Taylor Series Expansion of Function

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+....$$

Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

Question # 1

Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x)= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$

(ii) $\cos x= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$

(iii) $\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+.....$

(iv) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....$

(v) $e^{2x} = 1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+.....$

Solution

(i) Let $f(x) = \ln(1+x)$

$\Rightarrow f(0) = \ln(1+0) = 0$

$f'(x) = \frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$\Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$

$\Rightarrow f''(0) = -(1+0)^{-2} = -1$

$f'''(x) = \frac{d}{dx}\left[-(1+x)^{-2}\right] = +2(1+x)^{-3}$

$\Rightarrow f'''(0) = 2(1+0)^{-3} = 2$

$f^{(iv)}(x) = \frac{d}{dx}2(1+x)^{-3} = -6(1+x)^{-4}$

$\Rightarrow f^{(iv)}(0) = -6(1+0)^{-4} = -6$

By Maclaurin series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \ln(1+x) = 0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+......$$

$$= x-\frac{x^2}{2\cdot1}+\frac{x^3}{3\cdot2\cdot1}(2)-\frac{x^4}{4\cdot3\cdot2\cdot1}(6)+......$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$$

(ii) Let $f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x \qquad \Rightarrow f'(0) = -\sin(0) = 0$$

$$f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \qquad \Rightarrow f''(0) = -\cos(0) = -1$$

$$f'''(x) = \frac{d}{dx}(-\cos x) = +\sin x \qquad \Rightarrow f'''(0) = \sin(0) = 0$$

$$f^{(iv)}(x) = \frac{d}{dx}\sin x = \cos x \qquad \Rightarrow f^{(iv)}(x) = \cos(0) = 1$$

$$f^{(v)}(x) = \frac{d}{dx}\cos x = -\sin x \qquad \Rightarrow f^{(v)}(x) = -\sin(0) = 0$$

$$f^{(vi)}(0) = \frac{d}{dx}(-\sin x) = -\cos x \qquad \Rightarrow f^{(vi)}(0) = -\cos(0) = -1$$

Now by Maclaurin series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \cos x = 1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+\frac{x^5}{5!}(0)+\frac{x^6}{6!}(-1)+......$$

$$= 1+0-\frac{x^2}{2!}+0+\frac{x^4}{4!}+0-\frac{x^6}{6!}+......$$

$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$$

(iii) Let $f(x) = \sqrt{1+x}$

$$= (1+x)^{\frac{1}{2}} \qquad \Rightarrow f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(x) = \frac{d}{dx}(1+x)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1+x)^{-\frac{1}{2}}(1) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$\Rightarrow f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$

$$f''(x) = \frac{d}{dx}\left[\frac{1}{2}(1+x)^{-\frac{1}{2}}\right] = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$\Rightarrow f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\frac{d}{dx}\left[(1+x)^{-\frac{3}{2}}\right]$$

$$= -\frac{1}{4}\left[-\frac{3}{2}(1+x)^{-\frac{5}{2}}\right] = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$\Rightarrow f'''(0) = \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$$

Now by Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow \sqrt{1+x} = 1 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\cdot\frac{3}{8} + \ldots\ldots$$

$$= 1 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{6}\cdot\frac{3}{8} + \ldots\ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots\ldots$$

(iv) Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f''(0) = e^0 = 1$$

$$f'''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'''(0) = e^0 = 1$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

(v) Let $f(x) = e^{2x} \Rightarrow f(0) = e^{2(0)} = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow f'(0) = 2e^{2(0)} = 2(1) = 2$$

$$f''(x) = 2\frac{d}{dx}\left(e^{2x}\right) = 2\left(2e^{2x}\right) = 4e^{2x}$$

$$\Rightarrow f''(0) = 4e^{2(0)} = 4(1) = 4$$

$$f'''(x) = 4\frac{d}{dx}\left(e^{2x}\right) = 4\left(2e^{2x}\right) = 8e^{2x}$$

$$\Rightarrow f'''(0) = 8e^{2(0)} = 8$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \ldots..$$

$$= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots..$$

Question # 2

Show that

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

and evaluate $\cos 61^\circ$.

Solution Let $f(x) = \cos x$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x$$

$$f''(x) = -\frac{d}{dx}\sin x = -\cos x$$

$$f'''(x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x$$

By Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

$$\Rightarrow \cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \ldots\ldots$$

$$\Rightarrow \cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

Put $x = 60^\circ$ and $h = 1^\circ = \frac{\pi}{180} = 0.01745$ rad

$$\cos(60+1) = \cos 60 - (0.01745)\sin 60 - \frac{(0.01745)^2}{\lfloor 2}\cos 60 + \frac{(0.01745)^3}{\lfloor 3}\sin 60 + \ldots\ldots$$