Definite Integral

Let $\int f(x)dx = \varphi(x)+c$

Then $\int_a^b f(x)\,dx = \left|\varphi(x)\right|_a^b \quad or \quad [\varphi(x)]_a^b$

$= \varphi(b)-\varphi(a)$

Also

- $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

where $a<c<b$

Evaluate the following definite integrals:

Question # 1

$$\int_1^2 \left(x^2+1\right)dx$$

Solution

$$\int_1^2 \left(x^2+1\right)dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + \left|x\right|_1^2 = \left(\frac{2^3}{3}-\frac{1^3}{3}\right)+(2-1)$$

$$= \frac{8}{3}-\frac{1}{3}+1 \quad = \frac{10}{3}$$

Question # 2

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$$

Solution

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$$

$$= \int_{-1}^1 x^{\frac{1}{3}}dx + \int_{-1}^1 dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_{-1}^1 + \left|x\right|_{-1}^1$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_{-1}^1 + \left(1-(-1)\right)$$

$$= \frac{3}{4}\left((1)^{\frac{4}{3}}-(-1)^{\frac{4}{3}}\right)+(1+1)$$

$$= \frac{3}{4}(1-1)+2 \quad = 2$$

Question # 3

$$\int_{-2}^0 \frac{1}{(2x-1)^2}dx$$

Solution

$$\int_{-2}^0 \frac{1}{(2x-1)^2}dx$$

$$= \int_{-2}^0 (2x-1)^{-2}\,dx$$

$$= \left|\frac{(2x-1)^{-2+1}}{(-2+1)\cdot 2}\right|_{-2}^0$$

$$= \left|\frac{(2x-1)^{-1}}{(-1)\ 2}\right|_{-2}^0$$

$$= \frac{(2(0)-1)^{-1}}{-2} - \frac{(2(-2)-1)^{-1}}{-2}$$

$$= \frac{(0-1)^{-1}}{-2} - \frac{(-4-1)^{-1}}{-2}$$

$$= \frac{(-1)^{-1}}{-2} - \frac{(-5)^{-1}}{-2}$$

$$= \frac{1}{(-2)(-1)} - \frac{1}{(-2)(-5)}$$

$$= \frac{1}{2}-\frac{1}{10} \quad = \frac{2}{5}$$

Question # 4

$$\int_{-6}^2 \sqrt{3-x}\,dx$$

Solution

$$\int_{-6}^{2}\sqrt{3-x}\,dx$$

$$=\int_{-6}^{2}(3-x)^{\frac{1}{2}}\,dx$$

$$=\left|\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)}\right|_{-6}^{2}\quad=\left|\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left((3-2)^{\frac{3}{2}}-(3+6)^{\frac{3}{2}}\right)$$

$$=-\frac{2}{3}\left((1)^{\frac{3}{2}}-(9)^{\frac{3}{2}}\right)\quad=-\frac{2}{3}(1-27)\quad=\frac{52}{3}.$$

Question # 5

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt$$

Solution

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt$$

$$=\int_{1}^{\sqrt{5}}(2t-1)^{\frac{3}{2}}\,dt$$

$$=\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot 2}\right|_{1}^{\sqrt{5}}\quad=\left|\frac{(2t-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot 2}\right|_{1}^{\sqrt{5}}$$

$$=\left|\frac{(2t-1)^{\frac{5}{2}}}{5}\right|_{1}^{\sqrt{5}}\quad=\frac{\left(2\sqrt{5}-1\right)^{\frac{5}{2}}}{5}-\frac{(2(1)-1)^{\frac{5}{2}}}{5}$$

$$=\frac{\left(2\sqrt{5}-1\right)^{\frac{5}{2}}}{5}-\frac{1}{5}$$

$$=\frac{\sqrt{\left(2\sqrt{5}-1\right)^{5}}}{5}-\frac{1}{5}\qquad \textit{Ans.}$$

Question # 6

$$\int_{2}^{\sqrt{5}}x\sqrt{x^2-1}\,dx$$

Solution

$$\int_{2}^{\sqrt{5}}x\sqrt{x^2-1}\,dx$$

$$=\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot 2x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^2-1\right)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(x^2-1\right)\,dx$$

$$=\frac{1}{2}\left|\frac{\left(x^2-1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{\sqrt{5}}\quad=\frac{1}{2}\left|\frac{\left(x^2-1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{\sqrt{5}}$$

$$=\frac{1}{2}\cdot\frac{2}{3}\left[\left(\left(\sqrt{5}\right)^2-1\right)^{\frac{3}{2}}-\left((2)^2-1\right)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[(5-1)^{\frac{3}{2}}-(4-1)^{\frac{3}{2}}\right]\quad=\frac{1}{3}\left[(4)^{\frac{3}{2}}-(3)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[\left(2^2\right)^{\frac{3}{2}}-(3)^{1+\frac{1}{2}}\right]\quad=\frac{1}{3}\left[(2)^3-3(3)^{\frac{1}{2}}\right]$$

$$=\frac{1}{3}\left[8-3\sqrt{3}\right]$$

Question # 7

$$\int_{1}^{2}\frac{x}{x^2+2}\,dx$$

Solution

$$\int_{1}^{2}\frac{x}{x^2+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{2x}{x^2+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{\frac{d}{dx}\left(x^2+2\right)}{x^2+2}\,dx\quad=\frac{1}{2}\left|\ln\left|x^2+2\right|\right|_{1}^{2}$$

$$=\frac{1}{2}\left(\ln\left|2^2+2\right|-\ln\left|1^2+2\right|\right)$$

$$=\frac{1}{2}\left(\ln 6-\ln 3\right)$$

$$=\frac{1}{2}\ln\left(\frac{6}{3}\right)\quad=\frac{1}{2}\ln 2$$

Question # 8

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 dx$$

Solution

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 dx \quad = \int_2^3 \left(x^2+\frac{1}{x^2}-2\right) dx$$

$$= \int_2^3 x^2\,dx + \int_2^3 x^{-2}\,dx - 2\int_2^3 dx$$

Now do yourself

Question # 9

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\;dx$$

Solution

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\;dx$$

$$= \int_{-1}^{1} \left(\frac{2x+1}{2}\right)\left(x^2+x+1\right)^{\frac{1}{2}} dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}(2x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}\frac{d}{dx}(2x+1)\,dx$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{1}$$

NOTE
$$(3)^{\frac{3}{2}} = (3)^{1+\frac{1}{2}} = 3^1\cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left|\left(x^2+x+1\right)^{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left[\left((1)^2+(1)+1\right)^{\frac{3}{2}} - \left((-1)^2+(-1)+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(1+1+1)^{\frac{3}{2}} - (1-1+1)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(3)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] \quad = \frac{1}{3}\left[3\sqrt{3}-1\right]$$

$$= \sqrt{3}-\frac{1}{3}$$

Question # 10

$$\int_0^3 \frac{dx}{x^2+9}$$

Solution

$$\int_0^3 \frac{dx}{x^2+9} \quad = \int_0^3 \frac{dx}{x^2+3^2}$$

$$= \left|\frac{1}{3}Tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}Tan^{-1}\left(\frac{3}{3}\right) - \frac{1}{3}Tan^{-1}\left(\frac{0}{3}\right)$$

$$= \frac{1}{3}Tan^{-1}(1) - \frac{1}{3}Tan^{-1}(0)$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right) - \frac{1}{3}(0) \quad = \frac{\pi}{12}$$

Question # 11

$$\int_{\pi/6}^{\pi/3} \cos t\,dt$$

Solution *Do yourself*

Question # 12

$$\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)dx$$

Solution *Do yourself*

Question # 13

$$\int_1^2 \ln x\;dx$$

Solution

Let $I = \int_1^2 \ln x\;dx \quad = \int_1^2 \ln x\cdot 1\;dx$

Integrating by parts

$$I = \left|\ln x\cdot x\right|_1^2 - \int_1^2 x\cdot\frac{1}{x}dx$$

$$= \left|x\ln x\right|_1^2 - \int_1^2 dx$$

$$= \left(2\cdot\ln 2 - 1\cdot\ln 1\right) - \left|x\right|_1^2$$

$$= \left(2\cdot\ln 2 - 1\cdot(0)\right) - (2-1)$$

$$= (2\cdot\ln 2 - 0) - 1 = 2\ln 2 - 1$$

Question # 14

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx$$

Solution

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx$$

$$= \int_0^2 e^{\frac{x}{2}}\,dx - \int_0^2 e^{-\frac{x}{2}}\,dx$$

$$= \left|\frac{e^{\frac{x}{2}}}{1/2}\right|_0^2 - \left|\frac{e^{-\frac{x}{2}}}{-1/2}\right|_0^2 \quad = 2\left|e^{\frac{x}{2}}\right|_0^2 + 2\left|e^{-\frac{x}{2}}\right|_0^2$$

$$= 2\left(e^{\frac{2}{2}} - e^{\frac{0}{2}}\right) + 2\left(e^{-\frac{2}{2}} - e^{-\frac{0}{2}}\right)$$

$$= 2\left(e^1 - e^0\right) + 2\left(e^{-1} - e^0\right)$$

$$= 2\left(e - 1 + \frac{1}{e} - 1\right) \quad = 2\left(e + \frac{1}{e} - 2\right)$$

$$= 2\left(\frac{e^2 + 1 - 2e}{e}\right) \quad = 2\frac{(e-1)^2}{e}$$

Question # 15

$$\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}\,d\theta$$

Solution

Let $I = \int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}\,d\theta$

$$= \int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2\cos^2\theta}\,d\theta$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_0^{\pi/4} \left(\frac{\cos\theta}{2\cos^2\theta} + + \frac{\sin\theta}{2\cos^2\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2\cos\theta}\,d\theta + \int_0^{\pi/4} \frac{\sin\theta}{2\cos\theta\cdot\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec\theta\,d\theta + \frac{1}{2}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{2}\Big|\ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} + \frac{1}{2}\Big|\sec\theta\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec(0) + \tan(0)|\right) + \frac{1}{2}\left(\sec\frac{\pi}{4} - \sec(0)\right)$$

$$= \frac{1}{2}\left(\ln\left|\sqrt{2}+1\right| - \ln|1+0|\right) + \frac{1}{2}\left(\sqrt{2}-1\right)$$

$$= \frac{1}{2}\left(\ln\left|\sqrt{2}+1\right| - 0\right) + \frac{1}{2}\left(\sqrt{2}-1\right)$$

$$= \frac{1}{2}\left(\ln\left|\sqrt{2}+1\right| + \sqrt{2} - 1\right) \qquad \textit{Ans.}$$

Question # 16

$$\int_0^{\pi/6} \cos^3\theta\,d\theta$$

Solution

$$\int_0^{\pi/6} \cos^3\theta\,d\theta = \int_0^{\pi/6} \cos^2\theta\cdot\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \left(1 - \sin^2\theta\right)\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \cos\theta\,d\theta - \int_0^{\pi/6} \sin^2\theta\cos\theta\,d\theta$$

$$= \Big|\sin\theta\Big|_0^{\pi/6} - \int_0^{\pi/6} \sin^2\theta\frac{d}{d\theta}\sin\theta\,d\theta$$

$$= \left(\sin\frac{\pi}{6} - \sin(0)\right) - \left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$$

$$= \left(\frac{1}{2} - 0\right) - \frac{1}{3}\left(\sin^3\frac{\pi}{6} - \sin^3(0)\right)$$

$$= \frac{1}{2} - \frac{1}{3}\left(\left(\frac{1}{2}\right)^2 - (0)^3\right)$$

$$= \frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) \quad = \frac{1}{2} - \frac{1}{24} \quad = \frac{11}{24}$$

Question # 17

$$\int_{\pi/6}^{\pi/4} \cos^2\theta\cdot\cot^2\theta\,d\theta$$

Solution

$$\int_{\pi/6}^{\pi/4} \cos^2\theta\cdot\cot^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta\left(\operatorname{cosec}^2\theta-1\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\operatorname{cosec}^2\theta-\cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\frac{1}{\sin^2\theta}-\cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cot^2\theta\, d\theta-\int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\operatorname{cosec}^2\theta-1\right) d\theta-\int_{\pi/6}^{\pi/4} \left(\frac{1+\cos\theta}{2}\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta-\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta d\theta$$

$$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4}-\frac{3}{2}\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/4}$$

$$= \left(-\cot\frac{\pi}{4}+\cot\frac{\pi}{6}\right)-\frac{3}{2}\left|\theta\right|_{\pi/6}^{\pi/4} -\frac{1}{2}\left(\frac{\sin 2\left(\pi/4\right)}{2}-\frac{\sin 2\left(\pi/6\right)}{2}\right)$$

$$= \left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}/2}{2}\right)$$

$$= \left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{12}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)$$

$$= -1+\sqrt{3}-\frac{\pi}{8}-\frac{1}{4}+\frac{\sqrt{3}}{8} \quad = -\frac{5}{4}+\frac{9}{8}\sqrt{3}-\frac{\pi}{8}$$

$$= \frac{9\sqrt{3}-10-\pi}{8}$$

Question # 18

$$\int_0^{\pi/4} \cos^4 t\, dt$$

Solution

$$\int_0^{\pi/4} \cos^4 t\, dt \quad = \int_0^{\pi/4} \left(\cos^2 t\right)^2 dt$$

$$= \int_0^{\pi/4} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \int_0^{\pi/4} \left(\frac{1+2\cos 2t+\cos^2 2t}{4}\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1+2\cos 2t+\cos^2 2t\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right) dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(\frac{2+4\cos 2t+1+\cos 4t}{2}\right) dt$$

$$= \frac{1}{8}\int_0^{\pi/4} \left(3+4\cos 2t+\cos 4t\right) dt$$

$$= \frac{1}{8}\left|3t+4\frac{\sin 2t}{2}+\frac{\sin 4t}{4}\right|_0^{\pi/4}$$

$$= \frac{1}{8}\left(3\left(\frac{\pi}{4}\right)+2\sin 2\left(\pi/4\right)+\frac{\sin 4\left(\pi/4\right)}{4} -3(0)-2\sin 2(0)-\frac{\sin 4(0)}{4}\right)$$

$$= \frac{1}{8}\left(\frac{3\pi}{4}+2+\frac{0}{4}-0-0-\frac{0}{4}\right) \quad = \frac{1}{8}\left(\frac{3\pi}{4}+2\right)$$

$$= \frac{1}{8}\left(\frac{3\pi+8}{4}\right) \quad = \frac{3\pi+8}{32}$$

Question # 19

$$\int_0^{\pi/3} \cos^2\theta\sin\theta\, d\theta$$

Solution

Let $I = \int_0^{\pi/3} \cos^2\theta\sin\theta\, d\theta$

Put $t=\cos\theta \quad \Rightarrow \quad dt=-\sin\theta\, d\theta$

$\Rightarrow \quad -dt=\sin\theta\, d\theta$

When $\theta=0$ then $t=1$

And when $\theta = \pi/3$ then $t = 1/2$

So $I = \int_{1}^{1/2} t^2(-dt)$

$= -\int_{1}^{1/2} t^2\,dt \quad = -\left|\frac{t^3}{3}\right|_{1}^{1/2}$

$= -\left(\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{(1)^3}{3}\right) \quad = -\left(\frac{1/8}{3} - \frac{1}{3}\right)$

$= -\left(\frac{1}{24} - \frac{1}{3}\right) \quad = -\left(-\frac{7}{24}\right) \quad = \frac{7}{24}$

Question # 20

$\int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\; d\theta$

Solution

$\int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\; d\theta$

$= \int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\frac{\sin^2\theta}{\cos^2\theta}\; d\theta$

$= \int_{0}^{\pi/4} \left(\frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta\right) d\theta$

$= \int_{0}^{\pi/4} \left(\tan^2\theta + \sin^2\theta\right) d\theta$

$= \int_{0}^{\pi/4} \left(\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right) d\theta$

$= \int_{0}^{\pi/4} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2}\right) d\theta$

$= \frac{1}{2}\int_{0}^{\pi/4} \left(2\sec^2\theta - 1 - \cos 2\theta\right) d\theta$

$= \frac{1}{2}\left|2\tan\theta - \theta - \frac{\sin 2\theta}{2}\right|_{0}^{\pi/4}$

$= \frac{1}{2}\left(2\tan\frac{\pi}{4} - \frac{\pi}{4} - \frac{\sin 2\left(\pi/4\right)}{2}\right.$

$\left. -2\tan(0) + 0 + \frac{\sin 2(0)}{2}\right)$

$= \frac{1}{2}\left(2(1) - \frac{\pi}{4} - \frac{1}{2} - 2(0) + 0 + 0\right)$

$= \frac{1}{2}\left(\frac{3}{2} - \frac{\pi}{4}\right) \quad = \frac{1}{2}\left(\frac{6-\pi}{4}\right) \quad = \frac{6-\pi}{8}$

Question # 21

$\int_{0}^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$

Solution

Let $I = \int_{0}^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$

$= \int_{0}^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + 1\right)}\, d\theta$

$= \int_{0}^{\pi/4} \frac{\sec^2\theta}{\left(\tan\theta + 1\right)}\, d\theta$

Put $t = \tan\theta + 1 \quad \Rightarrow \quad dt = \sec^2\theta\, d\theta$

When $x = 0$ then $t = 1$

Also when $x = \frac{\pi}{4}$ then $t = 2$

So $I = \int_{1}^{2} \frac{dt}{t}$

$= \left|\ln t\right|_{1}^{2}$

$= \ln 2 - \ln 1 \quad = \ln 2 - 0 \quad = \ln 2$

Review

If $f(x) = \begin{cases} g(x) & : \quad a \le x \le b \\ h(x) & : \quad b \le x \le c \end{cases}$

Then

$\int_{a}^{c} f(x)\,dx = \int_{a}^{b} g(x) + \int_{b}^{c} h(x)$

Question # 22

$\int_{-1}^{5} \left|x-3\right| dx$

Solution

Let $I=\int_{-1}^{5}|x-3|\,dx$

Since

$$|x-3|=\begin{cases} x-3 & \text{if } x-3\ge 0 \Rightarrow x\ge 3 \\ -(x-3) & \text{if } x-3<0 \Rightarrow x<3 \end{cases}$$

So $\int_{-1}^{5}|x-3|\,dx=\int_{-1}^{3}[-(x-3)]\,dx+\int_{3}^{5}(x-3)\,dx$

$$=-\int_{-1}^{3}(x-3)\,dx+\int_{3}^{5}(x-3)\,dx$$

$$=-\left|\frac{(x-3)^2}{2}\right|_{-1}^{3}+\left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$=-\left(\frac{(3-3)^2}{2}-\frac{(-1-3)^2}{2}\right)+\left(\frac{(5-3)^2}{2}-\frac{(3-3)^2}{2}\right)$$

$$=-\left(\frac{0}{2}-\frac{16}{2}\right)+\left(\frac{4}{2}-\frac{0}{2}\right)\quad=8+2\quad=10$$

Question # 23

$$\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$$

Solution

Let $I=\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$

$$=\int_{1/8}^{1}\left(x^{1/3}+2\right)^2x^{-2/3}\,dx$$

Put $t=x^{\frac{1}{3}}+2$

$\Rightarrow\ dt=\frac{1}{3}x^{-\frac{2}{3}}dx \quad\Rightarrow\ 3dt=x^{-\frac{2}{3}}dx$

When $x=\frac{1}{8}$ then $t=\frac{5}{2}$

And when $x=1$ then $t=3$

So $I=\int_{5/2}^{3}(t)^2\,3dt \quad =3\left|\frac{t^3}{3}\right|_{5/2}^{3}$

$$=3\left(\frac{3^3}{3}-\frac{\left(5/2\right)^3}{3}\right)\quad=3\left(\frac{27}{3}-\frac{125/8}{3}\right)$$

$$=3\left(\frac{27}{3}-\frac{125}{24}\right)\quad=3\left(\frac{91}{24}\right)\quad=\frac{91}{8}$$

Question # 24

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

Solution

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2-2 \\ & \underline{\pm x^2 \pm x} \\ & -x-2 \\ & \underline{\mp x \mp 1} \\ & -1 \end{array}$$

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

$$=\int_{1}^{3}\left(x-1-\frac{1}{x+1}\right)dx$$

$$=\int_{1}^{3}x\,dx-\int_{1}^{3}dx-\int_{1}^{3}\frac{dx}{x+1}$$

$$=\left|\frac{x^2}{2}\right|_{1}^{3}-|x|_{1}^{3}-\left|\ln|x+1|\right|_{1}^{3}$$

$$=\left(\frac{3^2}{2}-\frac{1^2}{2}\right)-(3-1)-\left(\ln|3+1|-\ln|1+1|\right)$$

$$=\left(\frac{9}{2}-\frac{1}{2}\right)-(2)-(\ln 4-\ln 2)$$

$$=4-2-\ln\frac{4}{2}\quad=2-\ln 2$$

Question # 25

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

Solution

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

$$=\int_{2}^{3}\frac{3x^2-2x+1}{x^3-x^2+x-1}\,dx$$

$$=\int_{2}^{3}\frac{\frac{d}{dx}\left(x^3-x^2+x-1\right)}{x^3-x^2+x-1}\,dx$$

$$=\left|\ln\left|x^3-x^2+x-1\right|\right|_{2}^{3}$$

$$=\ln\left|3^3-3^2+3-1\right|-\ln\left|2^3-2^2+2-1\right|$$

$$=\ln|27-9+3-1|-\ln|8-4+2-1|$$

$$=\ln 20-\ln 5\quad=\ln\frac{20}{5}\quad=\ln 4$$

Question # 26

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx$$

Solution

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx = \int_0^{\pi/4} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\pi/4} \left(\frac{\sin x}{\cos x \cdot \cos x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\pi/4} \left(\sec x \tan x - \sec^2 x \right) dx$$

$$= \left| \sec x - \tan x \right|_0^{\pi/4}$$

$$= \left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4} \right) - \left(\sec(0) - \tan(0) \right)$$

$$= \sqrt{2} - 1 - 1 + 0 \quad = \sqrt{2} - 2$$

Question # 27

$$\int_0^{\pi/4} \frac{1}{1 + \sin x}\, dx$$

Solution

Let $I = \int_0^{\pi/4} \frac{1}{1 + \sin x}\, dx$

$$= \int_0^{\pi/4} \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}\, dx$$

$$= \int_0^{\pi/4} \frac{1 - \sin x}{1 - \sin^2 x}\, dx \quad = \int_0^{\pi/4} \frac{1 - \sin x}{\cos^2 x}\, dx$$

Now same as Question # 24

Question # 28

$$\int_0^{1} \frac{3x}{\sqrt{4 - 3x}}\, dx$$

Solution

Let $I = \int_0^{1} \frac{3x}{\sqrt{4 - 3x}}\, dx$

Put $t = 4 - 3x \quad \Rightarrow \quad 3x = 4 - t$

Also $dt = -3dx \quad \Rightarrow \quad -\frac{1}{3}dt = dx$

When $x = 0$ then $t = 4$

And when $x = 1$ then $t = 1$

So $I = \int_4^{1} \frac{4 - t}{\sqrt{t}} \left(-\frac{1}{3}dt \right)$

$$= -\frac{1}{3} \int_4^{1} \left(\frac{4}{t^{1/2}} - \frac{t}{t^{1/2}} \right) dt$$

$$= +\frac{1}{3} \int_1^{4} \left(4t^{-\frac{1}{2}} - t^{\frac{1}{2}} \right) dt$$

Now do yourself

Question # 29

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x \left(2 + \sin x \right)}\, dx$$

Solution

Let $I = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x \left(2 + \sin x \right)}\, dx$

Put $t = \sin x \quad \Rightarrow \quad dt = \cos x\, dx$

When $x = \frac{\pi}{6}$ then $t = \frac{1}{2}$

When $x = \frac{\pi}{2}$ then $t = 1$

So $I = \int_{1/2}^{1} \frac{dt}{t(2 + t)}$

Now consider

$$\frac{1}{t(2 + t)} = \frac{A}{t} + \frac{B}{2 + t}$$

$$\Rightarrow \ 1 = A(2 + t) + Bt \ \ldots\ldots \text{(i)}$$

Put $t = 0$ in (i)

$$1 = A(2 + 0) + B(0) \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$

Put $2 + t = 0 \Rightarrow \ t = -2$ in eq. (i)

$$1 = 0 + B(-2) \quad \Rightarrow \quad 1 = -2B \quad \Rightarrow \quad B = -\frac{1}{2}$$

So $\frac{1}{t(2 + t)} = \frac{1/2}{t} + \frac{-1/2}{2 + t}$

$$\Rightarrow \int_{1/2}^{1} \frac{1}{t(2 + t)}\, dt = \int_{1/2}^{1} \frac{1/2}{t}\, dt + \int_{1/2}^{1} \frac{-1/2}{2 + t}\, dt$$

$$= \frac{1}{2} \int_{1/2}^{1} \frac{1}{t}\, dt - \frac{1}{2} \int_{1/2}^{1} \frac{1}{2 + t}\, dt$$

$$= \frac{1}{2} \left| \ln|t| \right|_{1/2}^{1} - \frac{1}{2} \left| \ln|2 + t| \right|_{1/2}^{1}$$

$$=\frac{1}{2}\left[\ln|1|-\ln\left|\frac{1}{2}\right|\right]$$
$$-\frac{1}{2}\left[\ln|2+1|-\ln\left|2+\frac{1}{2}\right|\right]$$
$$=\frac{1}{2}\left[0-\ln\frac{1}{2}\right]-\frac{1}{2}\left[\ln 3-\ln\frac{5}{2}\right]$$
$$=\frac{1}{2}\left[-\ln\frac{1}{2}-\ln 3+\ln\frac{5}{2}\right]$$
$$=\frac{1}{2}\ln\left(\frac{5/2}{1/2\times 3}\right) \quad =\frac{1}{2}\ln\left(\frac{5}{3}\right)$$

Question # 30

$$I=\int_0^{\pi/2}\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$$

Solution

Let $I=\int_0^{\pi/2}\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$

Put $t=\cos x \quad\Rightarrow\quad dt=-\sin x\,dx$

$\Rightarrow\ -dt=\sin x\,dx$

When $x=0$ then $t=1$

And when $x=\frac{\pi}{2}$ then $t=0$

So $I=\int_1^0\frac{-dt}{(1+t)(2+t)}$

$$=-\int_1^0\frac{dt}{(1+t)(2+t)} \quad =\int_0^1\frac{dt}{(1+t)(2+t)}$$

Now consider

$$\frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}$$

$\Rightarrow\ 1=A(2+t)+B(1+t)$ …. (i)

Put $1+t=0 \quad\Rightarrow\quad t=-1$ in (i)

$1=A(2-1)+0 \quad\Rightarrow\quad A=1$

Put $2+t=0 \quad\Rightarrow\quad t=-2$ in (i)

$1=0+B(1-2) \quad\Rightarrow\quad 1=-B$ i.e. $B=-1$

So

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}+\frac{-1}{2+t}$$

$$\int_0^1\frac{1}{(1+t)(2+t)}\,dt=\int_0^1\frac{1}{1+t}\,dt-\int_0^1\frac{1}{2+t}\,dt$$
$$=\left|\ln|1+t|\right|_0^1-\left|\ln|2+t|\right|_0^1$$
$$=\left(\ln|1+1|-\ln|1+0|\right)$$
$$-\left(\ln|2+1|-\ln|2+0|\right)$$
$$=\ln 2-0-\ln 3+\ln 2$$
$$=\ln\left(\frac{2\times 2}{3}\right) \quad =\ln\left(\frac{4}{3}\right)$$
