Question # 1

Find δy and dy in the following cases:

(i) $y = x^2 - 1$ when x changes from 3 to 3.02

(ii) $y = x^2 + 2x$ when x changes from 2 to 1.8

(iii) $y = \sqrt{x}$ when x changes from 4 to 4.41

Solution

(i) $y = x^2 - 1$ …… (i)

$x = 3$ & $\delta x = 3.02 - 3 = 0.02$

$$y + \delta y = (x + \delta x)^2 - 1$$

$$\Rightarrow \delta y = (x + \delta x)^2 - 1 - x^2 + 1$$

$$= (x + \delta x)^2 - x^2$$

Put $x = 3$ & $\delta x = 0.02$

$$\delta y = (3 + 0.02)^2 - (3)^2$$

$$\Rightarrow \boxed{\delta y = 0.1204}$$

Taking differential of (i)

$$dy = d\left(x^2 - 1\right)$$

$$\Rightarrow dy = 2x\ dx$$

Put $x = 3$ & $dx = \delta x = 0.02$

$$dy = 2(3)(0.02)$$

$$\Rightarrow \boxed{dy = 0.12}$$

(ii) *Do yourself as above.*

(iii) $y = \sqrt{x} = x^{\frac{1}{2}}$ ……. (i)

$x = 4$ & $\delta x = 4.41 - 4 = 0.41$

$$y + \delta y = (x + \delta x)^{\frac{1}{2}}$$

$$\Rightarrow \delta y = (x + \delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

Put $x = 4$ & $\delta x = 0.41$

$$\delta y = (4 + 0.41)^{\frac{1}{2}} - (4)^{\frac{1}{2}}$$

$$= 2.1 - 2$$

$$\Rightarrow \boxed{\delta y = 0.1}$$

Taking differential of (i)

$$dy = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)dx$$

$$= \frac{1}{2}x^{-\frac{1}{2}}\ dx$$

$$= \frac{1}{2x^{\frac{1}{2}}}\ dx$$

Put $x = 4$ & $dx = \delta x = 0.41$

$$dy = \frac{1}{2(4)^{\frac{1}{2}}}(0.41)$$

$$= \frac{0.41}{4}$$

$$\Rightarrow \boxed{dy = 0.1025}$$

Question # 2

Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

(i) $xy + x = 4$ (ii) $x^2 + 2y^2 = 16$

(iii) $x^4 + y^2 = xy^2$ (iv) $xy - \ln x = c$

Solution

(i) $xy + x = 4$

Taking differential on both sides

$$d(xy) + dx = d(4)$$

$$\Rightarrow xdy + ydx + dx = 0$$

$$\Rightarrow xdy + (y+1)dx = 0$$

$$\Rightarrow xdy = -(y+1)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y+1}{x}$$

$$\& \quad \frac{dx}{dy} = -\frac{x}{y+1}$$

(ii) *Do yourself as above*

(iii) $x^4 + y^2 = xy^2$

Taking differential

$$d\left(x^4\right) + d\left(y^2\right) = d\left(xy^2\right)$$

$$\Rightarrow 4x^3dx + 2ydy = x \cdot 2ydy + y^2dx$$

$$\Rightarrow 2ydy - 2xydy = y^2dx - 4x^3dx$$

$$\Rightarrow 2y(1-x)dy = \left(y^2 - 4x^3\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 4x^3}{2y(1-x)}$$

$$\& \quad \frac{dx}{dy} = \frac{2y(1-x)}{y^2 - 4x^3}$$

(iv) $xy - \ln x = c$

Taking differential

$$d(xy)-d(\ln x) = d(c)$$

$$\Rightarrow xdy+ydx-\frac{1}{x}dx = 0$$

$$\Rightarrow xdy = \frac{1}{x}dx-ydx$$

$$= \left(\frac{1}{x}-y\right)dx$$

$$\Rightarrow xdy = \left(\frac{1-xy}{x}\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-xy}{x^2}$$

$$\frac{dx}{dy} = \frac{x^2}{1-xy}$$

Question # 3

Using differentials to approximate the values of

(i) $\sqrt[4]{17}$ (ii) $(31)^{\frac{1}{5}}$

(iii) $\cos 29^\circ$ (iv) $\sin 61^\circ$

Solution

(i) Let $y = f(x) = \sqrt[4]{x}$

where $x=16$ and $\delta x = dx = 1$

Taking differential of above

$$dy = d\left(\sqrt[4]{x}\right)$$

$$= d(x)^{\frac{1}{4}}$$

$$= \frac{1}{4}x^{\frac{1}{4}-1}dx$$

$$= \frac{1}{4}x^{-\frac{3}{4}}dx$$

$$= \frac{1}{4x^{\frac{3}{4}}}dx$$

Put $x=16$ and $dx=1$

$$dy = \frac{1}{4(16)^{\frac{3}{4}}}(1)$$

$$= \frac{1}{4\left(2^4\right)^{\frac{3}{4}}}$$

$$= \frac{1}{4(8)} = 0.03125$$

Now $f(x+dx) \approx y+dy$

$$= f(x)+dy$$

$$\because y = f(x)$$

$$\Rightarrow \sqrt[4]{16+1} \approx \sqrt[4]{16}+0.03125$$

$$\Rightarrow \sqrt[4]{17} \approx \left(2^4\right)^{\frac{1}{4}}+0.03125$$

$$= 2+0.03125$$

$$= 2.03125$$

(ii) Let $y = f(x) = x^{\frac{1}{5}}$

Where $x=32$ & $\delta x = dx = -1$

Try yourself as above.

(iii) Let $y = f(x) = \cos x$

Where $x = 30^\circ$ & $\delta x = -1^\circ = -\frac{\pi}{180}$ rad

$= -0.01745$ rad

Now $dy = d(\cos x)$

$$= -\sin x\, dx$$

Put $x=30^\circ$ and $dx = \delta x = -0.01745$

$$dy = -\sin 30^\circ(-0.01745)$$

$$= -(0.5)(-0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y+dy$

$$= f(x)+dy$$

$$\Rightarrow \cos(30-1) = \cos 30^\circ + 0.008725$$

$$\Rightarrow \cos 29^\circ = 0.866+0.008725$$

$$= 0.8747$$

(iv) Let $y = f(x) = \sin x$

Where $x = 60^\circ$ & $\delta x = 1^\circ = \frac{\pi}{180}$ rad

$= 0.01745$ rad

Now $dy = d(\sin x)$

$$= \cos x\, dx$$

Put $x = 60^\circ$ and $dx = \delta x = 0.01745$

$$dy = \cos 60^\circ(0.01745)$$

$$= (0.5)(0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y+dy$

$$= f(x)+dy$$

$$\Rightarrow \sin(60+1) = \sin 60^\circ + 0.008725$$

$$\Rightarrow \sin 61^\circ = 0.866+0.008725$$

$$= 0.8747$$

Question # 4

Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02...

Solution

Let x be the length of side of cube where

$x=5$ & $\delta x = 5.02-5 = 0.02$

Assume V denotes the volume of the cube.

Then $V = x \cdot x \cdot x$

$= x^3$

Taking differential

$dV = 3x^2 dx$

Put $x = 5$ & $dx = \delta x = 0.02$

$dV = 3(5)^2 (0.02)$

$= 1.5$

Hence increase in volume is 1.5 cubic unit.