Exercise 3.4 (Solutions)

Calculus and Analytic Geometry, MATHEMATICS 12

Integration by Parts

If u and v are function of x, then

$$\int uv\,dx = u\int v dx - \int\left(\left(\int v dx\right)\cdot u'\,dx\right)$$

Question # 1

Evaluate the following integrals by parts add a word representing all the functions are defined.

$u = x$
$v = \sin x$

(i) $\int x\sin x\,dx$
(ii) $\int \ln x\,dx$
(iii) $\int x\ln x\;dx$
(iv) $\int x^2\ln x\,dx$
(v) $\int x^3\ln x\,dx$
(vi) $\int x^4\ln x\,dx$
(vii) $\int \tan^{-1}x\;dx$
(viii) $\int x^2\sin x\;dx$
(ix) $\int x^2\tan^{-1}x\,dx$
(x) $\int x\tan^{-1}x\,dx$
(xi) $\int x^3\tan^{-1}x\;dx$
(xii) $\int x^3\cos x\,dx$
(xiii) $\int \sin^{-1}x\,dx$
(xiv) $\int x\;\sin^{-1}x\;dx$
(xv) $\int e^x\sin x\cos x\,dx$
(xvi) $\int x\sin x\cos x\,dx$
(xvii) $\int x\cos^2 x\,dx$
(xviii) $\int x\sin^2 x\;dx$
(xix) $\int (\ln x)^2\,dx$
(xx) $\int \ln(\tan x)\sec^2 x dx$
(xxi) $\int \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution

(i) Let $I = \int x\sin x\,dx$

Integration by parts

$$I = x\cdot(-\cos x) - \int(-\cos x)\cdot(1)\,dx$$
$$= -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + c$$

(ii) Let $I = \int \ln x\,dx$
$= \int \ln x\cdot 1\;dx$

$u = \ln x$
$v = 1$

Integrating by parts

$$I = \ln x\cdot x - \int x\cdot\frac{1}{x}\;dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

(iii) Let $I = \int x\ln x\;dx$

Integrating by parts

$u = \ln x$
$v = x$

$$I = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\;dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

(iv) *Do yourself*

(v) *Do yourself*

(vi) *Do yourself*

(vii) Let $I = \int \tan^{-1}x\;dx$
$= \int \tan^{-1}x\cdot 1\;dx$

$u = \tan^{-1}x$
$v = 1$

Integrating by parts

$$I = \tan^{-1}x\cdot x - \int x\cdot\frac{1}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

(viii) Let $I = \int x^2\sin x\;dx$

$u = x^2$
$v = \sin x$

Integrating by parts

$$I = x^2(-\cos x) - \int(-\cos x)\cdot 2x\;dx$$
$$= -x^2\cos x + 2\int x\;\cos x\,dx$$

$u = x$
$v = \cos x$

Again integrating by parts

$$I = -x^2\cos x + 2\left(x\;\sin x - \int\sin x\;(1)\;dx\right)$$
$$= -x^2\cos x + 2x\;\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\;\sin x + 2\cos x + c$$

(ix) Let $I = \int x^2\tan^{-1}x\,dx$

$u = \tan^{-1}x$
$v = x^2$

Integrating by parts

$$I = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx + -\frac{1}{3}\int\frac{x}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\cdot\frac{x^2}{2} + -\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + -\frac{1}{6}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + -\frac{1}{6}\ln\left|1+x^2\right| + c$$

(x) Let $I = \int x\tan^{-1}x\,dx$

$u = \tan^{-1}x$
$v = x$

Integrating by parts

$$I = \frac{x^2}{2}\tan^{-1}x - \int\frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c \quad \textit{Ans.}$$

(xi) Let $I = \int x^3\tan^{-1}x\,dx$

$u = \tan^{-1}x$
$v = x^3$

Integrating by parts

$$I = \tan^{-1}x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{1+x^2}\,dx$$

$$= \frac{x^4}{4}\tan^{-1}x - -\frac{1}{4}\int\frac{x^4}{1+x^2}\,dx$$

$$\begin{array}{r} x^2-1 \\ 1+x^2\overline{)\,x^4\qquad} \\ \underset{-}{x^4}+\underset{-}{x^2} \\ \hline -x^2 \\ \underset{+}{-1}\ \underset{+}{-x^2} \\ \hline 1 \end{array}$$

$$= \frac{x^4}{4}\tan^{-1}x$$
$$-\frac{1}{4}\int\left(x^2 - 1 + \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2dx + \frac{1}{4}\int dx - \frac{1}{4}\int\frac{1}{1+x^2}\,dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + c$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{x^3}{12} + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + c$$

(xii) *Do yourself as Question # 1(viii).*

(xiii) $I = \int\sin^{-1}x\,dx$

$= \int\sin^{-1}x\cdot 1\,dx$

$u = \sin^{-1}x$
$v = 1$

Integrating by parts

$$I = \sin^{-1}x\cdot x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x - \int(1-x^2)^{-\frac{1}{2}}(x)\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-x^2)\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\cancel{1/2}} + c$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

(xiv) Let $I = \int x\sin^{-1}x\,dx$

Integrating by parts

$$I = \sin^{-1}x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}I_1-\frac{1}{2}\sin^{-1}x\ \text{.....}\ \text{(i)}$$

Where $I_1=\int\sqrt{1-x^2}\,dx$

Put $x=\sin\theta \quad \Rightarrow \quad dx=\cos\theta\,d\theta$

$$\Rightarrow\ I_1=\int\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$

$$=\int\sqrt{\cos^2\theta}\ \cos\theta\ d\theta$$

$$=\int\cos^2\theta\ \ d\theta\ \ =\int\left(\frac{1+\cos 2\theta}{2}\right)\ d\theta$$

$$=\frac{1}{2}\int(1+\cos 2\theta)\ d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$$

$$=\frac{1}{2}\left[\theta+\frac{2\sin\theta\cos\theta}{2}\right]+c$$

$$=\frac{1}{2}\left[\ \theta+\sin\theta\sqrt{1-\sin^2\theta}\ \right]+c$$

$$=\frac{1}{2}\left[\ \sin^{-1}x+x\sqrt{1-x^2}\ \right]+c$$

Using value of I_1 in (i)

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}+c\right)\right]$$

$$-\frac{1}{2}\sin^{-1}x$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c$$

$$-\frac{1}{2}\sin^{-1}x$$

$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c$$

(xv) Let $I=\int e^x\sin x\cos x\,dx$

$$=\frac{1}{2}\int e^x\cdot 2\sin x\cos x\ dx$$

$$=\frac{1}{2}\int e^x\sin 2x\ dx\ \because\ \sin 2x=2\sin x\cos x$$

Integrating by parts

$$I=\frac{1}{2}\left[e^x\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^x dx\right]$$

$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\int e^x\cos 2x\ dx$$

Again integrating by parts

$$I=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\ e^x\right)$$

$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\frac{1}{2}\int\ e^x\sin 2x\right)$$

$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-I\right)+c$$

$$=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x-\frac{1}{4}I+c$$

$$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$$

$$\Rightarrow\ \frac{5}{4}I=-\frac{1}{4}\ e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$$

$$\Rightarrow\ I=-\frac{1}{5}\ e^x\cos 2x+\frac{1}{10}e^x\sin 2x+\frac{4}{5}c$$

(xvi) Let $I=\int x\sin x\cos x\,dx$

$$=\frac{1}{2}\int x\cdot 2\sin x\cos x\ dx$$

$$=\frac{1}{2}\int x\cdot\sin 2x\ dx$$

$u=x$
$v=\sin 2x$

Integrating by parts

$$I=\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-\int\left(\frac{-\cos 2x}{2}\right)(1)dx\right]$$

(xvii)Let $I=\int x\cos^2 x\,dx$

$$=\int x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$=\frac{1}{2}\int x(1+\cos 2x)\,dx$$

$$=\frac{1}{2}\int x\,dx+\frac{1}{2}\int x\cos 2x dx$$

$u=x$
$v=\cos 2x$

$$=\frac{1}{2}\cdot\frac{x^2}{2}+\frac{1}{2}\left[x\ \frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot(1)dx\right]$$

$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\int\sin 2x\ dx$$

$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$

$$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x+\frac{1}{8}\cos 2x+c$$

(xviii) Let $I=\int x\sin^2 x\ dx$

$$= \int x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int x\left(1-\cos 2x\right)\, dx$$

$$= \frac{1}{2}\int x\, dx - \frac{1}{2}\int x\cos 2x\, dx$$

$u = x$
$v = \cos 2x$

Integrating by parts

$$I = \frac{1}{2}\,\frac{x^2}{2} - \frac{1}{2}\left[\, x\cdot\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}.(1)\,dx\,\right]$$

$$= \frac{x^2}{4} - \frac{1}{4}x\sin 2x + \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{x^2}{4} - \frac{1}{4}x\sin 2x + \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$$

$$= \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$$

(xix) Let $I = \int (\ln x)^2\,dx$

$$= \int\left(\ln x\right)^2\cdot 1\;dx$$

$u = \left(\ln x\right)^2$
$v = 1$

Integrating by parts

$$I = \left(\ln x\right)^2\cdot x - \int x\cdot 2\left(\ln x\right)\cdot\frac{1}{x}\;dx$$

$$= x\left(\ln x\right)^2 - 2\int\left(\ln x\right)\;dx$$

Again integrating by parts

$$I = x\left(\ln x\right)^2 - 2\left[\;\ln x\cdot x - \int x\cdot\frac{1}{x}\;dx\;\right]$$

$$= x\left(\ln x\right)^2 - 2x\ln x + 2\int dx$$

$$= x\left(\ln x\right)^2 - 2x\ln x + 2x + c$$

(xx) Let $I = \int \ln(\tan x)\;\sec^2 x\,dx$

Integrating by parts

$$I = \ln(\tan x)\cdot\tan x - \int \tan x\cdot\frac{1}{\tan x}\cdot\sec^2 x\;dx$$

$$= \tan x\ln\left(\tan x\right) - \int\sec^2 x\;dx$$

$$= \tan x\ln\left(\tan x\right) - \tan x + c$$

(xxi) Let $I = \int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$u = \sin^{-1}x$
$v = \left(1-x^2\right)^{-\frac{1}{2}}(-2x)$

$$= \int\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}(x)\,dx$$

$$= -\frac{1}{2}\int\sin^{-1}x\cdot\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\,dx$$

Integrating by parts

$$I = -\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$= -\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\cdot\frac{1}{\sqrt{1-x^2}}\;dx\right]$$

$$= -\frac{1}{2}\left[\;2(1-x^2)^{\frac{1}{2}}\sin^{-1}x - 2\int\;dx\right]$$

$$= -\sqrt{1-x^2}\;\sin^{-1}x + \int dx$$

$$= -\sqrt{1-x^2}\;\sin^{-1}x + x + c$$

$$= x - \sqrt{1-x^2}\;\sin^{-1}x + c$$

Question # 2

Evaluate the following integrals.

(i) $\int\tan^4 x dx$ (ii) $\int\sec^4 x\;dx$

(iii) $\int e^x\sin 2x\;\cos x\,dx$ (iv) $\int\tan^3 x\cdot\sec x\;dx$

(iv) $\int\tan^3 x\cdot\sec x\;dx$ (v) $\int x^3 e^{5x}dx$

(vi) $\int e^{-x}\sin 2x\,dx$ (vii) $\int e^{2x}\cdot\cos 3x\,dx$

(viii) $\int\text{cosec}^3 x\,dx$

Solution

(i) Let $I = \int\tan^4 x dx$

$$= \int\tan^2 x\cdot\tan^2 x\;dx$$

$$= \int\tan^2 x\left(\sec^2 x - 1\right)\;dx$$

$$= \int\left(\tan^2 x\sec^2 x - \tan^2 x\right)\;dx$$

$$= \int\tan^2 x\sec^2 x\,dx - \int\tan^2 x\;dx$$

$$= \int\tan^2 x\frac{d}{dx}(\tan x)\;dx - \int(\sec^2 x - 1)\;dx$$

$$= \frac{\tan^{2+1}x}{2+1} - \int\sec^2 x\;dx + \int dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + c$$

(ii) Let $I = \int \sec^4 x\ dx$

$$= \int (\sec^2 x)\cdot(\sec^2 x)\ dx$$

$$= \int (1+\tan^2 x)\cdot(\sec^2 x)\ dx$$

$$= \int \sec^2 x\ dx + \int \tan^2 x \sec^2 x\ dx$$

$$= \tan x + \int (\tan x)^2 \frac{d}{dx}(\tan x)\ dx$$

$$= \tan x + \frac{\tan^3 x}{3} + c$$

(iii) Let $I = \int e^x \sin 2x \cos x\, dx$

$$= \frac{1}{2}\int e^x \left(2\sin 2x \cos x\right) dx$$

$$= \frac{1}{2}\int e^x \left(\sin(2x+x) + \sin(2x-x)\right) dx$$

$$= \frac{1}{2}\int e^x \left(\sin 3x + \sin x\right) dx$$

$$= \frac{1}{2}\int e^x \sin 3x\ dx + \frac{1}{2}\int e^x \sin x\, dx$$

$$= \frac{1}{2}I_1 + \frac{1}{2}I_2 \quad \text{..........} \ (\text{i})$$

Where $I_1 = \int e^x \sin 3x\, dx$ and $I_2 = \int e^x \sin x\, dx$

Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

(iv) $I = \int \tan^3 x \cdot \sec x\ dx$

$$= \int \tan^2 x \cdot \tan x \cdot \sec x\ dx$$

$$= \int \left(\sec^2 x - 1\right)\cdot \sec x \tan x\ dx$$

Put $t = \sec x \quad \Rightarrow \quad dt = \sec x \tan x\, dx$

So $I = \int \left(t^2 - 1\right) dt$

$$= \frac{t^3}{3} - t + c$$

$$= \frac{\sec^3 x}{3} - \sec + c$$

(v) Let $I = \int x^3 e^{5x} dx$

$u = x^3$
$v = e^x$

Integrating by parts

$$I = x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot 3x^2 dx$$

$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\int x^2 e^{5x} dx$$

$u = x^2$
$v = e^x$

Again integrating by parts

$$I = \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\left[\ x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot 2x\, dx\ \right]$$

$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\int x\, e^{5x}\ dx$$

Again integrating by parts

$$I = \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\left[\ x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot (1) dx\ \right]$$

$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x e^{5x} - \frac{6}{125}\int e^{5x}\ dx$$

$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x e^{5x} - \frac{6}{125}\cdot\frac{e^{5x}}{5} + c$$

$$= \frac{e^{5x}}{5}\left(x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}\right) + c$$

(vi) Let $I = \int e^{-x} \sin 2x\, dx$

Integrating by parts

$u = e^{-x}$
$v = \sin 2x$

$$I = e^{-x}\cdot\frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2}\cdot e^{-x}(-1)\, dx$$

$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\ dx$$

Again integrating by parts

$$I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left[\ e^{-x}\cdot\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\cdot e^{-x}(-1)\ dx\ \right]$$

$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}\int e^{-x}\sin 2x\, dx$$

$$\Rightarrow \ I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I + c$$

$$\Rightarrow \ I + \frac{1}{4}I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x + c$$

$$\Rightarrow \ \frac{5}{4}I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x + c$$

$$\Rightarrow \ I = -\frac{2}{5}e^{-x}\cos 2x - \frac{1}{5}e^{-x}\sin 2x + \frac{4}{5}c$$

$$= -\frac{1}{5}e^{-x}\left(2\cos 2x + \sin 2x\right) + \frac{4}{5}c$$

(vii) *Do yourself as above*

(viii) $I = \int \operatorname{cosec}^3 x\, dx$

$u = \operatorname{cosec} x$
$v = \operatorname{cosec}^2 x$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x \ dx$$

Integrating by parts

$$I = \csc x\left(-\cot x\right) i \int \left(-\cot x\right)\left(-\csc x \cot x\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \ dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x\left(\operatorname{cosec}^2 x - 1\right) \ dx$$

$$= -\operatorname{cosec} x \cot x - \int \left(\operatorname{cosec}^3 x - \operatorname{cosec} x\right) \ dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$= -\operatorname{cosec} x \cot x - I + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I + I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad 2I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln\left|\csc x - \cot x\right| + \frac{1}{2}c$$

Question # 3

Show that

$$\int e^{ax} \sin bx \ dx = \frac{1}{\sqrt{a^2+b^2}} \ e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Solution

Let $I = \int e^{ax} \sin bx \ dx$

$u = e^{ax}$
$v = \sin bx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right) \cdot e^{ax}(a)\,dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx \ dx$$

Again integrating by parts

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[e^{ax}\,\frac{\sin bx}{b} - \int \frac{\sin bx}{b} \cdot e^{ax} a\,dx\right]$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I + c_1$$

$$\Rightarrow \quad I + \frac{a^2}{b^2}I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx + c_1$$

$$\Rightarrow \quad \left(\frac{b^2+a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}\left(-b\cos bx + a\sin bx\right) + c_1$$

$$\Rightarrow \quad \left(b^2+a^2\right)I = e^{ax}\left(a\sin bx - b\cos bx\right) + b^2c_1$$

Put $a = r\cos\theta$ & $b = r\sin\theta$

Squaring and adding

$$a^2+b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow \quad a^2+b^2 = r^2(1) \quad \Rightarrow \quad r = \sqrt{a^2+b^2}$$

Also

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \quad \Rightarrow \quad \frac{b}{a} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{b}{a}$$

So

$$\left(b^2+a^2\right)I = e^{ax}\left(r\cos\theta\sin bx - r\sin\theta\cos bx\right) + b^2c_1$$

$$\left(b^2+a^2\right)I = e^{ax}r\left(\sin bx\cos\theta - \cos bx\sin\theta\right) + b^2c_1$$

$$\Rightarrow \quad \left(a^2+b^2\right)I = e^{ax}r\sin\left(bx-\theta\right) + b^2c_1$$

Putting value of r and θ

$$\left(a^2+b^2\right)I = e^{ax}\sqrt{a^2+b^2}\ \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + b^2c_1$$

$$\Rightarrow I = \frac{\sqrt{a^2+b^2}}{(a^2+b^2)}\ e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + \frac{b^2}{a^2+b^2}c_1$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{a^2+b^2}}\ e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Where $c = \dfrac{b^2}{a^2+b^2}c_1$

Question # 4

Evaluate the following indefinite integrals.

(i) $\int\sqrt{a^2-x^2}\ dx$ (ii) $\int\sqrt{x^2-a^2}\ dx$

(iii) $\int\sqrt{4-5x^2}\,dx$ (iv) $\int\sqrt{3-4x^2}\,dx$

(v) $\int\sqrt{x^2+4}\,dx$ (vi) $\int x^2e^{ax}dx$

Solution

(i) Let $I = \int\sqrt{a^2-x^2}\ dx$
$= \int\sqrt{a^2-x^2}\cdot 1\ dx$

$u = \sqrt{a^2-x^2}$
$v = 1$

Integrating by parts

$$I = \sqrt{a^2-x^2}\cdot x - \int x\cdot\frac{1}{2}\left(a^2-x^2\right)^{-\frac{1}{2}}\cdot(-2x)\ dx$$

$$= x\sqrt{a^2-x^2} - \int\frac{-x^2}{\left(a^2-x^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{a^2-x^2} - \int\frac{a^2-x^2-a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{a^2-x^2} - \int \left(\frac{a^2-x^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} - \frac{a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\ dx + \int \frac{a^2}{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow\ I = x\sqrt{a^2-x^2} - I + a^2 \int \frac{1}{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow\ I + I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ 2I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ I = \frac{1}{2} x\sqrt{a^2-x^2} + \frac{1}{2} a^2 Sin^{-1}\frac{x}{a} + \frac{1}{2} c$$

Review

- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left| x + \sqrt{x^2+a^2} \right| + c$

(ii)Let $I = \int \sqrt{x^2-a^2}\ dx$

$= \int \sqrt{x^2-a^2} \cdot 1\ dx$

$u = \sqrt{x^2-a^2}$
$v = 1$

Integrating by parts

$$I = \sqrt{x^2-a^2} \cdot x - \int x \cdot \frac{1}{2}\left(x^2-a^2\right)^{-\frac{1}{2}} \cdot (2x)\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \left(\frac{x^2-a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} + \frac{a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\ dx - \int \frac{a^2}{\sqrt{x^2-a^2}} dx$$

$$\Rightarrow\ I = x\sqrt{x^2-a^2} - I - a^2 \int \frac{1}{\sqrt{x^2-a^2}} dx$$

$$\Rightarrow\ I + I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\because\ \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ 2I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ I = \frac{1}{2} x\sqrt{x^2-a^2} - \frac{a^2}{2} \ln\left| x + \sqrt{x^2-a^2} \right| + \frac{1}{2} c$$

(iii) Let $I = \int \sqrt{4-5x^2}\ dx$

$= \int \sqrt{4-5x^2} \cdot 1\ dx$

Integrating by parts

$$I = \sqrt{4-5x^2} \cdot x - \int x \cdot \frac{1}{2}(4-5x^2)^{-\frac{1}{2}} \cdot (-10x)\ dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{-5x^2}{(4-5x^2)} dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{4-5x^2-4}{(4-5x^2)} dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \sqrt{4-5x^2}\, dx + 4 \int \frac{1}{\sqrt{4-5x^2}}\ dx$$

$$\Rightarrow\ I = \sqrt{4-5x^2} \cdot x - I + 4 \int \frac{1}{\sqrt{5\left(\frac{4}{5} - x^2\right)}}\ dx$$

$$\Rightarrow\ I + I = \sqrt{4-5x^2} \cdot x + 4 \int \frac{1}{\sqrt{5}\sqrt{\frac{4}{5} - x^2}}\ dx$$

$$\Rightarrow\ 2I = \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}\ dx$$

$$= \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} Sin^{-1}\left(\frac{x}{2/\sqrt{5}}\right) + c_1$$

$$\because\ \int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}$$

$$\Rightarrow\ I = \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}} Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + \frac{1}{2} c_1$$

$$=\frac{x}{2}\sqrt{4-5x^2}+\frac{2}{\sqrt{5}}Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right)+c$$

Where $c=\frac{1}{2}c_1$

(iv) *Same as above.*

(v) *Same as Q #* 4(ii)

Use $\int\frac{dx}{\sqrt{x^2+4}}=\ln\left|x+\sqrt{x^2+4}\right|+c$

(vi) *Do yourself as Question #* 2(v)

Important Formula

Since $\frac{d}{dx}\left(e^{ax}f(x)\right)=e^{ax}\frac{d}{dx}f(x)+f(x)\frac{d}{dx}e^{ax}$

$$=e^{ax}f'(x)+f(x)\cdot e^{ax}(a)$$

$$=e^{ax}\left[\,a\,f(x)+f'(x)\,\right]$$

On integrating

$$\int\frac{d}{dx}\left(e^{ax}f(x)\right)\,dx=\int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]\,dx$$

$$\Rightarrow\ e^{ax}f(x)=\int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]\,dx$$

$$\Rightarrow\ \boxed{\int e^{ax}\left[\,a\,f(x)+f'(x)\,\right]\,dx=e^{ax}f(x)+c}$$

Question # 5

Evaluate the following integrals.

(i) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$ (ii) $\int e^x\left(\cos x+\sin x\right)dx$

(iii) $\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

(iv) $\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

(v) $\int\frac{xe^x}{(1+x)^2}dx$ (vi) $\int\frac{xe^x}{(1+x)^2}dx$

(vii) $\int e^{-x}\left(\cos x-\sin x\right)dx$

(viii) $\int\frac{e^{m\tan^{-1}x}}{1+x^2}dx$ (ix) $\int\frac{2x}{1-\sin x}dx$

(x) $\int\frac{e^x(1+x)}{(2+x)^2}dx$ (xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

Solution

(i) Let $I=\int e^x\left(\frac{1}{x}+\ln x\right)dx$

$$=\int e^x\left(\ln x+\frac{1}{x}\right)dx$$

Put $f(x)=\ln x\ \Rightarrow\ f'(x)=\frac{1}{x}$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^xf(x)+c\quad=e^x\ln x+c$$

(ii) Let $I=\int e^x\left(\cos x+\sin x\right)dx$

$$=\int e^x\left(\sin x+\cos x\right)dx$$

Put $f(x)=\sin x\ \Rightarrow\ f'(x)=\cos x$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^xf(x)+c$$

$$=e^x\sin x+c$$

(iii) Let $I=\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

Put $f(x)=\sec^{-1}x\ \Rightarrow\ f'(x)=\frac{1}{x\sqrt{x^2-1}}$

So $I=\int e^{ax}\left[a\,f(x)+f'(x)\right]dx$

$$=e^{ax}f(x)+c$$

$$=e^{ax}\sec^{-1}x+c$$

(iv) Let $I=\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

$$=\int e^{3x}\left(\frac{3\sin x}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$$

$$=\int e^{3x}\left(3\frac{1}{\sin x}-\frac{\cos x}{\sin x\cdot\sin x}\right)dx$$

$$=\int e^{3x}\left(3\csc x-\csc x\cot x\right)dx$$

Put $f(x)=\csc x\ \Rightarrow\ f'(x)=-\csc x\cot x$

$\Rightarrow\ I=\int e^{3x}\left(3f(x)+f'(x)\right)dx$

$$=e^{3x}f(x)+c$$

$$=e^{3x}\csc x+c$$

(v) Let $I=\int e^{2x}\left(-\sin x+2\cos x\right)dx$

$$=\int e^{2x}\left(2\cos x-\sin x\right)dx$$

Put $f(x)=\cos x\ \Rightarrow\ f'(x)=-\sin x$

So $I=\int e^{2x}\left(2f(x)+f'(x)\right)dx$

$$= e^{2x} f(x) + c$$
$$= e^{2x} \cos x + c$$

(vi) Let $I = \int \frac{xe^x}{(1+x)^2} dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2} dx$$
$$= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx$$
$$= \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$$

So $I = \int e^x \left(f(x) + f'(x) \right) dx$

$$= e^x f(x) + c$$
$$= e^x \left(\frac{1}{1+x} \right) + c$$

(vii) Let $I = \int e^{-x} \left(\cos x - \sin x \right) dx$

$$= \int e^{-x} \left((-1)\sin x + \cos x \right) dx$$

Put $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

So $I = \int e^{-x} \left((-1) f(x) + f'(x) \right) dx$

$$= e^{-x} f(x) + c$$
$$= e^{-x} \sin x + c$$

(viii) Let $I = \int \frac{e^{m\tan^{-1} x}}{1+x^2} dx$

$$= \int e^{m\tan^{-1} x} \cdot \frac{1}{1+x^2} dx$$

Put $t = \tan^{-1} x \quad \Rightarrow \quad dt = \frac{1}{1+x^2} dx$

So $I = \int e^{mt} dt$

$$= \frac{e^{mt}}{m} + c$$
$$= \frac{1}{m} e^{m\tan^{-1} x} + c$$

Important Integral

Let $I = \int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

Put $t = \cos x \quad \Rightarrow \quad dt = -\sin x \, dx$

$$\Rightarrow \quad -dt = \sin x \, dx$$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$$= -\ln|t| + c$$
$$= -\ln|\cos x| + c$$
$$= \ln|\cos x|^{-1} + c \qquad \because \; m\ln x = \ln x^m$$
$$= \ln\left|\frac{1}{\cos x}\right| + c \quad = \ln|\sec x| + c$$
$$\Rightarrow \quad \boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

Similarly, we have

$$\boxed{\int \cot x \; dx = \ln|\sin x| + c}$$

(ix) Let $I = \int \frac{2x}{1-\sin x} dx$

$$= \int \frac{2x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx$$
$$= \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx$$
$$= \int \frac{2x + 2x\sin x)}{\cos^2 x} dx$$
$$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x} \right) dx$$
$$= \int \frac{2x}{\cos^2 x} dx + \int \frac{2x\sin x}{\cos x \cdot \cos x} dx$$
$$= 2\int x\sec^2 x \; dx + 2\int x\sec x \tan x \, dx$$

Integrating by parts

$$I = 2\left[x \cdot \tan x - \int \tan x \cdot 1 dx \right]$$
$$+ 2\left[x \cdot \sec x - \int \sec x(1) \, dx \right]$$
$$= 2\left[x \cdot \tan x - \ln|\sec x| \right]$$
$$+ 2\left[x \cdot \sec x - \ln|\sec x + \tan x| \right] + c$$
$$= 2x\tan x - 2\ln|\sec x|$$
$$+ 2x\sec x - 2\ln|\sec x + \tan x| + c$$

(x) Let $I = \int \frac{e^x(1+x)}{(2+x)^2} dx$

$$=\int\frac{e^x(2+x-1)}{(2+x)^2}dx$$

$$=\int e^x\left(\frac{2+x}{(2+x)^2}-\frac{1}{(2+x)^2}\right)dx$$

$$=\int e^x\left((2+x)^{-1}-(2+x)^{-2}\right)dx$$

Put $f(x)=(2+x)^{-1} \quad \Rightarrow \quad f'(x)=-(2+x)^{-2}$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^x f(x)+c$$

$$=e^x(2+x)^{-1}+c$$

$$=\frac{e^x}{2+x}+c$$

(xi) Let $I=\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

$$=\int\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2}\text{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)e^x\,dx$$

$$=\int e^x\left(-\cot\frac{x}{2}+\frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$$

Put $f(x)=-\cot\frac{x}{2} \quad \Rightarrow \quad f'(x)=\text{cosec}^2\frac{x}{2}\cdot\frac{1}{2}$

$$\Rightarrow \quad f'(x)=\frac{1}{2}\text{cosec}^2\frac{x}{2}$$

So $I=\int e^x\left(f(x)+f'(x)\right)$

$$=e^x f(x)+c$$

$$=e^x\left(-\cot\frac{x}{2}\right)+c$$

$$=-e^x\cot\frac{x}{2}+c.$$