

## 2.10 Derivative of General Exponential Function (Page 80)

A function define by

$$f(x) = a^x \text{ where } a > 0, a \neq 1$$

is called general exponential function.

Suppose

$$y = a^x$$

$$\begin{aligned} \Rightarrow y + \delta y &= a^{x+\delta x} \Rightarrow \delta y = a^{x+\delta x} - y \\ \Rightarrow \delta y &= a^{x+\delta x} - a^x \quad \text{Since } y = a^x \\ \Rightarrow \delta y &= a^x(a^{\delta x} - 1) \end{aligned}$$

Dividing by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{a^x(a^{\delta x} - 1)}{\delta x}$$

Taking limit as  $\delta x \rightarrow 0$

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{a^x(a^{\delta x} - 1)}{\delta x} \\ \Rightarrow \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} a^x \left( \frac{a^{\delta x} - 1}{\delta x} \right) \Rightarrow \frac{dy}{dx} = a^x \lim_{\delta x \rightarrow 0} \left( \frac{a^{\delta x} - 1}{\delta x} \right) \\ \Rightarrow \boxed{\frac{d}{dx}(a^x) = a^x \cdot \ln a} &\quad \text{Since } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \end{aligned}$$

## Derivative of Natural Exponential Function

The exponential function  $f(x) = e^x$ , where  $e = 2.71828\dots$ , is called Natural Exponential Function.

Suppose  $y = e^x$

*Do yourself ... Just Change a by e in above article. You'll get*

$$\boxed{\frac{d}{dx} e^x = e^x}$$

## 2.11 Derivative of General Logarithmic Function (page 81)

If  $a > 0, a \neq 1$  and  $x = a^y$ , then the function defined by  $y = \log_a x$  ( $x > 0$ ) is called General Logarithmic Function.

Suppose  $y = \log_a x$

$$\begin{aligned} \Rightarrow y + \delta y &= \log_a(x + \delta x) \Rightarrow \delta y = \log_a(x + \delta x) - y \\ \Rightarrow \delta y &= \log_a(x + \delta x) - \log_a x \\ &= \log_a \left( \frac{x + \delta x}{x} \right) \quad \text{Since } \log_a m - \log_a n = \log_a \frac{m}{n} \end{aligned}$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log_a \left( \frac{x + \delta x}{x} \right)$$

Taking limit as  $\delta x \rightarrow 0$

$$\begin{aligned}
 & \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} \log_a \left( \frac{x + \delta x}{x} \right) \\
 \Rightarrow \quad & \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} \log_a \left( 1 + \frac{\delta x}{x} \right) \\
 & = \lim_{\delta x \rightarrow 0} \frac{x}{x} \cdot \frac{1}{\delta x} \log_a \left( 1 + \frac{\delta x}{x} \right) \quad \text{÷ing and ×ing by } x \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \frac{x}{\delta x} \log_a \left( 1 + \frac{\delta x}{x} \right) \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \log_a \left( 1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} \quad \text{Since } m \log_a x = \log_a x^m \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \log_a \left[ \lim_{\delta x \rightarrow 0} \left( 1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} \right] \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \log_a e \quad \text{Since } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\
 \Rightarrow \quad & \frac{d}{dx} (\log_a x) = \frac{1}{x} \frac{1}{\log_e a} \quad \text{Since } \log_a b = \frac{1}{\log_b a} \\
 \Rightarrow \quad & \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \quad \text{Since } \log_e a = \ln a
 \end{aligned}$$

## Derivative of Natural Logarithmic Function

The logarithmic function  $f(x) = \log_e x$  where  $e = 2.71828\dots$  is called Natural Logarithmic Function. And we write  $\ln x$  instead of  $\log_e x$  for our ease.

Suppose  $y = \ln x$

$$\begin{aligned}
 \Rightarrow \quad & y + \delta y = \ln(x + \delta x) \Rightarrow \delta y = \ln(x + \delta x) - y \\
 \Rightarrow \quad & \delta y = \ln(x + \delta x) - \ln x \\
 \Rightarrow \quad & \delta y = \ln \left( \frac{x + \delta x}{x} \right) \quad \text{Since } \Rightarrow \ln m - \ln n = \ln \frac{m}{n} \\
 & = \ln \left( 1 + \frac{\delta x}{x} \right)
 \end{aligned}$$

Dividing both sides by  $\delta x$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} = \ln \left( 1 + \frac{\delta x}{x} \right)$$

Taking limit as  $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} \ln \left( 1 + \frac{\delta x}{x} \right)$$

$$\begin{aligned}
 \Rightarrow \quad & \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{x}{x} \cdot \frac{1}{\delta x} \ln \left( 1 + \frac{\delta x}{x} \right) && \text{dividing and multiplying by } x \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \frac{x}{\delta x} \ln \left( 1 + \frac{\delta x}{x} \right) && \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \ln \left( 1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} && \text{Since } m \ln x = \ln x^m \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \ln \left[ \lim_{\delta x \rightarrow 0} \left( 1 + \frac{\delta x}{x} \right)^{\frac{x}{\delta x}} \right] && \\
 \Rightarrow \quad & \frac{dy}{dx} = \frac{1}{x} \ln e && \text{Since } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\
 \Rightarrow \quad & \frac{d}{dx} (\ln x) = \frac{1}{x} \cdot 1 && \text{Since } \ln e = \log_e e = 1 \\
 \Rightarrow \quad & \frac{d}{dx} (\ln x) = \frac{1}{x} &&
 \end{aligned}$$


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### Exercise 2.6 (Questions)

#### Question # 1

Find  $f'(x)$  if

- (i)  $f(x) = e^{\sqrt{x}-1}$
- (ii)  $f(x) = x^3 e^{\frac{1}{x}}, (x \neq 0)$
- (iii)  $f(x) = e^x (1 + \ln x)$
- (iv)  $f(x) = \frac{e^x}{e^{-x} + 1}$
- (v)  $f(x) = \ln(e^x + e^{-x})$
- (vi)  $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$
- (vii)  $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$
- (viii)  $f(x) = \ln \sqrt{(e^{2x} + e^{-2x})}$

#### Solution

$$(i) \quad f(x) = e^{\sqrt{x}-1}$$

Diff. w.r.t  $x$

$$\begin{aligned}
 \frac{d}{dx} f(x) &= \frac{d}{dx} e^{\sqrt{x}-1} \\
 \Rightarrow f'(x) &= e^{\sqrt{x}-1} \frac{d}{dx} (\sqrt{x} - 1) \\
 &= e^{\sqrt{x}-1} \left( \frac{1}{2} x^{-\frac{1}{2}} - 0 \right) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \quad \text{Ans.}
 \end{aligned}$$

$$(ii) \quad f(x) = x^3 e^{\frac{1}{x}}$$

Diff. w.r.t  $x$

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} x^3 e^{\frac{1}{x}} \\
\Rightarrow f'(x) &= x^3 \frac{d}{dx} e^{\frac{1}{x}} + e^{\frac{1}{x}} \frac{d}{dx} x^3 \\
&= x^3 e^{\frac{1}{x}} \frac{d}{dx} \left( \frac{1}{x} \right) + e^{\frac{1}{x}} (3x^2) \\
&= x^3 e^{\frac{1}{x}} \left( -\frac{1}{x^2} \right) + e^{\frac{1}{x}} (3x^2) \quad \therefore \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} x^{-1} = -x^{-2} = -\frac{1}{x^2} \\
&= -x e^{\frac{1}{x}} + 3x^2 e^{\frac{1}{x}} = x e^{\frac{1}{x}} (3x - 1) \quad \text{Ans.}
\end{aligned}$$

(iii)  $f(x) = e^x (1 + \ln x)$

Diff. w.r.t  $x$ 

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} e^x (1 + \ln x) \\
\Rightarrow f'(x) &= e^x \frac{d}{dx} (1 + \ln x) + (1 + \ln x) \frac{d}{dx} e^x \\
&= e^x \left( 0 + \frac{1}{x} \right) + (1 + \ln x) e^x \\
\Rightarrow f'(x) &= e^x \left( \frac{1}{x} + 1 + \ln x \right) \quad \text{or} \quad f'(x) = e^x \left( \frac{1 + x(1 + \ln x)}{x} \right)
\end{aligned}$$

(iv)  $f(x) = \frac{e^x}{e^{-x} + 1}$

Diff. w.r.t  $x$ 

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} \left( \frac{e^x}{e^{-x} + 1} \right) \\
\Rightarrow f'(x) &= \frac{(e^{-x} + 1) \frac{d}{dx} e^x - e^x \frac{d}{dx} (e^{-x} + 1)}{(e^{-x} + 1)^2} \\
&= \frac{(e^{-x} + 1)e^x - e^x (e^{-x}(-1) + 0)}{(e^{-x} + 1)^2} = \frac{e^x (e^{-x} + 1 + e^{-x})}{(e^{-x} + 1)^2} \\
\Rightarrow f'(x) &= \frac{e^x (2e^{-x} + 1)}{(e^{-x} + 1)^2} \quad \text{Ans.}
\end{aligned}$$

$$(v) \quad f(x) = \ln(e^x + e^{-x})$$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} \ln(e^x + e^{-x}) \\ \Rightarrow f'(x) &= \frac{1}{(e^x + e^{-x})} \frac{d}{dx}(e^x + e^{-x}) \\ &= \frac{1}{(e^x + e^{-x})} (e^x + e^{-x}(-1)) \\ \Rightarrow f'(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{or} \quad f'(x) = \tanh x \quad \therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

$$(vi) \quad f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} \left( \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}} \right) \\ &= \frac{(e^{ax} + e^{-ax}) \frac{d}{dx}(e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax}) \frac{d}{dx}(e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{(e^{ax} + e^{-ax})(e^{ax}(a) - e^{-ax}(-a)) - (e^{ax} - e^{-ax})(e^{ax}(a) + e^{-ax}(-a))}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a(e^{ax} + e^{-ax})(e^{ax} + e^{-ax}) - a(e^{ax} - e^{-ax})(e^{ax} - e^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a[(e^{ax} + e^{-ax})^2 - (e^{ax} - e^{-ax})^2]}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a[(e^{2ax} + e^{-2ax} + 2e^{ax}e^{-ax}) - (e^{2ax} + e^{-2ax} - 2e^{ax}e^{-ax})]}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a[e^{2ax} + e^{-2ax} + 2 - e^{2ax} - e^{-2ax} + 2]}{(e^{ax} + e^{-ax})^2} \quad \therefore e^{ax}e^{-ax} = e^0 = 1 \\ \Rightarrow f'(x) &= \frac{4a}{(e^{ax} + e^{-ax})^2} \quad \text{Ans.} \end{aligned}$$


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$$\begin{aligned}
 \text{(vii)} \quad f(x) &= \sqrt{\ln(e^{2x} + e^{-2x})} \\
 \Rightarrow \frac{d}{dx} f(x) &= \frac{d}{dx} \left[ \ln(e^{2x} + e^{-2x}) \right]^{\frac{1}{2}} \\
 \Rightarrow f'(x) &= \frac{1}{2} \left[ \ln(e^{2x} + e^{-2x}) \right]^{-\frac{1}{2}} \frac{d}{dx} \ln(e^{2x} + e^{-2x}) \\
 &= \frac{1}{2 \left[ \ln(e^{2x} + e^{-2x}) \right]^{\frac{1}{2}}} \cdot \frac{1}{(e^{2x} + e^{-2x})} \frac{d}{dx} (e^{2x} + e^{-2x}) \\
 &= \frac{1}{2 \sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{(e^{2x} + e^{-2x})} (e^{2x}(2) + e^{-2x}(-2)) \\
 &= \frac{1}{2 \sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{2(e^{2x} - e^{-2x})}{(e^{2x} + e^{-2x})} = \frac{(e^{2x} - e^{-2x})}{(e^{2x} + e^{-2x}) \sqrt{\ln(e^{2x} + e^{-2x})}} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad f(x) &= \ln \sqrt{(e^{2x} + e^{-2x})} \\
 &= \ln(e^{2x} + e^{-2x})^{\frac{1}{2}} \Rightarrow f(x) = \frac{1}{2} \ln(e^{2x} + e^{-2x}) \quad \because \ln x^m = m \ln x
 \end{aligned}$$

Now diff. w.r.t  $x$

$$\frac{d}{dx} f(x) = \frac{1}{2} \frac{d}{dx} \ln(e^{2x} + e^{-2x})$$

*Now do yourself*

### Question # 2

Find  $\frac{dy}{dx}$  if

$$\text{(i)} \quad y = x^2 \ln \sqrt{x}$$

$$\text{(ii)} \quad y = x \sqrt{\ln x}$$

$$\text{(iii)} \quad y = \frac{x}{\ln x}$$

$$\text{(iv)} \quad y = x^2 \ln \frac{1}{x}$$

$$\text{(v)} \quad y = \ln \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\text{(vi)} \quad y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{(vii)} \quad y = \ln(9 - x^2)$$

$$\text{(viii)} \quad y = e^{-2x} \sin 2x$$

$$\text{(ix)} \quad y = e^{-x} (x^3 + 2x^2 + 1)$$

$$\text{(x)} \quad y = x e^{\sin x}$$

$$\text{(xi)} \quad y = 5e^{3x-4}$$

$$\text{(xii)} \quad y = (x+1)^x$$

$$\text{(xiii)} \quad y = (\ln x)^{\ln x}$$

$$\text{(xiv)} \quad y = \frac{\sqrt{x^2 - 1} (x+1)}{(x^3 + 1)^{3/2}}$$

### Solution

$$\text{(i)} \quad y = x^2 \ln \sqrt{x}$$

$$\Rightarrow y = x^2 \ln(x)^{\frac{1}{2}} \quad \Rightarrow y = \frac{1}{2}x^2 \ln x \quad \therefore \ln x^m = m \ln x$$

Now diff. w.r.t  $x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx} x^2 \ln x \\ &= \frac{1}{2} \left( x^2 \frac{d}{dx} \ln x + \ln x \frac{d}{dx} x^2 \right) \\ &= \frac{1}{2} \left( x^2 \cdot \frac{1}{x} + \ln x (2x) \right) = \frac{1}{2}x + x \ln x \text{ or } \frac{1}{2}x + 2x \ln \sqrt{x} \quad \text{Ans.}\end{aligned}$$

$$(ii) \quad y = x\sqrt{\ln x}$$

Diff. w.r.t  $x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} x(\ln x)^{\frac{1}{2}} \\ &= x \frac{d}{dx} (\ln x)^{\frac{1}{2}} + (\ln x)^{\frac{1}{2}} \frac{d}{dx}(x) \\ &= x \cdot \frac{1}{2} (\ln x)^{-\frac{1}{2}} \frac{d}{dx} (\ln x) + (\ln x)^{\frac{1}{2}} (1) = \frac{x}{2(\ln x)^{\frac{1}{2}}} \left( \frac{1}{x} \right) + (\ln x)^{\frac{1}{2}} \\ &= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1+2\ln x}{2\sqrt{\ln x}} \quad \text{Answer}\end{aligned}$$

$$(iii) \quad y = \frac{x}{\ln x}$$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \frac{d}{dx} \left( \frac{x}{\ln x} \right) \\ &= \frac{\ln x \frac{dx}{dx} - x \frac{d}{dx} \ln x}{(\ln x)^2} = \frac{\ln x (1) - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \quad \text{Answer}\end{aligned}$$

$$(iv) \quad y = x^2 \ln \frac{1}{x}$$

$$\Rightarrow y = x^2 \ln x^{-1} \quad \Rightarrow y = -x^2 \ln x$$

*Now do yourself.*

$$(v) \quad y = \ln \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\Rightarrow y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}} \Rightarrow y = \frac{1}{2} \ln\left(\frac{x^2-1}{x^2+1}\right)$$

Now diff. w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx} \ln\left(\frac{x^2-1}{x^2+1}\right) \\ &= \frac{1}{2} \cdot \frac{1}{\left(\frac{x^2-1}{x^2+1}\right)} \cdot \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) \\ &= \frac{x^2+1}{2(x^2-1)} \cdot \left( \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \right) \\ &= \frac{1}{2(x^2-1)} \cdot \left( \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)} \right) \\ &= \frac{1}{2(x^2-1)} \cdot \left( \frac{2x(x^2+1-x^2+1)}{(x^2+1)} \right) = \frac{1}{(x^2-1)} \cdot \left( \frac{x(2)}{(x^2+1)} \right) = \frac{2x}{(x^4-1)} \quad Ans. \end{aligned}$$

$$(vi) \quad y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \ln\left(x + \sqrt{x^2 + 1}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}\left(x + \sqrt{x^2 + 1}\right) = \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \frac{d}{dx}(x^2 + 1)\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{1}{2(x^2 + 1)^{\frac{1}{2}}} \cdot (2x)\right) = \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left( \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) = \frac{1}{\sqrt{x^2 + 1}} \quad Answer \end{aligned}$$

$$(vii) \quad y = \ln(9 - x^2)$$

Diff. w.r.t  $x$

$$\frac{dy}{dx} = \frac{d}{dx} \ln(9 - x^2)$$

$$= \frac{1}{9-x^2} \cdot \frac{d}{dx}(9-x^2) = \frac{1}{9-x^2} \cdot (-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{9-x^2}$$

(viii)  $y = e^{-2x} \sin 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} e^{-2x} \sin 2x$$

$$= e^{-2x} \frac{d}{dx} \sin 2x + \sin 2x \frac{d}{dx} e^{-2x}$$

$$= e^{-2x} \cos 2x (2) + \sin 2x e^{-2x} (-2) = 2e^{-2x} (\cos 2x - \sin 2x) \quad Answer$$

(ix)  $y = e^{-x} (x^3 + 2x^2 + 1)$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{-x} (x^3 + 2x^2 + 1) \\ &= e^{-x} \frac{d}{dx} (x^3 + 2x^2 + 1) + (x^3 + 2x^2 + 1) \frac{d}{dx} e^{-x} \\ &= e^{-x} (3x^2 + 4x + 0) + (x^3 + 2x^2 + 1) \cdot e^{-x} (-1) \\ &= e^{-x} (3x^2 + 4x) - (x^3 + 2x^2 + 1) \cdot e^{-x} = e^{-x} (3x^2 + 4x - x^3 - 2x^2 - 1) \\ &= e^{-x} (-x^3 + x^2 + 4x - 1) \quad Answer \end{aligned}$$

(x)  $y = xe^{\sin x}$

Diff w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} xe^{\sin x} \\ &= x \frac{d}{dx} e^{\sin x} + e^{\sin x} \frac{d}{dx} x \\ &= x \cdot e^{\sin x} \frac{d}{dx} \sin x + e^{\sin x} (1) = x \cdot e^{\sin x} \cos x + e^{\sin x} \\ &= e^{\sin x} (x \cos x + 1) \quad Answer \end{aligned}$$

(xi)  $Do\ yourself$

(xii)  $y = (x+1)^x$

Taking log on both sides

$$\ln y = \ln(x+1)^x \Rightarrow \ln y = x \ln(x+1)$$

Diff w.r.t  $x$

$$\begin{aligned} \frac{d}{dx} \ln y &= \frac{d}{dx} x \ln(x+1) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= x \frac{d}{dx} \ln(x+1) + \ln(x+1) \frac{dx}{dx} \\ &= x \cdot \frac{1}{x+1} \frac{d}{dx}(x+1) + \ln(x+1)(1) \\ \Rightarrow \frac{dy}{dx} &= y \left( \frac{x}{x+1}(1) + \ln(x+1) \right) \\ &= (x+1)^x \left( \frac{x}{x+1} + \ln(x+1) \right) \quad \text{Answer} \end{aligned}$$

$$(xiii) \quad y = (\ln x)^{\ln x}$$

Taking log on both sides

$$\ln y = \ln(\ln x)^{\ln x} \Rightarrow \ln y = (\ln x) \cdot \ln(\ln x)$$

Diff w.r.t  $x$

$$\begin{aligned} \frac{d}{dx} \ln y &= \frac{d}{dx} (\ln x) \cdot \ln(\ln x) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= (\ln x) \frac{d}{dx} \ln(\ln x) + \ln(\ln x) \frac{d}{dx} (\ln x) \\ &= (\ln x) \cdot \frac{1}{\ln x} \frac{d}{dx}(\ln x) + \ln(\ln x) \cdot \frac{1}{x} \\ &= \frac{1}{x} + \frac{\ln(\ln x)}{x} = \frac{1 + \ln(\ln x)}{x} \\ \Rightarrow \frac{dy}{dx} &= y \left( \frac{1 + \ln(\ln x)}{x} \right) \quad \Rightarrow \frac{dy}{dx} = (\ln x)^{\ln x} \left( \frac{1 + \ln(\ln x)}{x} \right) \end{aligned}$$

$$(xiv) \quad y = \frac{\sqrt{x^2 - 1} (x+1)}{(x^3 + 1)^{3/2}} \Rightarrow y = \frac{((x+1)(x-1))^{\frac{1}{2}} (x+1)}{\left[ (x+1)(x^2 - x + 1) \right]^{\frac{3}{2}}}$$

$$\Rightarrow y = \frac{(x+1)^{\frac{1}{2}} (x-1)^{\frac{1}{2}} (x+1)}{(x+1)^{\frac{3}{2}} (x^2 - x + 1)^{\frac{3}{2}}} \Rightarrow y = \frac{(x+1)^{\frac{3}{2}} (x-1)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}} (x^2 - x + 1)^{\frac{3}{2}}}$$

$$\Rightarrow y = \frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}}$$

Taking log on both sides

$$\begin{aligned}\ln y &= \ln \frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}} \\ &= \ln(x-1)^{\frac{1}{2}} - \ln(x^2-x+1)^{\frac{3}{2}} \\ \Rightarrow \ln y &= \frac{1}{2} \ln(x-1) - \frac{3}{2} \ln(x^2-x+1)\end{aligned}$$

Now diff. w.r.t  $x$

$$\begin{aligned}\frac{d}{dx} \ln y &= \frac{1}{2} \frac{d}{dx} \ln(x-1) - \frac{3}{2} \frac{d}{dx} \ln(x^2-x+1) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2} \frac{1}{x-1} \frac{d}{dx}(x-1) - \frac{3}{2} \frac{1}{(x^2-x+1)} \frac{d}{dx}(x^2-x+1) \\ &= \frac{1}{2(x-1)}(1) - \frac{3}{2(x^2-x+1)}(2x-1) = \frac{1}{2(x-1)} - \frac{3(2x-1)}{2(x^2-x+1)} \\ \Rightarrow \frac{dy}{dx} &= y \left[ \frac{x^2-x+1-3(2x-1)(x-1)}{2(x-1)(x^2-x+1)} \right] \\ &= \frac{(x-1)^{\frac{1}{2}}}{(x^2-x+1)^{\frac{3}{2}}} \cdot \left[ \frac{x^2-x+1-3(2x^2-x-2x+1)}{2(x-1)(x^2-x+1)} \right] \\ &= \left[ \frac{x^2-x+1-6x^2+3x+6x-3}{2(x-1)^{1-\frac{1}{2}}(x^2-x+1)^{\frac{3}{2}+1}} \right] = \frac{-5x^2+8x-2}{2(x-1)^{\frac{1}{2}}(x^2-x+1)^{\frac{5}{2}}} \\ \Rightarrow \frac{dy}{dx} &= -\frac{5x^2-8x+2}{2\sqrt{x-1}(x^2-x+1)^{\frac{5}{2}}} \text{ Ans.}\end{aligned}$$

$$(xv) \quad y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+x-2}}$$

$$\Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+2x-x-2}} \Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x(x+2)-1(x+2)}}$$

$$\Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{(x+2)(x-1)}} \Rightarrow y = (x+2)^{2-\frac{1}{2}} \Rightarrow y = (x+2)^{\frac{3}{2}}$$

Now diff. w.r.t  $x$

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)^{\frac{3}{2}}$$

*Do yourself*

### 2.1.3 Derivative of Hyperbolic Function (page 85)

The hyperbolic functions are define by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in R; \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in R$$

$$\text{and } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in R$$

The reciprocal of these functions are defined as;

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \in R - \{0\}; \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad x \in R$$

$$\text{and } \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \in R - \{0\}$$

and there derivatives are

(i) $\frac{d}{dx}(\sinh x) = \cosh x$	(ii) $\frac{d}{dx}(\cosh x) = \sinh x$
(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	(iv) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$	(vi) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

**Proof:**

$$\begin{aligned} \text{(i)} \quad \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right) = \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) \\ &= \frac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) = \frac{1}{2}(e^x - e^{-x})' \\ &= \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x \end{aligned}$$

(ii) Similar as above.

(iii) See the below (iv) proof.

$$\text{(iv)} \quad \frac{d}{dx} \coth x = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right)$$

$$\begin{aligned}
&= \frac{(e^x - e^{-x}) \frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x}) \frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2} \\
&= \frac{(e^x - e^{-x})(e^x + e^{-x}(-1)) - (e^x + e^{-x})(e^x - e^{-x}(-1))}{(e^x - e^{-x})^2} \\
&= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\
&= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2} \\
&= \frac{(e^{2x} + e^{-2x} - 2e^x e^{-x}) - (e^{2x} + e^{-2x} + 2e^x e^{-x})}{(e^x - e^{-x})^2} \\
&= \frac{e^{2x} + e^{-2x} - 2 - e^{2x} - e^{-2x} - 2}{(e^x - e^{-x})^2} \quad \because e^x e^{-x} = e^0 = 1 \\
&= \frac{-4}{(e^x - e^{-x})^2} = -\left(\frac{2}{e^x - e^{-x}}\right)^2 = -\operatorname{csch}^2 x
\end{aligned}$$

(v)  $\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = \frac{d}{dx}2(e^x + e^{-x})^{-1} = 2\frac{d}{dx}(e^x + e^{-x})^{-1}$

$$\begin{aligned}
&= 2\left[(-1)(e^x + e^{-x})^{-1-1} \frac{d}{dx}(e^x + e^{-x})\right] \\
&= -2(e^x + e^{-x})^{-2}(e^x + e^{-x}(-1)) = \frac{-2}{(e^x + e^{-x})^2}(e^x - e^{-x}) \\
&= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})(e^x + e^{-x})} = -\frac{2}{(e^x + e^{-x})}(e^x - e^{-x}) \\
&= -\operatorname{sech} x \tanh x
\end{aligned}$$

(vi) *Do yourself as above (v).*

## 2.14 Derivative of Inverse Hyperbolic Function (page 86)

$$\begin{array}{ll}
\text{(i)} \quad \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} & \text{(ii)} \quad \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} \\
\text{(iii)} \quad \frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2} & \text{(iv)} \quad \frac{d}{dx} \coth^{-1} x = \frac{1}{1-x^2}
\end{array}$$

$$(v) \frac{d}{dx} \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}} \quad (vi) \frac{d}{dx} \operatorname{csch}^{-1} x = \frac{-1}{x\sqrt{1+x^2}}$$

**Proof:**

- (i) Let  $y = \sinh^{-1} x \Rightarrow \sinh y = x$   
differentiate w.r.t.  $x$ .

$$\begin{aligned} \frac{d}{dx} \sinh y &= \frac{d}{dx} x \quad \Rightarrow \cosh y \frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1+\sinh^2 y}} \quad \therefore \cosh^2 x - \sinh^2 x = 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1+x^2}} \quad \therefore \sinh y = x \end{aligned}$$

- (ii) Do yourself as above.

- (iii) Do yourself as (iv) below or see book at page 88.

- (iv) Let  $y = \coth^{-1} x \Rightarrow \coth y = x$   
differentiate w.r.t.  $x$

$$\begin{aligned} \frac{d}{dx} \coth y &= \frac{d}{dx} x \quad \Rightarrow -\operatorname{csch}^2 y \frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{-\operatorname{csch}^2 y} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{-(\coth^2 y - 1)} \quad \therefore \coth^2 y - 1 = \operatorname{csch}^2 y \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{-\coth^2 y + 1} = \frac{1}{1 - \coth^2 y} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{1 - x^2} \quad \therefore \coth y = x \end{aligned}$$

- (v) Suppose  $y = \operatorname{sech}^{-1} x \Rightarrow \operatorname{sech} y = x$   
differentiate w.r.t.  $x$

$$\begin{aligned} \frac{d}{dx} \operatorname{sech} y &= \frac{d}{dx} x \quad \Rightarrow -\operatorname{sech} y \tanh y \frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{-\operatorname{sech} y \tanh y} \\ \Rightarrow \frac{dy}{dx} &= \frac{-1}{\operatorname{sech} y \sqrt{1 - \tanh^2 y}} \quad \therefore 1 - \tanh^2 y = \operatorname{sech}^2 y \\ \Rightarrow \frac{dy}{dx} &= \frac{-1}{x\sqrt{1-x^2}} \quad \therefore \operatorname{sech} y = x \end{aligned}$$

- (vi) Do yourself as above

**Question # 3**

Find  $\frac{dy}{dx}$  if

(i)  $y = \cosh 2x$

(ii)  $y = \sinh 3x$

(iii)  $y = \tanh^{-1}(\sin x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv)  $y = \sinh^{-1}(x^3)$

(v)  $y = (\ln \tanh x)$

(vi)  $y = \sinh^{-1}\left(\frac{x}{2}\right)$

**Solution**

(i)  $y = \cosh 2x$

Diff. w.r.t  $x$

$$\frac{dy}{dx} = \frac{d}{dx} \cosh 2x \Rightarrow \frac{dy}{dx} = \sinh 2x \frac{d}{dx}(2x) \Rightarrow \frac{dy}{dx} = 2 \sinh 2x$$

(ii)

*Do yourself*

(iii)  $y = \tanh^{-1}(\sin x) \Rightarrow \tanh y = \sin x$

Diff. w.r.t  $x$

$$\frac{d}{dx} \tanh y = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{\operatorname{sech}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{1 - \tanh^2 y}$$

$$\because \cosh^2 \theta - \sinh^2 \theta = 1$$

$$= \frac{\cos x}{1 - \sin^2 x}$$

$$\therefore 1 - \tanh^2 \theta = \operatorname{sech}^2 \theta$$

$$= \frac{\cos x}{\cos^2 x} \Rightarrow \frac{dy}{dx} = \sec x$$

$$\therefore \sin x = \tanh y$$

(iv)  $y = \sinh^{-1}(x^3) \Rightarrow \sinh y = x^3$

$$\Rightarrow \frac{d}{dx} \sinh y = \frac{d}{dx} x^3 \Rightarrow \cosh y \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{\cosh y}$$

$$= \frac{3x^2}{\sqrt{1 + \sinh^2 y}}$$

$$\therefore \cosh^2 y - \sinh^2 y = 1$$

$$= \frac{3x^2}{\sqrt{1+(x^3)^2}} = \frac{3x^2}{\sqrt{1+x^6}}. \quad \text{Answer}$$

(v) *Do yourself*

$$(vi) \quad y = \sinh^{-1}\left(\frac{x}{2}\right) \Rightarrow \sinh y = \frac{x}{2}$$

Now diff w.r.t  $x$

$$\begin{aligned} \frac{d}{dx} \sinh y &= \frac{d}{dx} \left( \frac{x}{2} \right) \Rightarrow \cosh y \frac{dy}{dx} = \frac{1}{2} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2 \cosh y} && \because \cosh^2 y - \sinh^2 y = 1 \\ &= \frac{1}{2\sqrt{1+\sinh^2 y}} && \therefore \cosh^2 y = 1 + \sinh^2 y \\ &= \frac{1}{2\sqrt{1+(x/2)^2}} = \frac{1}{2\sqrt{(4+x^2)/2}} = \frac{1}{\sqrt{4+x^2}} \quad \text{Answer.} \end{aligned}$$


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