Question # 1
Find two positive integers whose sum is 30 and their product will be maximum.
Solution
Let x and $30-x$ be two positive integers and P denotes product integers then

$$P = x(30-x)$$
$$= 30x - x^2$$

Diff. w.r.t. x

$$\frac{dP}{dx} = 30-2x \quad \ldots\ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = -2 \quad \ldots\ldots\ (ii)$$

For critical points, put $\frac{dP}{dx}=0$

$$\Rightarrow 30-2x=0$$
$$\Rightarrow -2x=-30 \qquad \Rightarrow x=15$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -2 < 0$$

$\Rightarrow P$ is maximum at $x=15$
Other $+$tive integer $= 30-x$
$= 30-15 = 15$
Hence 15 and 15 are the required positive numbers.

Question # 2
Divide 20 into two parts so that the sum of their squares will be minimum.
Solution
Let x be the part of 20 then other is $20-x$.
Let S denotes sum of squares then

$$S = x^2 + (20-x)^2$$
$$= x^2 + 400 - 40x + x^2$$
$$= 2x^2 - 40x + 400$$

Diff. w.r.t x

$$\frac{dS}{dx} = 4x-40 \quad \ldots\ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 4 \quad \ldots\ldots\ (ii)$$

For stationary points put $\frac{dS}{dx}=0$

$$\Rightarrow 4x-40=0 \qquad \Rightarrow 4x=40$$
$$\Rightarrow x=10$$

Putting value of x in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=10} = 4 > 0$$

$\Rightarrow S$ is minimum at $x=10$
Other integer $= 20-x = 20-10 = 10$
Hence 10, 10 are the two parts of 20.

Question # 3
Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.
Solution
Let x and $12-x$ be two $+$tive integers and P denotes product of one with square of the other then

$$P = x(12-x)^2$$
$$\Rightarrow P = x\left(144-24x+x^2\right)$$
$$= x^3 - 24x^2 + 144x$$

Diff. w.r.t x

$$\frac{dP}{dx} = 3x^2 - 48x + 144 \quad \ldots\ (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 6x-48 \quad \ldots\ (ii)$$

For critical points put $\frac{dP}{dx}=0$

$$3x^2 - 48x + 144 = 0$$
$$\Rightarrow x^2 - 16x + 48 = 0$$
$$\Rightarrow x^2 - 4x - 12x + 48 = 0$$
$$\Rightarrow x(x-4) - 12(x-4) = 0$$
$$\Rightarrow (x-4)(x-12) = 0$$
$$\Rightarrow x=4 \quad \text{or} \quad x=12$$

We can not take $x=12$ as sum of integers is 12. So put $x=4$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=4} = 6(4)-48$$
$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=4$.

So the other integer $= 12-4 = 8$

Hence 4, 8 are the required integers.

Alternative Method: (by Irfan Mehmood: Fazaia Degree College Risalpur)

Let x and $12-x$ be two positive integers and P denotes product of one with square of the other then

$$P = x^2(12-x)$$
$$\Rightarrow P = 12x^2 - x^3$$

Diff. w.r.t x

$$\frac{dP}{dx} = 24x-3x^2 \;\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 24-6x \;\ldots\ldots\; \text{(ii)}$$

For critical points put $\frac{dP}{dx}=0$

$$24x-3x^2 = 0$$
$$\Rightarrow 3x(x-8) = 0$$
$$\Rightarrow x=0 \text{ or } x=8$$

We cannot take $x=0$ as given integers are positive. So put $x=8$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=8} = 24-6(8)$$
$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=8$.

So the other integer $= 12-8 = 4$

Hence 4, 8 are the required integers.

Question # 4

The perimeter of a triangle is $16cm$. If one side is of length $6cm$, What are length of the other sides for maximum area of the triangle.

Solution

Let the remaining sides of the triangles are x and y

Perimeter $= 16$

$$\Rightarrow 6+x+y = 16$$
$$\Rightarrow x+y = 16-6 \qquad \Rightarrow x+y = 10$$
$$\Rightarrow y = 10-x \;\ldots\ldots\; \text{(i)}$$

Now suppose A denotes the square of the area of triangle then

$$A = s(s-a)(s-b)(s-c)$$

Where $s = \frac{a+b+c}{2} = \frac{6+x+y}{2}$

$$= \frac{6+x+10-x}{2} \qquad \text{from (i)}$$
$$= \frac{16}{2} = 8$$

So $A = 8(8-6)(8-x)(8-y)$

$$= 8(2)(8-x)(8-10+x)$$
$$= 16(8-x)(-2+x)$$
$$= 16(-16+2x+8x-x^2)$$
$$\Rightarrow A = 16(-16+10x-x^2)$$

Diff. w.r.t x

$$\frac{dA}{dx} = 16(10-2x) \;\ldots\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = 16(-2) = -32$$

For critical points put $\frac{dA}{dx}=0$

$$16(10-2x) = 0$$
$$\Rightarrow (10-2x) = 0 \quad \Rightarrow -2x = -10$$
$$\Rightarrow x = 5$$

Putting value of x in (*ii*)

$$\left.\frac{d^2A}{dx^2}\right|_{x=5} = -32 < 0$$

$\Rightarrow$ A is maximum at $x=5$

Putting value of x in (i)

$$y = 10-5 = 5$$

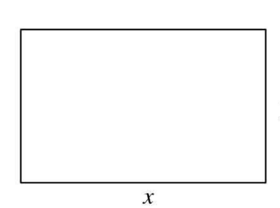
Hence length of remaining sides of triangles are $5cm$ and $5cm$.

Question # 5

Find the dimensions of a rectangle of largest area having perimeter $120cm$.

Solution

Let x and y be the length and breadth of rectangle, then



Area $= A = xy \;\ldots..\; \text{(i)}$

Perimeter $= 120$

$\Rightarrow x+x+y+y = 120$

$\Rightarrow 2x+2y = 120$

$\Rightarrow x+y = 60$

$\Rightarrow y = 60-x$ .... (ii)

Putting in (i)

$$A = x(60-x)$$

$$\Rightarrow A = 60x-x^2$$

Diff. w.r.t x

$$\frac{dA}{dx} = 60-2x \text{ .... (iii)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = -2 \text{ .... (iv)}$$

For critical points put $\frac{dA}{dx}=0$

$$60-2x = 0 \quad \Rightarrow -2x = -60$$

$$\Rightarrow x = 30$$

Putting value of x in (iv)

$$\left.\frac{d^2A}{dx^2}\right|_{x=30} = -2 < 0$$

$\Rightarrow A$ is maximum at $x=30$

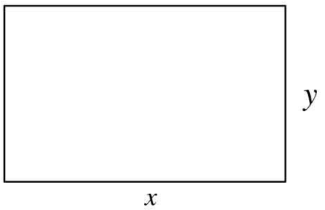
Putting value of x in (ii)

$$y = 60-30 = 30$$

Hence dimension of rectangle is $30cm$, $30cm$.

Question # 6

Find the lengths of the sides of a variable rectangle having area $36cm^2$ when its perimeter is minimum.



Solution

Let x and y be the length and breadth of the rectangle then

Area $= xy$

$\Rightarrow 36 = xy$

$\Rightarrow y = 36/x$ ... (i)

Now perimeter $= 2x+2y$

$$\Rightarrow P = 2x+2\left(36/x\right)$$

$$= 2\left(x+36x^{-1}\right)$$

Diff. P w.r.t x

$$\frac{dP}{dx} = 2\left(1-36x^{-2}\right) \text{ ... (ii)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 2\left(0-36\left(-2x^{-3}\right)\right)$$

$$= 2\left(72x^{-3}\right) = \frac{144}{x^3}$$

For critical points put $\frac{dP}{dx}=0$

$$2\left(1-36x^{-2}\right)=0 \quad \Rightarrow 1-\frac{36}{x^2}=0$$

$$\Rightarrow 1=\frac{36}{x^2} \quad \Rightarrow x^2 = 36 \quad \Rightarrow x = \pm 6$$

Since length can not be negative therefore

$$x=6$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{(6)^3} > 0$$

Hence P is minimum at $x=6$.

Putting in eq. (i)

$$y = \frac{36}{6} = 6$$

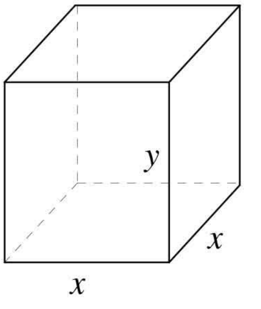
Hence $6cm$ and $6cm$ are the lengths of the sides of the rectangle.

Question # 7

A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.

Solution

Let x be the lengths of the sides of the base and y be the height of the box.



Then Volume

$= x \cdot x \cdot y$

$\Rightarrow 4 = x^2y$

$\Rightarrow y = \frac{4}{x^2}$ ... (i)

Suppose S denotes the surface area of the box, then

$$S = x^2+4xy$$

$$\Rightarrow S = x^2+4x\left(\frac{4}{x^2}\right)$$

$$\Rightarrow S = x^2 + 16x^{-1}$$

Diff. S w.r.t x

$$\frac{dS}{dx} = 2x - 16x^{-2} \ldots \text{ (ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 16\left(-2x^{-3}\right)$$

$$= 2 + \frac{32}{x^3} \ldots \text{ (iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 16x^{-2} = 0 \quad \Rightarrow 2x - \frac{16}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 16}{x^2} = 0$$

$$\Rightarrow 2x^3 - 16 = 0 \quad \Rightarrow 2x^3 = 16$$

$$\Rightarrow x^3 = 8 \quad \Rightarrow x = 2$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=2} = 2 + \frac{32}{(2)^3} > 0$$

$\Rightarrow S$ is min. when $x = 2$

Putting value of x in (i)

$$y = \frac{4}{(2)^2} = 1$$

Hence $2dm$, $2dm$ and $1dm$ are the dimensions of the box.

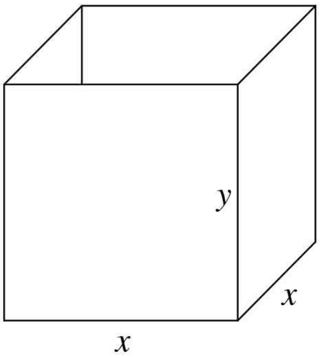
Question # 8

Find the dimensions of a rectangular garden having perimeter 80 meters if its area is to be maximum.

Solution

Do yourself as question # 5.

Question # 9



An open tank of square base of side x and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with lead will be least.

Solution

Let y be the height of the open tank.

Then Volume $= x \cdot x \cdot y$

$$\Rightarrow V = x^2 y$$

$$\Rightarrow y = \frac{V}{x^2} \;\ldots\ldots\ldots\ldots \text{ (i)}$$

If S denotes the surface area the open tank, then

$$S = x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + 4Vx^{-1}$$

Diff. w.r.t x

$$\frac{dS}{dx} = 2x - 4Vx^{-2} \;\ldots\ldots\ldots \text{ (ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 4V\left(-2x^{-3}\right)$$

$$= 2 + \frac{8V}{x^3} \;\ldots\ldots\ldots\ldots\ldots \text{ (iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 4Vx^{-2} = 0 \quad \Rightarrow 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 4V}{x^2} = 0 \quad \Rightarrow 2x^3 - 4V = 0$$

$$\Rightarrow 2x^3 = 4V \quad \Rightarrow x^3 = 2V$$

$$\Rightarrow x = (2V)^{\frac{1}{3}}$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=(2V)^{\frac{1}{3}}} = 2 + \frac{8V}{\left((2V)^{\frac{1}{3}}\right)^3}$$

$$= 2 + \frac{8V}{2V} = 2 + 4 = 6 > 0$$

$\Rightarrow S$ is minimum when $x = (2V)^{\frac{1}{3}}$

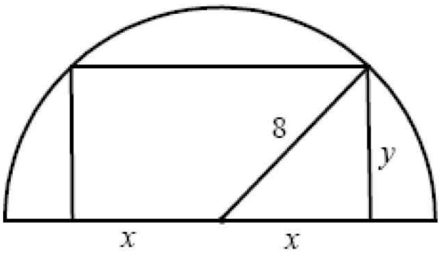
i.e. $x^3 = 2V \quad \Rightarrow V = \frac{x^3}{2}$

Putting in (i)

$$y = \frac{x^3/2}{x^2} = \frac{x}{2}$$

Hence height of the open tank is $\frac{x}{2}$.

Question # 10
Find the dimensions of the rectangular of maximum area which fits inside the semi-circle of radius $8cm$



Solution

Let $2x$ & y be dimension of rectangle.
Then from figure, using Pythagoras theorem

$$x^2 + y^2 = 8^2$$

$$\Rightarrow y^2 = 64 - x^2 \text{ ……….. (i)}$$

Now Area of the rectangle is given by

$$A = 2x \cdot y$$

Squaring both sides

$$A^2 = 4x^2y^2$$
$$= 4x^2(64 - x^2)$$
$$= 256x^2 - 4x^4$$

Now suppose

$$f = A^2 = 256x^2 - 4x^4 \text{ ……….. (ii)}$$

Diff. w.r.t x

$$\frac{df}{dx} = 512x - 16x^3 \text{ ……….. (iii)}$$

Again diff. w.r.t x

$$\frac{d^2f}{dx^2} = 512 - 48x^2 \text{ ……… (iv)}$$

For critical points, put $\frac{df}{dx} = 0$

$$\Rightarrow 512x - 16x^3 = 0$$
$$\Rightarrow 16x(32 - x^2) = 0$$
$$\Rightarrow 16x = 0 \quad \text{or} \quad 32 - x^2 = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 32$$
$$\Rightarrow x = \pm 4\sqrt{2}$$

Since x can not be zero or $-$ive, therefore

$$x = 4\sqrt{2}$$

Putting in (iv)

$$\left.\frac{d^2f}{dx^2}\right|_{x=4\sqrt{2}} = 512 - 48(4\sqrt{2})^2$$
$$= 512 - 48(32) = 512 - 1536$$
$$= -1024 < 0$$

$\Rightarrow$ Area is max. for $x = 4\sqrt{2}$

Hence length $= 2x = 2(4\sqrt{2})$

Breadth $= y = \sqrt{64 - (4\sqrt{2})^2}$

$$= \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$$

Hence dimension is $8\sqrt{2}$ cm and $4\sqrt{2}$ cm.

Question # 11
Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3, -1)$

Solution

Let $P(x, y)$ be point and let $A(3, -1)$.

Then $d = |AP| = \sqrt{(x-3)^2 + (y+1)^2}$

$$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$$
$$= (x-3)^2 + (x^2 - 1 + 1)^2$$

$\because y = x^2 - 1$ (given)

$$\Rightarrow d^2 = (x-3)^2 + x^4$$

Let $f = d^2 = (x-3)^2 + x^4$.

Diff. w.r.t x

$$\frac{df}{dx} = 2(x-3) + 4x^3 \text{ ………… (i)}$$

Again diff. w.r.t x

$$\frac{d^2f}{dx^2} = 2 + 12x^2 \text{ ……… (ii)}$$

For stationary points, put $\frac{df}{dx} = 0$

$$2(x-3) + 4x^3 = 0$$

$\Rightarrow\ 2x-6+4x^3 = 0$

$\Rightarrow\ 4x^3+2x-6 = 0$

$\Rightarrow\ 2x^3+x-3 = 0 \quad \div$ing by 2

By synthetic division

1	2	0	1	-3
	↓	2	2	3
	2	2	3	0

$\Rightarrow\ x = 1 \quad$ or $\quad 2x^2+2x+3 = 0$

$$\Rightarrow\ x = \frac{-2\pm\sqrt{4-4(2)(3)}}{4}$$

$$= \frac{-2\pm\sqrt{-20}}{4}$$

This is complex and not acceptable.

Now put $x=1$ in (ii)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 2+12(1)^2 = 14 > 0$$

$\Rightarrow\ d$ is minimum at $x=1$.

Also $y = 1^2-1 = 0$.

Hence $(1,0)$ is the required point.

Question # 12

Find the point on the curve $y=x^2+1$ that is closest to the point $(18,1)$

Solution

Do yourself as Q # 11