Exercise 2.9 (Solutions)

Calculus and Analytic Geometry, MATHEMATICS 12

Increasing and Decreasing Function (Page 104)

Let f be defined on an interval (a,b) and let $x_1, x_2 \in (a,b)$. Then

1. f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$
2. f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Theorem (Page 105)

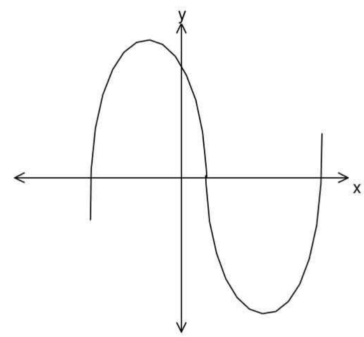
Let f be differentiable on the open interval (a,b).

1- f is increasing on (a,b) if $f'(x) > 0$ for each $x \in (a,b)$.

2- f is decreasing on (a,b) if $f'(x) < 0$ for each $x \in (a,b)$.

First Derivative Test (Page 109)

Let f be differentiable in neighbourhood of c, where $f'(c) = 0$.



1. The function has relative maxima at $x = c$ if $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$.
2. The function has relative minima at $x = c$ if $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$.

Second Derivative Test (Page 111)

Let f be differential function in a neighbourhood of c, where $f'(c) = 0$. Then

1- f has relative maxima at c if $f''(c) < 0$.

2- f has relative minima at c if $f''(c) > 0$.

Question # 1

Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

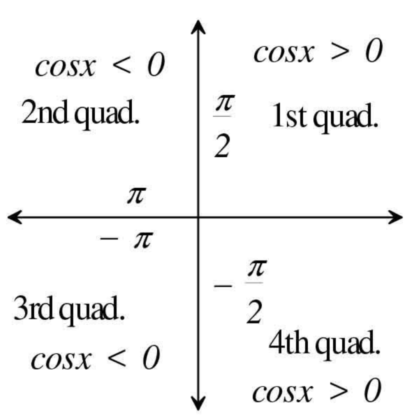
(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

(ii) $f(x) = \cos x$; $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(iii) $f(x) = 4 - x^2$; $x \in [-2, 2]$

(iv) $f(x) = x^2 + 3x + 2$; $x \in [-4, 1]$

Solution



(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

$\Rightarrow f'(x) = \cos x$

Put $f'(x) = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

So we have sub-intervals $\left(-\pi,-\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\pi\right)$

$f'(x)=\cos x<0$ whenever $x\in\left(-\pi,-\frac{\pi}{2}\right)$

So f is decreasing on the interval $\left(-\pi,-\frac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

So f is increasing on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(\frac{\pi}{2},\pi\right)$

So f is decreasing on the interval $\left(\frac{\pi}{2},\pi\right)$.

(ii) $f(x)=\cos x$; $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\Rightarrow f'(x)=-\sin x$

Put $f'(x)=0 \Rightarrow -\sin x=0 \Rightarrow \sin x=0 \Rightarrow x = 0$

So we have sub-intervals $\left(-\frac{\pi}{2},0\right)$ and $\left(0,\frac{\pi}{2}\right)$.

Now $f'(x)=-\sin x>0$ whenever $x\in\left(-\frac{\pi}{2},0\right)$

So f is increasing on $\left(-\frac{\pi}{2},0\right)$

$f'(x)=-\sin x<0$ whenever $x\in\left(0,\frac{\pi}{2}\right)$

So f is decreasing on $\left(0,\frac{\pi}{2}\right)$.

(iii) $f(x)=4-x^2$; $x\in[-2,2]$

$\Rightarrow f'(x) = -2x$

Put $f'(x)=0 \Rightarrow -2x=0 \Rightarrow x=0$

So we have subintervals $(-2,0)$ and $(0,2)$

$\because f'(x)=-2x>0$ whenever $x\in(-2,0)$

$\therefore f$ is increasing on the interval $(-2,0)$

Also $f'(x)=-2x<0$ whenever $x\in(0,2)$

$\therefore f$ is decreasing on $(0,2)$

(iv) $f(x) = x^2+3x+2$; $x\in[-4,1]$

$\Rightarrow f'(x) = 2x+3$

Put $f'(x)=0 \Rightarrow 2x+3=0 \Rightarrow x=-\frac{3}{2}$

So we have sub-intervals $\left(-4,-\frac{3}{2}\right)$ and $\left(-\frac{3}{2},1\right)$

Now $f'(x)=2x+3<0$ whenever $x\in\left(-4,-\frac{3}{2}\right)$

So f is decreasing on $\left(-4,-\frac{3}{2}\right)$

Also $f'(x)>0$ whenever $x\in\left(-\frac{3}{2},1\right)$

Therefore f is increasing on $\left(-\frac{3}{2},1\right)$.

Question # 2

Ind the extreme values of the following functions defined as:

(i) $f(x)=1-x^3$ (ii) $f(x) = x^2-x-2$

(iii) $f(x) = 5x^2-6x+2$ (iv) $f(x) = 3x^2$

(v) $f(x) =3x^2-4x+5$ (vi) $f(x) = 2x^3-2x^2-36x+3$

(vii) $f(x) = x^4-4x^2$ (viii) $f(x) = (x-2)^2(x-1)$

(ix) $f(x) = 5+3x-x^3$

Solution

(i) $f(x)=1-x^3$

Diff. w.r.t x

$f'(x) =-3x^2$ ……. (i)

For stationary points, put $f'(x) = 0$

$\Rightarrow -3x^2 = 0 \Rightarrow x=0$

Diff (i) w.r.t x

$f''(x)=-6x$ ………. (ii)

Now put $x=0$ in (ii)

$f''(0)=-6(0)=0$

So second derivative test fails to determinate the extreme points.

Put $x=0-\varepsilon=-\varepsilon$ in (i)

$f'(x)=-3(-\varepsilon)^2=-3\varepsilon^2<0$

Put $x=0+\varepsilon=\varepsilon$ in (i)

$f'(x)=-3(\varepsilon)^2 =-3\varepsilon^2 <0$

As $f'(x)$ doest not change its sign before and after $x=0$.

Since at $x=0$, $f(x)=1$ therefore $(0,1)$ is the point of inflexion.

(ii) $f(x) = x^2 - x - 2$

Diff. w.r.t. x

$f'(x) = 2x-1$ .... (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 2x-1=0 \quad \Rightarrow 2x=1 \quad \Rightarrow x=\frac{1}{2}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(2x-1) = 2$$

As $f''\left(\frac{1}{2}\right) = 2 > 0$

Thus $f(x)$ is minimum at $x=\frac{1}{2}$

Now $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$

(iii) $f(x) = 5x^2 - 6x + 2$

Diff. w.r.t. x

$f'(x) = 10x-6$ .... (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 10x-6=0 \quad \Rightarrow 10x=6 \quad \Rightarrow x=\frac{6}{10} \quad \Rightarrow x=\frac{3}{5}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(10x-6) = 10$$

As $f''\left(\frac{3}{5}\right) = 10 > 0$

Thus $f(x)$ is minimum at $x=\frac{3}{5}$

And $f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$

(iv) $f(x) = 3x^2$

Diff. w.r.t x

$f'(x) = 6x$ .... (i)

For stationary points, put $f'(x)=0$

$$\Rightarrow 6x=0 \quad \Rightarrow x=0$$

Diff. (i) w.r.t x

$$f''(x) = 6$$

At $x=0$

$$f''(0) = 6 > 0$$

$\Rightarrow f$ has minimum value at $x=0$

And $f(0) = 3(0)^2 = 0$

(v) *Do yourself*

(vi) $f(x) = 2x^3 - 2x^2 - 36x + 3$

Diff. w.r.t x

$$f'(x) = \frac{d}{dx}\left(2x^3 - 2x^2 - 36x + 3\right) = 6x^2 - 4x - 36 \text{ ....(i)}$$

For stationary points, put $f'(x) = 0$

$$\Rightarrow 6x^2 - 4x - 36 = 0$$

$$\Rightarrow 3x^2 - 2x - 12 = 0 \qquad \div\text{ing by 2}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(3)(-18)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4+216}}{6} = \frac{2 \pm \sqrt{220}}{6} = \frac{2 \pm 2\sqrt{55}}{6} = \frac{1 \pm \sqrt{55}}{3}$$

Diff. (i) w.r.t x

$$f''(x) = \frac{d}{dx}\left(6x^2 - 4x - 36\right) = 12x - 4$$

Now $$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$$

$$= 4\left(1+\sqrt{55}\right) - 4 = 4 + 4\sqrt{55} - 4 = 4\sqrt{55} > 0$$

$\Rightarrow f(x)$ has relative minima at $x = \dfrac{1+\sqrt{55}}{3}$.

And $$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$= \frac{2}{27}\left(1+\sqrt{55}\right)^3 - \frac{2}{9}\left(1+\sqrt{55}\right)^2 - 12\left(1+\sqrt{55}\right) + 3$$

$$= \frac{2}{27}\left(1 + 3\sqrt{55} + 3\cdot 55 + 55\sqrt{55}\right) - \frac{2}{9}\left(1 + 2\sqrt{55} + 55\right) - 12\left(1+\sqrt{55}\right) + 3$$

$$= \frac{2}{27}\left(166 + 58\sqrt{55}\right) - \frac{2}{9}\left(56 + 2\sqrt{55}\right) - 12\left(1+\sqrt{55}\right) + 3$$

$$= \frac{332}{27} + \frac{116}{27}\sqrt{55} - \frac{112}{9} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$= -\frac{247}{27} - \frac{220}{27}\sqrt{55} \quad = -\frac{1}{27}\left(247 + 220\sqrt{55}\right)$$

Also $f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right)-4$

$$= 4\left(1-\sqrt{55}\right)-4 = 4-4\sqrt{55}-4 = -4\sqrt{55} < 0$$

$\Rightarrow f(x)$ has relative maxima at $x = \frac{1+\sqrt{55}}{3}$.

And Since $f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}\left(247+220\sqrt{55}\right)$

Therefore by replacing $\sqrt{55}$ by $-\sqrt{55}$, we have

$$f\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{1}{27}\left(247-220\sqrt{55}\right)$$

(vii) $f(x) = x^4-4x^2$

Diff. w.r.t. x

$$f'(x) = 4x^3-8x \text{ .... (i)}$$

For critical points put $f'(x)=0$

$$\Rightarrow 4x^3-8x=0 \quad \Rightarrow 4x\left(x^2-2\right)=0$$

$$\Rightarrow 4x=0 \quad \text{or} \quad x^2-2=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x^2=2 \quad \Rightarrow x=\pm\sqrt{2}$$

Now diff. (i) w.r.t x

$$f''(x) = 12x^2-8$$

For $x=-\sqrt{2}$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=-\sqrt{2}$

And $f\left(-\sqrt{2}\right) = \left(-\sqrt{2}\right)^4-4\left(-\sqrt{2}\right)^2 = 4-8 = -4$

For $x=0$

$$f''(0) = 12(0)-8 = -8 < 0$$

$\Rightarrow f$ has relative maxima at $x=0$

And $f(0) = (0)^4-4(0)^2 = 0$

For $x=\sqrt{2}$

$$f''(\sqrt{2}) = 12(\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=\sqrt{2}$

And $f\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^4-4\left(\sqrt{2}\right)^2 = 4-8 = -4$

(viii) $f(x) = (x-2)^2(x-1)$

$= (x^2-4x+4)(x-1) = x^3-4x^2+4x-x^2+4x-4$

$= x^3-5x^2+8x-4$

Diff. w.r.t. x

$f'(x) = 3x^2-10x+8$

For critical (stationary) points, put $f'(x) = 0$

$\Rightarrow 3x^2-10x+8=0 \quad \Rightarrow 3x^2-6x-4x+8=0$

$\Rightarrow 3x(x-2)-4(x-2)=0 \quad \Rightarrow (x-2)(3x-4)=0$

$\Rightarrow (x-2)=0$ or $(3x-4)=0$

$\Rightarrow x=2$ or $x = \dfrac{4}{3}$

Now diff. (i) w.r.t x

$f''(x) = 6x-10$

For $x = 2$

$f''(2) = 6(2)-10 = 2 > 0$

$\Rightarrow f$ has relative minima at $x=2$

And $f(2) = (2-2)^2(2-1) = 0$

For $x=\dfrac{4}{3}$

$f''\left(\dfrac{4}{3}\right) = 6\left(\dfrac{4}{3}\right)-10 = 8-10 = -2 < 0$

$\Rightarrow f$ has relative maxima at $x=\dfrac{4}{3}$

And $f\left(\dfrac{4}{3}\right) = \left(\dfrac{4}{3}-2\right)^2\left(\dfrac{4}{3}-1\right) = \left(-\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right) = \left(\dfrac{4}{9}\right)\left(\dfrac{1}{3}\right) = \dfrac{4}{27}$

(ix) $f(x) = 5+3x-x^3$

Diff. w.r.t x

$f'(x) = 3-3x^2$ ……. (i)

For stationary points, put $f'(x)=0$

$\Rightarrow 3-3x^2=0 \quad \Rightarrow 3x^2=3 \quad \Rightarrow x^2=1 \quad \Rightarrow x=\pm 1$

Diff. (i) w.r.t x

$f''(x) = -6x$

For $x=1$

$f''(1) = -6(1) = -6 < 0$

$\Rightarrow f$ has relative maxima at $x=1$

And $f(1) = 5+3(1)-(1)^3 = 5+3-1 = 7$

For $x=-1$

$f''(-1) = -6(-1) = 6 > 0$

$\Rightarrow f$ has relative minima at $x = -1$, and

$f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$

Question # 3

Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$

$$f(x) = \sin x + \cos x$$

Solution $f(x) = \sin x + \cos x$ where $x \in [0, 2\pi]$

Diff. w.r.t x

$$f'(x) = \cos x - \sin x \quad \ldots\ldots\ldots \text{(i)}$$

For stationary points, put $f'(x) = 0$

$$\cos x - \sin x = 0$$

$$\Rightarrow -\sin x = -\cos x \quad \Rightarrow \frac{\sin x}{\cos x} = 1 \quad \Rightarrow \tan x = 1$$

$$\Rightarrow x = \tan^{-1}(1) \quad \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \quad \text{when } x \in [0, 2\pi]$$

Now diff. (i) w.r.t x

$$f''(x) = -\sin x - \cos x$$

For $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) < 0$$

$\Rightarrow f$ has relative maxima at $x = \frac{\pi}{4}$

And $f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) = \left(\sqrt{2}\right)^2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

For $x = \frac{5\pi}{4}$

$$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) > 0$$

$\Rightarrow f$ has relative minima at $x = \frac{5\pi}{4}$

And $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$

Question # 4

Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$

Solution $y = \dfrac{\ln x}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot\frac{1}{x} - \ln x\cdot(1)}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\ln x}{x^2} \quad \text{........ (i)}$$

For critical points, put $\dfrac{dy}{dx}=0$

$$\Rightarrow \frac{1-\ln x}{x^2} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow \ln x = 1$$

$$\Rightarrow \ln x = \ln e \quad \Rightarrow x = e \qquad \because \ln e = 1$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-\ln x}{x^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2\cdot\left(-\frac{1}{x}\right) - (1-\ln x)\cdot(2x)}{(x^2)^2} = \frac{-x-2x+2x\ln x}{x^4} = \frac{-3x+2x\ln x}{x^4}$$

At $x = e$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3e+2e\cdot\ln e}{e^4}$$

$$= \frac{-3e+2e\cdot(1)}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

$\Rightarrow$ y has a maximum value at $x = e$.

Question # 5

Show that $y = x^x$ has maximum value at $x = \dfrac{1}{e}$.

Solution $y = x^x$

Taking log on both sides

$$\ln y = \ln x^x \Rightarrow \ln y = x\ln x$$

Diff. w.r.t x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}x\ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\cdot\frac{d}{dx}\ln x + \ln x\cdot\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x} + \ln x\cdot(1)$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) \quad \Rightarrow \frac{dy}{dx} = x^x(1+\ln x) \quad \text{......... (i)}$$

For critical point, put $\frac{dy}{dx}=0$

$$\Rightarrow x^x(1+\ln x)=0 \quad \Rightarrow 1+\ln x=0 \text{ as } x^x\neq 0$$

$$\Rightarrow \ln x=-1 \quad \Rightarrow \ln x=-\ln e \qquad \because \ln e=1$$

$$\Rightarrow \ln x = \ln e^{-1} \quad \Rightarrow x = e^{-1} \quad \Rightarrow x = \frac{1}{e}$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}x^x(1+\ln x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = x^x\frac{d}{dx}(1+\ln x)+(1+\ln x)\frac{d}{dx}x^x$$

$$= x^x\cdot\frac{1}{x}+(1+\ln x)\cdot x^x(1+\ln x) \qquad \text{from (i)}$$

$$= x^x\left(\frac{1}{x}+(1+\ln x)^2\right)$$

At $x=\frac{1}{e}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1/e} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(\frac{1}{1/e}+\left(1+\ln\frac{1}{e}\right)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+\left(1+\ln e^{-1}\right)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-\ln e)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-1)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\cdot e > 0$$

$\Rightarrow$ y has a minimum value at $x=\frac{1}{e}$
