

Evaluate the following integrals:

Question # 1

$$\int \frac{-2x}{\sqrt{4-x^2}} dx$$

Solution

$$\text{Let } I = \int \frac{-2x}{\sqrt{4-x^2}} dx$$

$$\text{Put } t = 4 - x^2 \Rightarrow dt = -2x dx$$

$$\text{So } I = \int \frac{dt}{\sqrt{t}} = \int (t)^{-\frac{1}{2}} dt$$

$$= \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(t)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{t} + c = 2\sqrt{4-x^2} + c$$

Important Integrals

$$\text{Since } \frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$$

By Integrating, we have

$$\begin{aligned} \tan^{-1}\left(\frac{x}{a}\right) &= \int \frac{a}{a^2 + x^2} dx \\ &= a \cdot \int \frac{1}{a^2 + x^2} dx \end{aligned}$$

$$\Rightarrow \boxed{\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)}$$

Similarly

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}}$$

Question # 2

$$\int \frac{dx}{x^2 + 4x + 13}$$

Solution

$$\text{Let } I = \int \frac{dx}{x^2 + 4x + 13}$$

$$= \int \frac{dx}{x^2 + 2(x)(2) + (2)^2 - (2)^2 + 13}$$

$$= \int \frac{dx}{(x+2)^2 - 4 + 13}$$

$$= \int \frac{dx}{(x+2)^2 + 9} = \int \frac{dx}{(x+2)^2 + (3)^2}$$

$$\text{Put } t = x + 2 \Rightarrow dt = dx$$

$$\text{So } I = \int \frac{dt}{t^2 + 3^2}$$

$$= \frac{1}{3} \tan^{-1} \frac{t}{3} + c$$

$$= \frac{1}{3} \tan^{-1} \frac{x+2}{3} + c$$

Question # 3

$$\int \frac{x^2}{4+x^2} dx$$

Solution

$$\begin{aligned} \int \frac{x^2}{4+x^2} dx &= \int \left(1 - \frac{4}{4+x^2}\right) dx \quad \begin{array}{l} 4+x^2 \overline{) x^2} \\ \underline{-x^2+4} \\ -4 \end{array} \\ &= \int dx - 4 \int \frac{dx}{4+x^2} \end{aligned}$$

$$= x - 4 \int \frac{dx}{(2)^2 + x^2}$$

$$= x - 4 \int \frac{dx}{(2)^2 + x^2}$$

$$= x - 4 \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x - 2 \tan^{-1}\left(\frac{x}{2}\right) + c$$

Question # 4

$$\int \frac{1}{x \ln x} dx$$

Solution

$$\text{Suppose } I = \int \frac{1}{x \ln x} dx$$

$$= \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$

$$\text{Put } t = \ln x \Rightarrow dt = \frac{1}{x} dx$$

$$\text{So } I = \int \frac{1}{t} dt = \ln|t| + c$$

$$= \ln|\ln x| + c$$

Question # 5

$$\int \frac{e^x}{e^x + 3} dx$$

Solution

Suppose $I = \int \frac{e^x}{e^x + 3} dx$

Put $t = e^x + 3 \Rightarrow dt = e^x dx$

So $I = \int \frac{dt}{t} = \ln|t| + c$
 $= \ln|e^x + 3| + c$

Question # 6

$$\int \frac{x+b}{(x^2 + 2bx + c)^{\frac{1}{2}}} dx$$

Solution

Let $I = \int \frac{x+b}{(x^2 + 2bx + c)^{\frac{1}{2}}} dx$

Put $t = x^2 + 2bx + c$

$\Rightarrow dt = (2x + 2b)dx \Rightarrow dt = 2(x + b)dx$

$\Rightarrow \frac{1}{2} dt = (x + b)dx$

So $I = \int \frac{\frac{1}{2} dt}{t^{\frac{1}{2}}} = \frac{1}{2} \int t^{-\frac{1}{2}} dt$

$$= \frac{1}{2} \cdot \frac{t^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + c_1 = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c_1$$

$$= (x^2 + 2bx + c)^{\frac{1}{2}} + c_1$$

$$= \sqrt{x^2 + 2bx + c} + c_1$$

Question # 7

$$\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

Solution

Let $I = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

Put $t = \tan x \Rightarrow dt = \sec^2 x dx$

So $I = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt$

$$= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2(\tan x)^{\frac{1}{2}} + c = 2\sqrt{\tan x} + c$$

Important Integral

$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{\sec \theta (\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} d\theta \end{aligned}$$

Take $t = \sec \theta + \tan \theta$

$\Rightarrow dt = (\sec^2 \theta + \sec \theta \tan \theta) d\theta$

So $\int \sec \theta d\theta = \int \frac{1}{t} dt$

$$= \ln|t| + c$$

$$= \ln|\sec \theta + \tan \theta| + c$$

$$\Rightarrow \boxed{\int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + c}$$

Similarly

$$\boxed{\int \operatorname{cosec} \theta d\theta = \ln|\operatorname{cosec} \theta - \cot \theta| + c}$$

*See proof at page 133***Question # 8**

(a) Show that

$$\frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

(b) Show that

$$\sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

Solution

(a) Let $I = \frac{dx}{\sqrt{x^2 - a^2}}$

Put $x = a \sec \theta \Rightarrow dx = a \sec \theta \tan \theta d\theta$

So $I = \int \frac{a \sec \theta \tan \theta d\theta}{\sqrt{(a \sec \theta)^2 - a^2}}$

$$= \int \frac{a \sec \theta \tan \theta d\theta}{\sqrt{a^2 (\sec^2 \theta - 1)}}$$

$$= \int \frac{a \sec \theta \tan \theta d\theta}{\sqrt{a^2 \tan^2 \theta}}$$

$\therefore 1 + \tan^2 \theta = \sec^2 \theta$

$$= \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} = \int \sec \theta d\theta$$

$$= \ln|\sec \theta + \tan \theta| + c_1$$

$$= \ln \left| \sec \theta + \sqrt{\sec^2 \theta - 1} \right| + c_1$$

$$\begin{aligned}
&= \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + c_1 \quad \left| \begin{array}{l} x = a \sec \theta \\ \frac{x}{a} = \sec \theta \end{array} \right. \\
&= \ln \left| \frac{x}{a} + \sqrt{\frac{x^2 - a^2}{a^2}} \right| + c_1 \\
&= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + c_1 \\
&= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c_1 \\
&= \ln \left| x + \sqrt{x^2 - a^2} \right| - \ln a + c_1 \\
&= \ln \left| x + \sqrt{x^2 - a^2} \right| + c
\end{aligned}$$

where $c = -\ln a + c_1$

(b) Let $I = \int \sqrt{a^2 - x^2} dx$

Put $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$

$$\begin{aligned}
\text{So } I &= \int \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta \\
&= \int \sqrt{a^2 (1 - \sin^2 \theta)} \cdot a \cos \theta d\theta \\
&= \int \sqrt{a^2 \cos^2 \theta} \cdot a \cos \theta d\theta \quad \because 1 - \sin^2 \theta = \cos^2 \theta \\
&= \int a \cos \theta \cdot a \cos \theta d\theta \\
&= a^2 \int \cos^2 \theta d\theta \quad \because \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \\
&= a^2 \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta) d\theta \\
&= \frac{a^2}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + c \\
&= \frac{a^2}{2} \left(\theta + \frac{2 \sin \theta \cos \theta}{2} \right) + c \\
&= \frac{a^2}{2} \left(\theta + \sin \theta \sqrt{1 - \sin^2 \theta} \right) + c \\
&= \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right) + c \\
&= \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{\frac{a^2 - x^2}{a^2}} \right) + c \\
&= \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} \right) + c \\
&= \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{x}{a^2} \sqrt{a^2 - x^2} \right) + c
\end{aligned}$$

$x = a \sin \theta$
 $\frac{x}{a} = \sin \theta$
 $\sin^{-1} \frac{x}{a} = \theta$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

Evaluate the following integrals:

Question # 9

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

Solution

$$\text{Let } I = \int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

$$\text{Put } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\begin{aligned}
I &= \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^{\frac{3}{2}}} \\
&= \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^{\frac{3}{2}}} \because 1 + \tan^2 \theta = \sec^2 \theta \\
&= \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} \\
&= \int \frac{d\theta}{\sec \theta} = \int \cos \theta d\theta = \sin \theta + c \\
&= \frac{\sin \theta}{\cos \theta} \cdot \cos \theta + c = \tan \theta \cdot \frac{1}{\sec \theta} + c \\
&= \tan \theta \cdot \frac{1}{\sqrt{1 + \tan^2 \theta}} + c \\
&= \frac{x}{\sqrt{1 + x^2}} + c \quad \because x = \tan \theta
\end{aligned}$$

Question # 10

$$\int \frac{1}{(1+x^2) \tan^{-1} x} dx$$

Solution

$$\text{Let } I = \int \frac{1}{(1+x^2) \tan^{-1} x} dx$$

$$= \int \frac{1}{\tan^{-1} x} \cdot \frac{1}{(1+x^2)} dx$$

$$\text{Put } t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$$

$$\text{So } I = \int \frac{1}{t} dt = \ln |t| + c$$

$$= \ln |\tan^{-1} x| + c$$

Question # 11

$$\int \sqrt{\frac{1+x}{1-x}} dx$$

Solution

$$\text{Let } I = \int \sqrt{\frac{1+x}{1-x}} dx$$

$$\text{Put } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\begin{aligned} \text{So } I &= \int \sqrt{\frac{1+\sin \theta}{1-\sin \theta}} \cdot \cos \theta d\theta \\ &= \int \sqrt{\frac{1+\sin \theta}{1-\sin \theta} \cdot \frac{1+\sin \theta}{1+\sin \theta}} \cdot \cos \theta d\theta \\ &= \int \sqrt{\frac{(1+\sin \theta)^2}{1-\sin^2 \theta}} \cdot \cos \theta d\theta \\ &= \int \sqrt{\frac{(1+\sin \theta)^2}{\cos^2 \theta}} \cdot \cos \theta d\theta \\ &= \int \frac{1+\sin \theta}{\cos \theta} \cdot \cos \theta d\theta = \int (1+\sin \theta) d\theta \\ &= \theta - \cos \theta + c \\ &= \theta - \sqrt{1-\sin^2 \theta} + c \quad \left| \begin{array}{l} \because x = \sin \theta \\ \therefore \sin^{-1} x = \theta \end{array} \right. \\ &= \sin^{-1} x - \sqrt{1-x^2} + c \end{aligned}$$

Question # 12

$$\int \frac{\sin \theta}{1+\cos^2 \theta} d\theta$$

Solution

$$\text{Let } I = \int \frac{\sin \theta}{1+\cos^2 \theta} d\theta$$

$$\text{Put } t = \cos \theta$$

$$\Rightarrow dt = -\sin \theta d\theta \Rightarrow -dt = \sin \theta d\theta$$

$$\begin{aligned} \text{So } I &= \int \frac{-dt}{1+t^2} = -\int \frac{dt}{1+t^2} \\ &= -\tan^{-1} t + c \\ &= -\tan^{-1} (\cos \theta) + c \end{aligned}$$

Question # 13

$$\int \frac{ax}{\sqrt{a^2-x^4}} dx$$

Solution

$$\text{Let } I = \int \frac{ax}{\sqrt{a^2-x^4}} dx$$

$$= a \int \frac{x}{\sqrt{a^2-x^4}} dx$$

$$\text{Put } t = x^2 \text{ then } t^2 = x^4$$

$$dt = 2x dx \Rightarrow \frac{1}{2} dt = x \cdot dx$$

$$\text{So } I = a \int \frac{\frac{1}{2} dt}{\sqrt{a^2-t^2}}$$

$$= \frac{a}{2} \int \frac{dt}{\sqrt{a^2-t^2}}$$

$$= \frac{a}{2} \sin^{-1} \left(\frac{t}{a} \right) + c \quad \because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

$$= \frac{a}{2} \sin^{-1} \left(\frac{x^2}{a} \right) + c$$

Question # 14

$$\int \frac{dx}{\sqrt{7-6x-x^2}}$$

Solution

$$\text{Let } I = \int \frac{dx}{\sqrt{7-6x-x^2}}$$

$$= \int \frac{dx}{\sqrt{-(x^2+6x-7)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2+2(3)(x)+(3)^2-(3)^2-7)}}$$

$$= \int \frac{dx}{\sqrt{-(x+3)^2-16}}$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

$$\text{Put } t = x+3 \Rightarrow dx = dt$$

$$\text{So } I = \frac{dt}{\sqrt{16-t^2}} = \int \frac{dt}{\sqrt{(4)^2-(t)^2}}$$

$$= \sin^{-1} \left(\frac{t}{4} \right) + c$$

$$= \sin^{-1} \left(\frac{x+3}{4} \right) + c$$

Question # 15

$$\int \frac{\cos x}{\sin x \cdot \ln \sin x} dx$$

Solution

$$\text{Let } I = \int \frac{\cos x}{\sin x \cdot \ln \sin x} dx$$

$$= \int \frac{1}{\ln \sin x} \cdot \frac{\cos x}{\sin x} dx$$

$$\text{Put } t = \ln \sin x \Rightarrow dt = \frac{1}{\sin x} \cdot \cos x dx$$

$$\text{So } I = \int \frac{1}{t} dt = \ln |t| + c$$

$$= \ln |\ln \sin x| + c$$

Question # 16

$$\int \cos x \left(\frac{\ln \sin x}{\sin x} \right) dx$$

Solution

$$\text{Let } I = \int \cos x \left(\frac{\ln \sin x}{\sin x} \right) dx$$

$$= \int \ln \sin x \cdot \frac{\cos x}{\sin x} dx$$

$$\text{Put } t = \ln \sin x \Rightarrow dt = \frac{1}{\sin x} \cdot \cos x dx$$

No do yourself

Question # 17

$$\int \frac{x dx}{4 + 2x + x^2}$$

Solution

$$\text{Let } I = \int \frac{x dx}{4 + 2x + x^2}$$

$$= \frac{1}{2} \int \frac{2x dx}{x^2 + 2x + 4}$$

+ing and -ing 2 in numerator.

$$\Rightarrow I = \frac{1}{2} \int \frac{(2x+2)-2}{x^2 + 2x + 4} dx$$

$$= \frac{1}{2} \int \left(\frac{2x+2}{x^2 + 2x + 4} - \frac{2}{x^2 + 2x + 4} \right) dx$$

$$= \frac{1}{2} \int \frac{2x+2}{x^2 + 2x + 4} dx - \frac{1}{2} \int \frac{2}{x^2 + 2x + 4} dx$$

$$= \frac{1}{2} \int \frac{d(x^2 + 2x + 4)}{x^2 + 2x + 4} dx - \frac{2}{2} \int \frac{dx}{x^2 + 2x + 4}$$

$$= \frac{1}{2} \ln |x^2 + 2x + 4| - \int \frac{dx}{(x+1)^2 + (\sqrt{3})^2}$$

$$= \frac{1}{2} \ln |x^2 + 2x + 4| - \frac{1}{\sqrt{3}} \tan^{-1} \frac{x+1}{\sqrt{3}} + c$$

Question # 18

$$\int \frac{x}{x^4 + 2x^2 + 5} dx$$

Solution

$$\text{Let } I = \int \frac{x}{x^4 + 2x^2 + 5} dx$$

$$\text{Put } t = x^2 \text{ then } t^2 = x^4$$

$$dt = 2x dx \Rightarrow \frac{1}{2} dt = x dx$$

$$\text{So } I = \int \frac{\frac{1}{2} dt}{t^2 + 2t + 5} = \frac{1}{2} \int \frac{dt}{t^2 + 2t + 5}$$

$$= \frac{1}{2} \int \frac{dt}{(t+1)^2 + (2)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left(\frac{t+1}{2} \right) + c$$

$$= \frac{1}{4} \tan^{-1} \left(\frac{x^2 + 1}{2} \right) + c$$

Question # 19

$$\int \left[\cos \left(\sqrt{x} - \frac{x}{2} \right) \right] \times \left(\frac{1}{\sqrt{x}} - 1 \right) dx$$

Solution

$$\text{Let } I = \int \left[\cos \left(\sqrt{x} - \frac{x}{2} \right) \right] \times \left(\frac{1}{\sqrt{x}} - 1 \right) dx$$

$$\text{Put } t = \sqrt{x} - \frac{x}{2}$$

$$\Rightarrow dt = \left(\frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2} \right) dx \Rightarrow dt = \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 1 \right) dx$$

$$\Rightarrow 2 dt = \left(\frac{1}{\sqrt{x}} - 1 \right) dx$$

$$\text{So } I = \int \cos t \cdot 2 dt$$

$$= 2 \int \cos t dt$$

$$= 2 \sin t + c$$

Question # 20

$$\int \frac{x+2}{\sqrt{x+3}} dx$$

Solution

$$\text{Let } I = \int \frac{x+2}{\sqrt{x+3}} dx$$

$$\text{Put } t = x+3 \text{ then } t-3 = x$$

$$\Rightarrow dt = dx$$

$$\text{So } I = \int \frac{t-3+2}{\sqrt{t}} dt$$

$$= \int \frac{t-1}{(t)^{\frac{1}{2}}} dt = \int \left(\frac{t}{(t)^{\frac{1}{2}}} - \frac{1}{(t)^{\frac{1}{2}}} \right) dt$$

$$= \int \left((t)^{\frac{1}{2}} - (t)^{-\frac{1}{2}} \right) dt$$

$$= \frac{(t)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(t)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(t)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2(x+3)^{\frac{3}{2}}}{3} - 2(x+3)^{\frac{1}{2}} + c$$

$$= \frac{2(x+3)^{\frac{3}{2}}}{3} - 2\sqrt{x+3} + c$$

Question # 21

$$\int \frac{\sqrt{2}}{\sin x + \cos x} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{\sqrt{2}}{\sin x + \cos x} dx \\ &= \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} dx \end{aligned}$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} dx$$

$$\text{Put } \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$$\begin{aligned} \text{So } I &= \int \frac{1}{\sin \frac{\pi}{4} \cdot \sin x + \cos \frac{\pi}{4} \cdot \cos x} dx \\ &= \int \frac{1}{\cos \left(x - \frac{\pi}{4} \right)} dx = \int \sec \left(x - \frac{\pi}{4} \right) dx \\ &= \ln \left| \sec \left(x - \frac{\pi}{4} \right) + \tan \left(x - \frac{\pi}{4} \right) \right| + c \end{aligned}$$

Question # 22

$$\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x}$$

Solution

$$\text{Let } I = \int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x}$$

$$\because \cos \frac{\pi}{3} = \frac{1}{2} \quad \& \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore I = \int \frac{dx}{\cos \frac{\pi}{3} \sin x + \sin \frac{\pi}{3} \cos x}$$

$$= \int \frac{dx}{\sin \left(x + \frac{\pi}{3} \right)} = \int \operatorname{cosec} \left(x + \frac{\pi}{3} \right) dx$$

$$= \ln \left| \operatorname{cosec} \left(x + \frac{\pi}{3} \right) - \cot \left(x + \frac{\pi}{3} \right) \right| + c$$
