Question # 1

Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x+5)^{\frac{3}{2}}$ (iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 3x^4 + 4x^3 + x - 2\right)$$

$$\Rightarrow y_1 = 2\left(5x^4\right) - 3\left(4x^3\right) + 4\left(3x^2\right) + 1 - 0$$

$$= 10x^4 - 12x^3 + 12x^2 + 1$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}\left(10x^4 - 12x^3 + 12x^2 + 1\right)$$

$$\Rightarrow y_2 = 10\left(4x^3\right) - 12\left(3x^2\right) + 12(2x) + 0$$

$$= 40x^3 - 36x^2 + 24x \quad Ans.$$

(ii) $y = (2x+5)^{\frac{3}{2}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{\frac{3}{2}}$$

$$\Rightarrow y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1}\frac{d}{dx}(2x+5)$$

$$= \frac{3}{2}(2x+5)^{\frac{1}{2}}(2) = 3(2x+5)^{\frac{1}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}(2x+5)^{\frac{1}{2}}$$

$$\Rightarrow y_2 = 3\cdot\frac{1}{2}(2x+5)^{-\frac{1}{2}}(2) \quad \Rightarrow y_2 = \frac{3}{\sqrt{2x+5}}$$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\Rightarrow y = (x)^{\frac{1}{2}} + (x)^{-\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x)^{\frac{1}{2}}+(x)^{-\frac{1}{2}}\right] \quad \Rightarrow y_1 = \frac{1}{2}(x)^{-\frac{1}{2}}-\frac{1}{2}(x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{1}{2}\frac{d}{dx}\left[(x)^{-\frac{1}{2}}-(x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = \frac{1}{2}\left[-\frac{1}{2}(x)^{-\frac{3}{2}}+\frac{3}{2}(x)^{-\frac{5}{2}}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{x^{\frac{3}{2}}}+\frac{3}{x^{\frac{5}{2}}}\right] = \frac{1}{4}\left[\frac{-x+3}{x^{\frac{5}{2}}}\right] \quad \text{or} \quad y_2 = \frac{3-x}{4x^{\frac{5}{2}}}$$

Question # 2

Find y_2 if

(i) $y = x^2e^{-x}$ (ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

Solution

(i) $y = x^2e^{-x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2e^{-x}$$

$$\Rightarrow y_1 = x^2\frac{d}{dx}e^{-x}+e^{-x}\frac{d}{dx}x^2$$

$$= x^2e^{-x}(-1)+e^{-x}(2x)$$

$$= e^{-x}\left(-x^2+2x\right)$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}e^{-x}\left(-x^2+2x\right)$$

$$y_2 = e^{-x}\frac{d}{dx}\left(-x^2+2x\right)+\left(-x^2+2x\right)\frac{d}{dx}e^{-x}$$

$$= e^{-x}(-2x+2)+\left(-x^2+2x\right)e^{-x}(-1)$$

$$= e^{-x}\left(-2x+2+x^2-2x\right)$$

$$= e^{-x}\left(x^2-4x+2\right)$$

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$\Rightarrow y = \ln(2x+3)-\ln(3x+2)$$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\ln(2x+3)-\frac{d}{dx}\ln(3x+2)$$

$$\Rightarrow y_1 = \frac{1}{2x+3}(2)-\frac{1}{3x+2}(3)$$

$$= 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 2\frac{d}{dx}(2x+3)^{-1} - 3\frac{d}{dx}(3x+2)^{-1}$$

$$\Rightarrow y_2 = 2\left[-(2x+3)^{-2}(2)\right] - 3\left[-(3x+2)^{-2}(3)\right]$$

$$\Rightarrow y_2 = -\frac{4}{(2x+3)^2} + \frac{9}{(3x+2)^2} \quad \textit{Ans.}$$

$$\text{OR} \quad y_2 = \frac{-4(3x+2)^2 + 9(3x+2)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-4\left(9x^2+12x+4\right) + 9\left(4x^2+12x+9\right)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-36x^2 - 48x - 16 + 36x^2 + 108x + 81}{(2x+3)^2(3x+2)^2} = \frac{60x+65}{(2x+3)^2(3x+2)^2} \quad \textit{Ans.}$$

(iii) $\quad y = \sqrt{\dfrac{1-x}{1+x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

By solving, you will get (differentiate here)

$$\Rightarrow y_1 = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}} = -(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left[(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = -(1-x)^{-\frac{1}{2}}\frac{d}{dx}(1+x)^{-\frac{3}{2}} - (1+x)^{-\frac{3}{2}}\frac{d}{dx}(1-x)^{-\frac{1}{2}}$$

$$= -(1-x)^{-\frac{1}{2}}\left(-\frac{3}{2}(1+x)^{-\frac{5}{2}}(1)\right) - (1+x)^{-\frac{3}{2}}\left(-\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\right)$$

$$= \frac{3}{2(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}} - \frac{1}{2(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}$$

$$= \frac{3(1-x)-(1+x)}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{3-3x-1-x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}}$$

$$= \frac{2-4x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{1-2x}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} \quad \textit{Ans.}$$

Question # 3

Find y_2 if

(i) $x^2 + y^2 = a^2$ (ii) $x^3 - y^3 = a^3$ (iii) $x = a\cos\theta,\ y = a\sin\theta$

(iv) $x = at^2,\ y = bt^4$ (v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution

(i) $x^2 + y^2 = a^2$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\,y_1 = -2x \Rightarrow y_1 = -\frac{x}{y}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right) \Rightarrow y_2 = -\left(\frac{y\frac{dx}{dx} - x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow y_2 = -\left(\frac{y(1) - x\left(-\frac{x}{y}\right)}{y^2}\right) \qquad \because \frac{dy}{dx} = -\frac{x}{y}$$

$$= -\left(\frac{y + \frac{x^2}{y}}{y^2}\right) = -\left(\frac{\frac{y^2 + x^2}{y}}{y^2}\right)$$

$$= -\left(\frac{x^2 + y^2}{y^3}\right) \quad Ans.$$

OR $\quad y_2 = -\dfrac{a^2}{y^3} \qquad \because x^2 + y^2 = a^2$

(ii) $x^3 - y^3 = a^3$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^3 - y^3\right) = \frac{d}{dx}a^3$$

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow -3y^2 y_1 = -3x^2 \Rightarrow y_1 = \frac{x^2}{y^2}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\Rightarrow y_2 = \frac{y^2\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(y^2)}{\left(y^2\right)^2}$$

$$= \frac{y^2(2x) - x^2\left(2y\frac{dy}{dx}\right)}{y^4}$$

$$= \frac{2xy^2 - 2x^2y\left(\frac{x^2}{y^2}\right)}{y^4} \qquad \because \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

$$= \frac{-2x\left(x^3 - y^3\right)}{y^5} \quad \textit{Ans.}$$

OR $$y_2 = \frac{-2x\left(a^3\right)}{y^5} \qquad \because x^3 - y^3 = a^3$$

$$\Rightarrow y_2 = -\frac{2a^3x}{y^5}$$

(iii) $x = a\cos\theta \;,\quad y = a\sin\theta$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\cos\theta$$

$$= -a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dx} = -\frac{1}{a\sin\theta}$$

Diff y w.r.t θ

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}\sin\theta$$

$$= a\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= a\cos\theta\cdot\frac{-1}{a\sin\theta} \qquad \Rightarrow y_1 = -\cot\theta$$

Now diff. y_1 w.r.t θ

$$\frac{dy_1}{dx} = -\frac{d}{dx}\cot\theta$$

$$\Rightarrow y_2 = +\operatorname{cosec}^2\theta\frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta . \left(-\frac{1}{a\sin\theta} \right)$$

$$\Rightarrow y_2 = \frac{-1}{a\sin^3\theta}$$

(iv) $x = at^2 \quad , \quad y = bt^4$

Diff. x w.r.t t

$$\frac{dx}{dt} = a\frac{d}{dt}t^2$$
$$= 2at$$
$$\Rightarrow \frac{dt}{dx} = \frac{1}{2at}$$

Diff. y w.r.t t

$$\frac{dy}{dt} = b\frac{d}{dt}(t^4)$$
$$= 4bt^3$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= 4bt^3 \cdot \frac{1}{2at} \qquad \Rightarrow y_1 = \frac{2b}{a}t^2$$

Now diff. y_1 w.r.t x

$$\frac{dy_1}{dx} = \frac{2b}{a}\frac{d}{dx}\left(t^2\right) = \frac{2b}{a}\frac{d}{dt}\left(t^2\right) \cdot \frac{dt}{dx}$$
$$\Rightarrow y_2 = \frac{2b}{a}(2t) \cdot \frac{1}{2at} \qquad \Rightarrow y_2 = \frac{2b}{a^2}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \frac{d}{dx}\left(x^2 + y^2 + 2gx + 2fy + c\right) = \frac{d}{dx}(0)$$
$$\Rightarrow 2x + 2y\frac{dy}{dx} + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$
$$\Rightarrow (2y + 2f)\frac{dy}{dx} + (2x + 2g) = 0$$
$$\Rightarrow (2y + 2f)\frac{dy}{dx} = -(2x + 2g)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{(2x + 2g)}{(2y + 2f)} \Rightarrow y_1 = -\frac{x + g}{y + f}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x + g}{y + f}\right)$$
$$\Rightarrow y_2 = -\left(\frac{(y + f)\frac{d}{dx}(x + g) - (x + g)\frac{d}{dx}(y + f)}{(y + f)^2} \right)$$

$$= -\frac{(y+f)(1)-(x+g)\frac{dy}{dx}}{(y+f)^2} = -\frac{(y+f)-(x+g)\left(-\frac{x+g}{y+f}\right)}{(y+f)^2}$$

$$= -\frac{\frac{(y+f)^2+(x+g)^2}{y+f}}{(y+f)^2} = -\frac{(y+f)^2+(x+g)^2}{(y+f)^3} \qquad \textit{Ans.}$$

OR $$y_2 = -\frac{y^2+2yf+f^2+x^2+2xg+g^2}{(y+f)^3}$$

$$= -\frac{(x^2+y^2+2gx+2fy+c)-c+f^2+g^2}{(y+f)^3}$$

$$= -\frac{0-c+f^2+g^2}{(y+f)^3} \qquad \because x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow y_2 = \frac{c-f^2-g^2}{(y+f)^3} \qquad \textit{Ans.}$$

Question # 4

Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2-9)$

Solution

(i) $y = \sin 3x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$\Rightarrow y_1 = \cos 3x\,(3) \quad \Rightarrow y_1 = 3\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}\cos 3x \quad \Rightarrow y_2 = 3(-\sin 3x\,(3)) \quad \Rightarrow y_2 = -9\sin 3x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -9\frac{d}{dx}\sin 3x$$

$$\Rightarrow y_3 = -9\cos 3x\,(3) \quad \Rightarrow y_3 = -27\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = -27\frac{d}{dx}\cos 3x \quad \Rightarrow y_4 = -27(-\sin 3x\,(3))$$

$$\Rightarrow \boxed{y_4 = 81\sin 3x}$$

(ii) $y = \cos^3 x$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^3 x\right)$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\frac{d}{dx}\cos x$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\left(-\sin x\right)$$

$$\Rightarrow y_1 = 3\left(1-\sin^2 x\right)\left(-\sin x\right) \qquad \Rightarrow y_1 = -3\sin x + 3\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -3\frac{d}{dx}\sin x + 3\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_2 = -3\cos x + 9\sin^2 x\frac{d}{dx}\sin x$$

$$\Rightarrow y_2 = -3\cos x + 9\left(1-\cos^2 x\right)\cos x$$

$$= -3\cos x + 9\cos x - 9\cos^3 x \quad = 6\cos x - 9\cos^3 x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = 6\frac{d}{dx}\cos x - 9\frac{d}{dx}\cos^3 x$$

$$\Rightarrow y_3 = 6\left(-\sin x\right) - 9\left(-3\sin x + 3\sin^3 x\right) \qquad \because \frac{d}{dx}(\cos^3 x) = -3\sin x + 3\sin^3 x$$

$$= -6\sin x + 27\sin x - 27\sin^3 x \quad = 21\sin x - 27\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 21\frac{d}{dx}\sin x - 27\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_4 = 21\left(\cos x\right) - 27\left(3\sin^2 x\right)\frac{d}{dx}\sin x$$

$$= 21\cos x - 81\sin^2 x\left(\cos x\right) \quad = 21\cos x - 81\left(1-\cos^2 x\right)\left(\cos x\right)$$

$$= 21\cos x - 81\cos x + 81\cos^3 x \quad = -60\cos x + 54\cos^3 x$$

Alternative:

$$y = \cos^3 x$$

Since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\Rightarrow \cos 3x - 3\cos x = 4\cos^3 x \quad \Rightarrow \cos^3 x = \frac{1}{4}\left(\cos 3x - 3\cos x\right)$$

Therefore

$$y = \frac{1}{4}\left(\cos 3x - 3\cos x\right)$$

Now diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(\frac{d}{dx}\cos 3x - 3\frac{d}{dx}\cos x\right)$$

Do yourself

(iii) $y = \ln\left(x^2-9\right)$

$= \ln\left[(x+3)(x-3)\right] = \ln(x+3)+\ln(x-3)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x+3)+\frac{d}{dx}\ln(x-3)$$

$$\Rightarrow y_1 = \frac{1}{x+3}+\frac{1}{x-3}$$

$$= (x+3)^{-1}+(x-3)^{-1}$$

Again diff w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}(x+3)^{-1}+\frac{d}{dx}(x-3)^{-1}$$

$$\Rightarrow y_2 = -(x+3)^{-2}-(x-3)^{-2}$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -\frac{d}{dx}(x+3)^{-2}-\frac{d}{dx}(x-3)^{-2} \quad \Rightarrow y_3 = 2(x+3)^{-3}+2(x-3)^{-3}$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 2\frac{d}{dx}(x+3)^{-3}+2\frac{d}{dx}(x-3)^{-3}$$

$$\Rightarrow y_4 = 2\left(-3(x+3)^{-4}\right)+2\left(-3(x-3)^{-4}\right)$$

$$= \frac{-6}{(x+3)^4}+\frac{-6}{(x+3)^4} = -6\left[\frac{1}{(x+3)^4}+\frac{1}{(x+3)^4}\right] \quad \textit{Ans.}$$

Question # 5

If $x=\sin\theta,\ y=\sin m\theta$, Show that $\left(1-x^2\right)y_2 - xy_1 + m^2y = 0$

Solution $x = \sin\theta$ …… (i), $y = \sin m\theta$ ……..(ii)

From (i) $\theta=\sin^{-1}x$, putting in (ii)

$$y = \sin\left(m\sin^{-1}x\right)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin m\left(\sin^{-1}x\right)$$

$$\Rightarrow y_1 = \cos\left(m\sin^{-1}x\right)\frac{d}{dx}m\sin^{-1}x$$

$$= \cos\left(m\sin^{-1}x\right)\cdot m\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1\sqrt{1-x^2} = m\cos\left(m\sin^{-1}x\right)$$

Taking square on both sides.

$$y_1^2\left(1-x^2\right) = m^2\cos^2\left(m\sin^{-1}x\right)$$

$$\Rightarrow y_1^2\left(1-x^2\right) = m^2\left(1-\sin^2(m\sin^{-1}x)\right) \qquad \because \cos^2x = 1-\sin^2x$$

$$\Rightarrow\ y_1^2\left(1-x^2\right)=m^2\left(1-y^2\right) \qquad \text{From (ii)}$$

Now again diff. w.r.t x

$$\frac{d}{dx}y_1^2\left(1-x^2\right)=m^2\frac{d}{dx}\left(1-y^2\right)$$

$$\Rightarrow\ y_1^2\frac{d}{dx}\left(1-x^2\right)+\left(1-x^2\right)\frac{d}{dx}y_1^2=m^2\left(0-2y\frac{dy}{dx}\right)$$

$$\Rightarrow\ y_1^2(-2x)+\left(1-x^2\right)2y_1\frac{dy_1}{dx}=-2m^2y\frac{dy}{dx}$$

$$\Rightarrow\ -2xy_1^2+\left(1-x^2\right)2y_1y_2=-2m^2y\,y_1$$

$$\Rightarrow\ 2y_1\left(-xy_1+\left(1-x^2\right)y_2\right)=2y_1\left(-m^2y\right)$$

$$\Rightarrow\ -xy_1+\left(1-x^2\right)y_2=-m^2y$$

$$\Rightarrow\ \left(1-x^2\right)y_2-xy_1+m^2y=0 \qquad \textit{Proved}$$

Question # 6

If $y = e^x\sin x$, show that $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0$

Solution $\quad y = e^x\sin x$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}e^x\sin x$$

$$=e^x\frac{d}{dx}\sin x+\sin x\frac{d}{dx}e^x$$

$$=e^x\cos x+\sin x\,e^x \quad = e^x\left(\cos x+\sin x\right)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}e^x\left(\cos x+\sin x\right)$$

$$\Rightarrow\ \frac{d^2y}{dx^2}=e^x\frac{d}{dx}\left(\cos x+\sin x\right)+\left(\cos x+\sin x\right)\frac{d}{dx}e^x$$

$$=e^x\left(-\sin x+\cos x\right)+\left(\cos x+\sin x\right)e^x \quad = e^x\left(-\sin x+\cos x+\cos x+\sin x\right)$$

$$=e^x\left(2\cos x\right) = 2e^x\cos x$$

Now

$$\text{L.H.S}=\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y$$

$$=2e^x\cos x-2e^x\left(\cos x+\sin x\right)+2e^x\sin x$$

$$=2e^x\left(\cos x-\cos x-\sin x+\sin x\right)$$

$$=0$$

i.e. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0 \qquad$ *Proved*

Question # 7

If $y = e^{ax}\sin bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$

Solution $y = e^{ax}\sin bx$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}e^{ax}\sin bx$$

$$= e^{ax}\frac{d}{dx}\sin bx + \sin bx\frac{d}{dx}e^{ax} = e^{ax}\cos bx(b) + \sin bx\, e^{ax}(a)$$

$$= e^{ax}(b\cos bx + a\sin bx)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}e^{ax}(b\cos bx + a\sin bx)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{ax}\frac{d}{dx}(b\cos bx + a\sin bx) + (b\cos bx + a\sin bx)\frac{d}{dx}e^{ax}$$

$$= e^{ax}(-b\sin bx(b) + a\cos bx(b)) + (b\cos bx + a\sin bx)e^{ax}(a)$$

$$= e^{ax}\left(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + 2a^2\sin bx - a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left[2a(b\cos bx + a\sin bx) - (a^2+b^2)\sin bx\right]$$

$$= 2ae^{ax}(b\cos bx + a\sin bx) - (a^2+b^2)e^{ax}\sin bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - (a^2+b^2)y \quad \Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$$

Question # 8

If $y = (Cos^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

Solution $y = (Cos^{-1}x)^2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(Cos^{-1}x)^2 \quad \Rightarrow y_1 = 2(Cos^{-1}x)\frac{d}{dx}Cos^{-1}x$$

$$\Rightarrow y_1 = 2(Cos^{-1}x)\cdot\frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow y_1\sqrt{1-x^2} = -2(Cos^{-1}x)$$

On squaring both sides

$$y_1^2(1-x^2) = 4(Cos^{-1}x)^2$$

$$\Rightarrow y_1^2(1-x^2) = 4y \qquad \because y = (Cos^{-1}x)^2$$

Again diff. w.r.t x

$$\frac{d}{dx}y_1^2(1-x^2) = 4\frac{dy}{dx}$$

$$\Rightarrow \left(1-x^2\right)\frac{d}{dx}y_1^2 + y_1^2\frac{d}{dx}\left(1-x^2\right) = 4y_1$$

$$\Rightarrow \left(1-x^2\right)\cdot 2y_1\frac{dy_1}{dx} + y_1^2(-2x) = 4y_1 \quad \Rightarrow 2y_1\left[\left(1-x^2\right)y_2 - xy_1\right] = 4y_1$$

$$\Rightarrow \left(1-x^2\right)y_2 - xy_1 - 2 = 0$$

Question # 9

If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution $y = a\cos(\ln x) + b\sin(\ln x)$

Diff. w.r.t x

$$\frac{dy}{dx} = a\frac{d}{dx}\cos(\ln x) + b\frac{d}{dx}\sin(\ln x)$$

$$= a\left[-\sin(\ln x)\right]\frac{d}{dx}(\ln x) + b\cos(\ln x)\frac{d}{dx}(\ln x)$$

$$= -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = -a\frac{d}{dx}\sin(\ln x) + b\frac{d}{dx}\cos(\ln x)$$

$$\Rightarrow x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\left(\frac{dx}{dx}\right) = -a\cos(\ln x)\frac{d}{dx}(\ln x) + b\left(-\sin(\ln x)\right)\frac{d}{dx}(\ln x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(1) = -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left(a\cos(\ln x) + b\sin(\ln x)\right) \quad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \textit{Proved}$$
