \Rightarrow x &= \frac{\ln(1+y)}{\ln a}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} &= \lim_{y \rightarrow 0} \frac{y}{\frac{\ln(1+y)}{\ln a}} \\
 &= \lim_{y \rightarrow 0} \frac{y \ln a}{\ln(1+y)} = \lim_{y \rightarrow 0} \frac{\ln a}{\frac{1}{y} \ln(1+y)}
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{y \rightarrow 0} \frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} = \frac{\ln a}{\lim_{y \rightarrow 0} \ln(1+y)^{\frac{1}{y}}} \quad \because \ln x^m = m \ln x \\
&= \frac{\ln a}{\ln \left( \lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}} \right)} = \frac{\ln a}{\ln(e)} \quad \because \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\
&= \frac{\ln a}{1} = \ln a \quad \because \ln e = 1
\end{aligned}$$

**Question # 4**

$$(i) \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{2n} = \left[ \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \right]^2 = e^2$$

$$(ii) \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{\frac{n}{2}} = \left[ \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

$$(iii) \quad \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^n = \left[ \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^{-n} \right]^{-1} = e^{-1} = \frac{1}{e}$$

$$(iv) \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^n \\ = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^{\frac{3n}{3}} = \left[ \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^{3n} \right]^{\frac{1}{3}} = e^{\frac{1}{3}}$$

$$(v) \quad \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^n = \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^{\frac{4n}{4}} = \left[ \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^{\frac{n}{4}} \right]^4 = e^4.$$

$$(vi) \quad \lim_{x \rightarrow 0} (1+3x)^{\frac{2}{x}} \\ = \lim_{x \rightarrow 0} (1+3x)^{\frac{6}{3x}} = \left[ \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right]^6 = e^6$$

$$(vii) \quad \lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} (1+2x^2)^{\frac{2}{2x^2}} = \left[ \lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{2x^2}} \right]^2 = e^2$$

$$(viii) \quad \lim_{h \rightarrow 0} (1-2h)^{\frac{1}{h}} = \lim_{h \rightarrow 0} (1-2h)^{\frac{-2}{-2h}} = \left[ \lim_{h \rightarrow 0} (1-2h)^{\frac{1}{-2h}} \right]^{-2} = e^{-2} = \frac{1}{e^2}$$

$$(ix) \quad \lim_{x \rightarrow \infty} \left(\frac{x}{1+x}\right)^x \\ = \lim_{x \rightarrow \infty} \left(\frac{1+x}{x}\right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{x}{x}\right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + 1\right)^{-x}$$

$$= \left[ \lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^x \right]^{-1} = e^{-1} = \frac{1}{e}$$

(x)  $\lim_{x \rightarrow 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} ; \quad x < 0$

Put  $x = -t$  where  $t > 0$

When  $x \rightarrow 0$  then  $t \rightarrow 0$ , so

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} &= \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t}} - 1}{e^{-\frac{1}{t}} + 1} = \frac{e^{-\frac{1}{0}} - 1}{e^{-\frac{1}{0}} + 1} \\ &= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0 - 1}{0 + 1} \quad \therefore e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 \\ &= -1 \end{aligned}$$

(xi)  $\lim_{x \rightarrow 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} ; \quad x > 0$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x}} \left( 1 - \frac{1}{e^{\frac{1}{x}}} \right)}{e^{\frac{1}{x}} \left( 1 + \frac{1}{e^{\frac{1}{x}}} \right)} = \lim_{x \rightarrow 0} \frac{\left( 1 - \frac{1}{e^{\frac{1}{x}}} \right)}{\left( 1 + \frac{1}{e^{\frac{1}{x}}} \right)} \\ &= \frac{1 - \frac{1}{e^{\frac{1}{0}}}}{1 + \frac{1}{e^{\frac{1}{0}}}} = \frac{1 - \frac{1}{e^\infty}}{1 + \frac{1}{e^\infty}} = \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \end{aligned}$$

Error Analyst		
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Book: *Exercise 1.3 (Page 27)*  
*Calculus and Analytic Geometry Mathematic I2*  
*Punjab Textbook Board, Lahore.*  
*Edition: May 2013.*

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Page Setup used A4.

Printed: December 19, 2014.