Question # 1

Find the definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2+1$ (ii) $2-\sqrt{x}$ (iii) $\dfrac{1}{\sqrt{x}}$ (iv) $\dfrac{1}{x^3}$

(v) $\dfrac{1}{x-a}$ (vi) $x(x-3)$ (vii) $\dfrac{2}{x^4}$ (viii) $(x+4)^{\frac{1}{3}}$

(ix) $x^{\frac{3}{2}}$ (x) $x^{5/2}$ (xi) $x^m, m\in N$ (xii) $\dfrac{1}{x^m}, m\in N$

(xiii) x^{40} (xiv) x^{-100}

Solution

(i) Let $y=2x^2+1$

$\Rightarrow\ y+\delta y=2(x+\delta x)^2+1 \qquad \Rightarrow\ \delta y=2(x+\delta x)^2+1-y$

$\Rightarrow\ \delta y=2\left(x^2+2x\delta x+\delta x^2\right)+1-2x^2-1 \qquad \because\ y=2x^2+1$

$\Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2 \qquad \Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2$

$\Rightarrow\ \delta y=4x\delta x+2\delta x^2$

$=\delta x(4x+2\delta x)$

Dividing by δx

$$\frac{\delta y}{\delta x}=4x+2\delta x$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(4x+2\delta x)$$

$$\Rightarrow\ \frac{dy}{dx}=4x+2(0)$$

$$\Rightarrow\ \frac{dy}{dx}=4x \quad \text{i.e.} \quad \boxed{\frac{d}{dx}\left(2x^2+1\right)=4x}$$

(ii) Let $y=2-\sqrt{x}$

$\Rightarrow\ y+\delta y=2-\sqrt{x+\delta x} \qquad \Rightarrow\ \delta y=2-\sqrt{x+\delta x}-y$

$\Rightarrow\ \delta y=2-\sqrt{x+\delta x}-2+\sqrt{x} \qquad \Rightarrow\ \delta y=x^{\frac{1}{2}}-(x+\delta x)^{\frac{1}{2}}$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{1}{2}\cdot\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\right)$$

$$= x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{1}{2}}\left(\frac{\delta x}{2x} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{\delta x^2}{x^2} + ...\right)$$

$$= -x^{\frac{1}{2}}\delta x\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

Taking limit as

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\lim_{\delta x \to 0}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + ...\right)$$

$$\Rightarrow \frac{dy}{dx} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - 0 + 0 - ....\right)$$

$$= -x^{\frac{1}{2}} \cdot \frac{1}{2x} = -\frac{1}{2}x^{\frac{1}{2}-1} \Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}}$$

(iii) Let $y = \frac{1}{\sqrt{x}} \quad \Rightarrow \quad y = x^{-\frac{1}{2}}$

Now do yourself as above

(iv) Let $y = \frac{1}{x^3} \quad \Rightarrow \quad y = x^{-3}$

$$\Rightarrow y + \delta y = (x + \delta x)^{-3}$$

$$\Rightarrow \delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\Rightarrow \delta y = x^{-3}\left[\left(1 + \frac{\delta x}{x}\right)^{-3} - 1\right]$$

$$= x^{-3}\left[\left(1 - \frac{3\delta x}{x} + \frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + ...\right) - 1\right]$$

$$= x^{-3}\left[1 - \frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + ... - 1\right]$$

$$= x^{-3}\left[-\frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + ...\right]$$

$$= x^{-3} \cdot \frac{\delta x}{x}\left[-3 + 6\left(\frac{\delta x}{x}\right) - .....\right]$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = x^{-3-1}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

Taking limit on both sided, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{-4}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{-4}\left[-3+0-0+.....\right]$$

$$\Rightarrow \frac{dy}{dx} = -3x^{-4} \quad \text{or} \quad \boxed{\frac{dy}{dx} = -\frac{3}{x^4}}$$

(v) Let $y = \frac{1}{x-a}$

$$\Rightarrow y = (x-a)^{-1}$$

$$\Rightarrow y+\delta y = (x+\delta x-a)^{-1}$$

$$\Rightarrow \delta y = (x-a+\delta x)^{-1} - y$$

$$\Rightarrow \delta y = (x-a+\delta x)^{-1} - (x-a)^{-1}$$

$$= (x-a)^{-1}\left[\left(1+\frac{\delta x}{x-a}\right)^{-1} - 1\right]$$

$$= (x-a)^{-1}\left[\left(1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....\right)-1\right]$$

$$\Rightarrow \delta y = (x-a)^{-1}\left[1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....-1\right]$$

$$= (x-a)^{-1}\left[-\frac{\delta x}{x-a}+\frac{-1(-2)}{2}\left(\frac{\delta x}{x-a}\right)^2+....\right]$$

$$= (x-a)^{-1}\cdot\frac{\delta x}{x-a}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = (x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Taking limit when $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

$$\Rightarrow \frac{dy}{dx} = (x-a)^{-2}\left[-1+0-0+....\right] \quad \Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{(x-a)^2}}$$

(vi) Let $y = x(x-3)$

$= x^2 - 3x$

Do yourself

(vii) Let $y = \dfrac{2}{x^4} = 2x^{-4}$

$\Rightarrow y+\delta y = 2(x+\delta x)^{-4}$

Do yourself

(viii) Let $y = (x+4)^{\frac{1}{3}}$

$$\Rightarrow y+\delta y=(x+\delta x+4)^{\frac{1}{3}}$$

$$\Rightarrow \delta y=(x+\delta x+4)^{\frac{1}{3}}-y$$

$$=(x+4+\delta x)^{\frac{1}{3}}-(x+4)^{\frac{1}{3}}$$

$$=(x+4)^{\frac{1}{3}}\left[\left(1+\frac{\delta x}{x+4}\right)^{\frac{1}{3}}-1\right]$$

$$=(x+4)^{\frac{1}{3}}\left[\left(1+\frac{1}{3}\frac{\delta x}{x+4}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2+\dots\right)-1\right]$$

$$=(x+4)^{\frac{1}{3}}\left[\frac{\delta x}{3(x+4)}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)^2+\dots\right]$$

$$=(x+4)^{\frac{1}{3}}\cdot\frac{\delta x}{x+4}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=(x+4)^{\frac{1}{3}-1}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+\dots\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-0+0-\dots\right] \quad\Rightarrow\quad \boxed{\frac{dy}{dx}=\frac{1}{3}(x+4)^{-\frac{2}{3}}}$$

(ix) Let $y = x^{\frac{3}{2}}$

$$\Rightarrow y+\delta y=(x+\delta x)^{\frac{3}{2}}$$

$$\Rightarrow \ \delta y=(x+\delta x)^{\frac{3}{2}}-x^{\frac{3}{2}}$$

$$=x^{\frac{3}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{3}{2}}-1\right]$$

$$=x^{\frac{3}{2}}\left[\left(1+\frac{3}{2}\frac{\delta x}{x}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^{2}+....\right)-1\right]$$

$$=x^{\frac{3}{2}}\left[\frac{3\delta x}{2x}+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2}\left(\frac{\delta x}{x}\right)^{2}+....\right]$$

$$=x^{\frac{3}{2}}\cdot\frac{\delta x}{x}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=x^{\frac{3}{2}-1}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{1}{2}}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

$$\Rightarrow \ \frac{dy}{dx}=x^{\frac{1}{2}}\left[\frac{3}{2}-0+0-....\right] \qquad \Rightarrow \ \boxed{\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}}$$

(x) Let $y = x^{5/2}$

Do yourself as above.

(xi) Let $y = x^{m}$

$$\Rightarrow \ y+\delta y = (x+\delta x)^{m}$$

$$\Rightarrow \ \delta y = (x+\delta x)^{m}-x^{m}$$

$$= x^{m}\left[\left(1+\frac{\delta x}{x}\right)^{m}-1\right]$$

$$= x^{m}\left[\left(1+m\cdot\frac{\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^{2}+....\right)-1\right]$$

$$= x^{m}\left[\frac{m\delta x}{x}+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)^{2}+....\right]$$

$$= x^{m}\cdot\frac{\delta x}{x}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = x^{m-1}\left[m + \frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right) + ...\right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{m-1}\left[m + \frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right) + ...\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{m-1}\left[m+0+0....\right] \quad \Rightarrow \boxed{\frac{dy}{dx} = m x^{m-1}}$$

(xii) Let $y = \frac{1}{x^m} = x^{-m}$

Do yourself as above, just change the m by $-m$ in above question.

(xiii) Let $y = x^{40}$

$$\Rightarrow y + \delta y = (x+\delta x)^{40}$$

$$\Rightarrow \delta y = (x+\delta x)^{40} - x^{40}$$

$$= \left[\binom{40}{0}x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ... + \binom{40}{40}\delta x^{40}\right] - x^{40}$$

$$= (1)x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ... + \binom{40}{40}\delta x^{40} - x^{40}$$

$$= \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + ... + \binom{40}{40}\delta x^{40}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + ... + \binom{40}{40}\delta x^{39}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + ... + \binom{40}{40}\delta x^{39}\right]$$

$$\frac{dy}{dx} = \left[\binom{40}{1}x^{39} + 0 + 0 + ... + 0\right]$$

$$\Rightarrow \frac{dy}{dx} = \binom{40}{1}x^{39} \quad \text{or} \quad \boxed{\frac{dy}{dx} = 40x^{39}}$$

(xiv) Let $y = x^{-100}$

Do yourself Question # 1(xii), Replace m by -100.

Question # 2

Find $\frac{dy}{dx}$ from the first principles if

(i) $\sqrt{x+2}$ (ii) $\frac{1}{\sqrt{x+a}}$

Solution

(i) Let $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$

Now do yourself as Question # 1(ix)

(ii) Let $y = \frac{1}{\sqrt{x+a}} = (x+a)^{-\frac{1}{2}}$

Now do yourself as Question # 1 (ix)