

### Example 4

Find the area bounded by the curve

$$f(x) = x^3 - 2x^2 + 1$$

and the x-axis in the first quadrant.

#### Solution

$$\text{Put } f(x) = 0$$

$$\Rightarrow x^3 - 2x^2 + 1 = 0$$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ \downarrow & 1 & -1 & -1 & \\ \hline 1 & -1 & -1 & \boxed{0} \end{array}$$

$$\Rightarrow (x-1)(x^2-x-1) = 0$$

$$\Rightarrow x-1=0 \quad \text{or} \quad x^2-x-1=0$$

$$\begin{aligned} \Rightarrow x = 1 &\quad \text{or} \quad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Thus the curve cuts the x-axis at  $x=1, \frac{1 \pm \sqrt{5}}{2}$

Since we are taking area in the first quad. only

$\therefore x = 1, \frac{1+\sqrt{5}}{2}$  ignoring  $\frac{1-\sqrt{5}}{2}$  as it is negative.

Intervals in 1<sup>st</sup> quad. are  $[0,1]$  &  $\left[1, \frac{1+\sqrt{5}}{2}\right]$

Since  $f(x) \geq 0$  whenever  $x \in [0,1]$

and  $f(x) \leq 0$  whenever  $x \in \left[1, \frac{1+\sqrt{5}}{2}\right]$

$$\begin{aligned} \therefore \text{Area in 1}^{\text{st}} \text{ quad.} &= \int_0^1 (x^3 - 2x^2 + 1) dx \\ &= \left| \frac{x^4}{4} - 2 \frac{x^3}{3} + x \right|_0^1 \\ &= \left( \frac{1}{2} - \frac{2}{3} + 1 \right) - 0 \\ &= \frac{7}{12} \text{ sq. unit} \end{aligned}$$

### Solution

$$y = x^2 + 1 \quad ; \quad x=1 \text{ to } x=2$$

$$\therefore y \geq 0 \text{ whenever } x \in [1,2]$$

$$\begin{aligned} \therefore \text{Area} &= \int_1^2 (x^2 + 1) dx \\ &= \int_1^2 x^2 dx + \int_1^2 1 dx \\ &= \left| \frac{x^3}{3} \right|_1^2 + \left| x \right|_1^2 \\ &= \left( \frac{(2)^3}{3} - \frac{(1)^3}{3} \right) + (2-1) \\ &= \left( \frac{8}{3} - \frac{1}{3} \right) + 1 \\ &= \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. unit.} \end{aligned}$$

### Question # 2

Find the area above the x-axis and under the curve  $y = 5 - x^2$  from  $x=-1$  to  $x=2$ .

#### Solution

$$y = 5 - x^2 \quad ; \quad x=-1 \text{ to } x=2$$

$$\therefore y > 0 \text{ whenever } x \in (-1,2)$$

$$\begin{aligned} \therefore \text{Area} &= \int_{-1}^2 (5 - x^2) dx \\ &= \left| 5x - \frac{x^3}{3} \right|_{-1}^2 \\ &= \left( 5(2) - \frac{(2)^3}{3} \right) - \left( 5(-1) - \frac{(-1)^3}{3} \right) \\ &= \left( 10 - \frac{8}{3} \right) - \left( -5 + \frac{1}{3} \right) \\ &= \frac{22}{3} - \left( -\frac{14}{3} \right) = \frac{22}{3} + \frac{14}{3} \\ &= \frac{36}{3} = 12 \text{ sq. unit} \end{aligned}$$

### Question # 1

Find the area between the x-axis and the curve

$$y = x^2 + 1 \text{ from } x=1 \text{ to } x=2.$$

### Question # 3

Find the area below the curve  $y = 3\sqrt{x}$  and above the x-axis between  $x=1$  to  $x=4$ .

**Solution**

$$y = 3\sqrt{x} ; \quad x=1 \text{ to } x=4$$

Since  $y \geq 0$  when  $x \in [1, 4]$

$$\begin{aligned}\therefore \text{Area} &= \int_1^4 3\sqrt{x} dx \\ &= \int_1^4 3x^{\frac{1}{2}} dx = 3 \int_1^4 x^{\frac{1}{2}} dx \\ &= 3 \left| \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_1^4 = 3 \left| \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^4 \\ &= 3 \times \frac{2}{3} \left| x^{\frac{3}{2}} \right|_1^4 = 2 \left( (4)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \right) \\ &= \frac{3}{4} \left( (4)^{\frac{4}{3}} - (1)^{\frac{4}{3}} \right) = 2 \left( (2^2)^{\frac{3}{2}} - 1 \right) \\ &= 2(8-1) = 14 \text{ sq. unit}\end{aligned}$$

**Question # 4**

Find the area bounded by cos function from

$$x = -\frac{\pi}{2} \text{ to } x = \frac{\pi}{2}$$

**Solution**

$$y = \cos x ; \quad x = -\frac{\pi}{2} \text{ to } x = \frac{\pi}{2}$$

$$\because y > 0 \text{ whenever } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\begin{aligned}\therefore \text{Area} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx \\ &= \left| \sin x \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \\ &= 1 + 1 = 2 \text{ sq. unit}\end{aligned}$$

**Question # 5**

Find the area between the x-axis and the curve

$$y = 4x - x^2$$

**Solution**

$$y = 4x - x^2$$

Putting  $y=0$ , we have

$$4x - x^2 = 0$$

$$\Rightarrow x(4-x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 4$$

Now  $y > 0$  when  $x \in (0, 4)$

$$\begin{aligned}\therefore \text{Area} &= \int_0^4 (4x - x^2) dx \\ &= \left| \frac{4x^2}{2} - \frac{x^3}{3} \right|_0^4 = \left| 2x^2 - \frac{x^3}{3} \right|_0^4 \\ &= \left( 2(4)^2 - \frac{(4)^3}{3} \right) - \left( 2(0)^2 - \frac{(0)^3}{3} \right) \\ &= \left( 32 - \frac{64}{3} \right) - (0-0) \\ &= \frac{32}{3} \text{ sq. unit.}\end{aligned}$$

**Question # 6**

Determine the area bounded by the parabola  $y = x^2 + 2x - 3$  and the x-axis.

**Solution**

$$y = x^2 + 2x - 3$$

Putting  $y = 0$ , we have

$$x^2 + 2x - 3 = 0$$

$$\Rightarrow x^2 + 3x - x - 2 = 0$$

$$\Rightarrow x(x+3) - 1(x+3) = 0$$

$$\Rightarrow (x+3)(x-1) = 0$$

$$\Rightarrow x = -3 \text{ or } x = 1$$

Now  $y \leq 0$  whenever  $x \in [-3, 1]$

$$\begin{aligned}\therefore \text{Area} &= - \int_{-3}^1 (x^2 + 2x - 3) dx \\ &= - \left| \frac{x^3}{3} + \frac{2x^2}{2} - 3x \right|_{-3}^1 \\ &= - \left| \frac{x^3}{3} + x^2 - 3x \right|_{-3}^1 \\ &= - \left( \frac{(1)^3}{3} + (1)^2 - 3(1) \right) \\ &\quad + \left( \frac{(-3)^3}{3} + (-3)^2 - 3(-3) \right) \\ &= - \left( \frac{1}{3} + 1 - 3 \right) + \left( \frac{-27}{3} + 9 + 9 \right) \\ &= - \left( -\frac{5}{3} \right) + (-9 + 18) \\ &= \frac{5}{3} + 9 = \frac{32}{3} \text{ sq. unit}\end{aligned}$$

**Question # 7**

Find the area bounded by the curve  $y = x^3 + 1$ , the x-axis and line  $x=2$ .

**Solution**

$$y = x^3 + 1$$

Putting  $y = 0$ , we have

$$x^3 + 1 = 0$$

$$\Rightarrow (x+1)(x^2 - x + 1) = 0$$

$$\Rightarrow x+1=0 \quad \text{or} \quad x^2 - x + 1 = 0$$

$$\Rightarrow x=-1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$\Rightarrow x = \frac{1 \pm \sqrt{-3}}{2}$$

Which is not possible.

Now  $y \geq 0$  when  $x \in [-1, 2]$

$$\therefore \text{Area} = \int_{-1}^2 (x^3 + 1) dx$$

$$= \left| \frac{x^4}{4} + x \right|_{-1}^2$$

$$= \left( \frac{(2)^4}{4} + 2 \right) - \left( \frac{(-1)^4}{4} - 1 \right)$$

$$= \left( \frac{16}{4} + 2 \right) - \left( \frac{1}{4} - 1 \right)$$

$$= 6 - \frac{3}{4} = \frac{27}{4} \text{ sq. unit}$$

**Question # 8**

Find the area bounded by the curve

$$y = x^3 - 2x + 4$$
 and the x-axis.

**Solution**

$$y = x^3 - 2x + 4 ; \quad x=1$$

Putting  $y = 0$ , we have

$$x^3 - 2x + 4 = 0$$

By synthetic division

$$\begin{array}{r} -2 \\ \hline 1 & 0 & -2 & 4 \\ \downarrow & -2 & 4 & -4 \\ \hline 1 & -2 & 2 & \underline{0} \end{array}$$

$$\Rightarrow (x+2)(x^2 - 2x + 2) = 0$$

$$\Rightarrow x+2=0 \quad \text{or} \quad x^2 - 2x + 2 = 0$$

$$\Rightarrow x = -2 \quad \text{or} \quad x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

This is imaginary.

Now  $y \geq 0$  when  $x \in [-2, 1]$

$$\therefore \text{Area} = \int_{-2}^1 (x^3 - 2x + 4) dx$$

$$= \int_{-2}^1 x^3 dx - 2 \int_{-2}^1 x dx + 4 \int_{-2}^1 dx$$

$$= \left| \frac{x^4}{4} \right|_{-2}^1 - 2 \left| \frac{x^2}{2} \right|_{-2}^1 + 4 \left| x \right|_{-2}^1$$

$$= \left( \frac{(1)^4}{4} - \frac{(-2)^4}{4} \right) - 2 \left( \frac{(1)^2}{2} - \frac{(-2)^2}{2} \right) + 4(1 - (-2))$$

$$= \left( \frac{1}{4} - \frac{16}{4} \right) - 2 \left( \frac{1}{2} - \frac{4}{2} \right) + 4(1+2)$$

$$= \left( \frac{1}{4} - 4 \right) - 2 \left( \frac{1}{2} - 2 \right) + 4(3)$$

$$= \left( -\frac{15}{4} \right) - 2 \left( -\frac{3}{2} \right) + 12$$

$$= -\frac{15}{4} + 3 + 12 = \frac{45}{4} \text{ sq. unit}$$

**Question # 9**

Find the area between the curve

**Solution**

$$y = x^3 - 4x$$

Putting  $y = 0$ , we have

$$x^3 - 4x = 0$$

$$\Rightarrow x(x^2 - 4) = 0$$

$$\Rightarrow x(x+2)(x-2) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2$$

Now  $y \geq 0$  whenever  $x \in [-2, 0]$

And  $y \leq 0$  whenever  $x \in [0, 2]$

$$\therefore \text{Area} = \int_{-2}^0 y dx - \int_0^2 y dx$$

$$\begin{aligned}
&= \int_{-2}^0 (x^3 - 4x) dx - \int_0^2 (x^3 - 4x) dx \\
&= \left| \frac{x^4}{4} - 4 \frac{x^2}{2} \right|_{-2}^0 - \left| \frac{x^4}{4} - 4 \frac{x^2}{2} \right|_0^2 \\
&= \left| \frac{x^4}{4} - 2x^2 \right|_{-2}^0 - \left| \frac{x^4}{4} - 2x^2 \right|_0^2 \\
&= \left( \frac{(0)^4}{4} - 2(0)^2 \right) - \left( \frac{(-2)^4}{4} - 2(-2)^2 \right) \\
&\quad - \left( \frac{(2)^4}{4} - 2(2)^2 \right) + \left( \frac{(0)^4}{4} - 2(0)^2 \right) \\
&= (0-0) - \left( \frac{16}{4} - 8 \right) \\
&\quad - \left( \frac{16}{4} - 8 \right) + (0-0) \\
&= -(4-8) - (4-8) = -(-4) - (-4) \\
&= 4+4 = 8 \text{ sq. unit.}
\end{aligned}$$


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**Question # 9**

Find the area between the curve

$$y = x(x-1)(x+1)$$

**Solution**

$$y = x(x-1)(x+1)$$

Putting  $y = 0$ , we have

$$x(x-1)(x+1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1$$

Now  $y \geq 0$  whenever  $x \in [-1, 0]$ And  $y \leq 0$  whenever  $x \in [0, 1]$ 

$$\begin{aligned}
\therefore \text{Area} &= \int_{-1}^0 y dx - \int_0^1 y dx \\
&= \int_{-1}^0 x(x-1)(x+1) dx \\
&\quad - \int_0^1 x(x-1)(x+1) dx \\
&= \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx \\
&= \left| \frac{x^4}{4} - \frac{x^2}{2} \right|_{-1}^0 - \left| \frac{x^4}{4} - \frac{x^2}{2} \right|_0^1
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{(0)^4}{4} - \frac{(0)^2}{2} \right) - \left( \frac{(-1)^4}{4} - \frac{(-1)^2}{2} \right) \\
&\quad - \left( \frac{(1)^4}{4} - \frac{(1)^2}{2} \right) + \left( \frac{(0)^4}{4} - \frac{(0)^2}{2} \right) \\
&= (0-0) - \left( \frac{1}{4} - \frac{1}{2} \right) \\
&\quad - \left( \frac{1}{4} - \frac{1}{2} \right) + (0-0) \\
&= 0 - \left( -\frac{1}{4} \right) - \left( -\frac{1}{4} \right) + 0 \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \text{ sq. unit}
\end{aligned}$$


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**Question # 11**

Find the area between the x-axis and the curve

$$y = \cos \frac{1}{2}x \text{ from } x = -\pi \text{ to } x = \pi$$

**Solution**

$$g(x) = \cos \frac{1}{2}x ; x = -\pi \text{ to } x = \pi$$

$$\because g(x) \geq 0 \text{ when } x \in [-\pi, \pi]$$

$$\therefore \text{Area} = \int_{-\pi}^{\pi} \cos \frac{1}{2}x dx$$

$$\begin{aligned}
&= \left| \frac{\sin \frac{x}{2}}{\frac{1}{2}} \right|_{-\pi}^{\pi} = 2 \left| \sin \frac{x}{2} \right|_{-\pi}^{\pi} \\
&= 2 \left( \sin \left( \frac{\pi}{2} \right) - \sin \left( \frac{-\pi}{2} \right) \right) \\
&= 2(1 - (-1)) = 2(1+1) \\
&= 2(2) = 4 \text{ sq. unit.}
\end{aligned}$$

**Question # 12**

Find the area between the x-axis and the curve

$$y = \sin 2x \text{ from } x = 0 \text{ to } x = \frac{\pi}{3}$$

**Solution**

$$y = \sin 2x ; x = 0 \text{ to } x = \frac{\pi}{3}$$

$$\because y \geq 0 \text{ when } x \in \left[ 0, \frac{\pi}{3} \right]$$

$$\therefore \text{Area} = \int_0^{\frac{\pi}{3}} \sin 2x dx$$

$$= \left| -\frac{\cos 2x}{2} \right|_0^{\frac{\pi}{3}} = -\frac{1}{2} \left( \cos \frac{2\pi}{3} - \cos(0) \right)$$

$$= -\frac{1}{2} \left( -\frac{1}{2} - 1 \right) = -\frac{1}{2} \left( -\frac{3}{2} \right) = \frac{3}{4} \text{ sq. unit.}$$

**Question # 13**

Find the area between the x-axis and the curve  $y = \sqrt{2ax - x^2}$  when  $a > 0$

**Solution**

$$y = \sqrt{2ax - x^2}$$

Putting  $y = 0$ , we have

$$\sqrt{2ax - x^2} = 0$$

On squaring

$$2ax - x^2 = 0$$

$$\Rightarrow x(2a - x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad 2a - x = 0 \Rightarrow x = 2a$$

$$\because y \geq 0 \quad \text{when } x \in [0, 2a]$$

$$\begin{aligned} \therefore \text{Area} &= \int_0^{2a} \sqrt{2ax - x^2} \, dx \\ &= \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2} \, dx \\ &= \int_0^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)} \, dx \\ &= \int_0^{2a} \sqrt{a^2 - (a-x)^2} \, dx \end{aligned}$$

$$\text{Put } a-x = a \sin \theta$$

$$\Rightarrow -dx = a \cos \theta \, d\theta$$

$$\Rightarrow dx = -a \cos \theta \, d\theta$$

When  $x = 0$

$$a - 0 = a \sin \theta \Rightarrow a \sin \theta = a$$

$$\Rightarrow \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

When  $x = 2a$

$$a - 2a = a \sin \theta \Rightarrow -a = a \sin \theta$$

$$\Rightarrow -1 = \sin \theta \Rightarrow \theta = -\frac{\pi}{2}$$

$$\text{So area} = \int_{\pi/2}^{-\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} (-a \cos \theta \, d\theta)$$

$$= -a \int_{\pi/2}^{-\pi/2} \sqrt{a^2 (1 - \sin^2 \theta)} \cos \theta \, d\theta$$

$$= -a \int_{\pi/2}^{-\pi/2} \sqrt{a^2 \cos^2 \theta} \cos \theta \, d\theta$$

$$= -a \int_{\pi/2}^{-\pi/2} a \cos \theta \cdot \cos \theta \, d\theta$$

$$= -a^2 \int_{\pi/2}^{-\pi/2} \cos^2 \theta \, d\theta$$

$$= -a^2 \int_{\pi/2}^{-\pi/2} \left( \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= -\frac{a^2}{2} \int_{\pi/2}^{-\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= -\frac{a^2}{2} \left| \theta + \frac{\sin 2\theta}{2} \right|_{\pi/2}^{-\pi/2}$$

$$= -\frac{a^2}{2} \left( -\frac{\pi}{2} + \sin(-\pi) - \frac{\pi}{2} - \sin \pi \right)$$

$$= -\frac{a^2}{2} (-\pi - 0 - 0)$$

$$= -\frac{a^2}{2} (-\pi) = \frac{a^2 \pi}{2} \text{ sq. unit}$$