Question # 1

Check each of the following equations written against the differential equation is its solution.

(i) $x\dfrac{dy}{dx}=1+y \;,\quad y=cx-1$

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0 \;,\quad y^2+y=c-\dfrac{1}{x}$

(iii) $y\dfrac{dy}{dx}-e^{2x}=1, \quad y^2 = 2x+e^{2x}+c$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0, \quad y = ce^{x^2}$

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}}, \quad y = Tan\left(e^x+c\right)$

Solution

(i) $x\dfrac{dy}{dx}=1+y$

$\Rightarrow x\,dy=(1+y)\,dx \Rightarrow \dfrac{dy}{1+y}=\dfrac{dx}{x}$

Integrating both sides

$$\int\frac{dy}{1+y}=\int\frac{dx}{x}$$

$\Rightarrow \ln(1+y)=\ln x+\ln c$

$=\ln cx$

$\Rightarrow 1+y=cx$

$\Rightarrow y=cx-1$ *Proved*

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0$

$\Rightarrow x^2(2y+1)\dfrac{dy}{dx}=1 \Rightarrow x^2(2y+1)\,dy=dx$

$\Rightarrow (2y+1)\,dy=\dfrac{1}{x^2}dx$

On integrating

$$\int(2y+1)\,dy=\int\frac{1}{x^2}dx$$

$\Rightarrow 2\int y\,dy+\int dy=\int x^{-2}\,dx$

$\Rightarrow 2\cdot\dfrac{y^2}{2}+y=\dfrac{x^{-2+1}}{-2+1}+c$

$\Rightarrow y^2+y=\dfrac{x^{-1}}{-1}+c$

$\Rightarrow y^2+y=c-\dfrac{1}{x}$ *Proved*

(iii) $y\dfrac{dy}{dx}-e^{2x}=1$

$\Rightarrow y\dfrac{dy}{dx} = 1+e^{2x} \Rightarrow ydy = \left(1+e^{2x}\right)dx$

On integrating

$$\int ydy = \int\left(1+e^{2x}\right)dx$$

$\Rightarrow \dfrac{y^2}{2} = x+\dfrac{e^{2x}}{2}+\dfrac{c}{2} \Rightarrow y^2 = 2x+e^{2x}+c$

$\Rightarrow y^2 = 2x+e^{2x}+c$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0$

$\Rightarrow \dfrac{1}{x}\dfrac{dy}{dx} = 2y \Rightarrow \dfrac{dy}{dx} = 2xy$

$\Rightarrow \dfrac{dy}{y} = 2xdx$

On integrating

$$\int\frac{dy}{y} = 2\int xdx$$

$\Rightarrow \ln y = 2\cdot\dfrac{x^2}{2}+\ln c$

$= x^2+\ln c$

$= x^2\ln e+\ln c \qquad \because \ln e=1$

$= \ln e^{x^2}+\ln c$

$\Rightarrow \ln y = \ln ce^{x^2}$

$\Rightarrow y = ce^{x^2}$ *Proved*

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}} \Rightarrow \dfrac{dy}{y^2+1} = e^x dx$

Integrating both sides

$\Rightarrow \displaystyle\int\frac{dy}{y^2+1} = \int e^x dx$

$\Rightarrow Tan^{-1}y = e^x+c$

$\Rightarrow y = Tan\left(e^x+c\right)$

Solve the following differential equations:

Question # 2

$$\frac{dy}{dx} = -y$$

Solution

$\dfrac{dy}{dx} = -y \Rightarrow \dfrac{dy}{y} = -dx$

On integrating

$$\int\frac{dy}{y} = -\int dx$$

$$\ln y = -x + \ln c$$
$$= -x\ln e + \ln c \qquad \because \ln e = 1$$
$$= \ln e^{-x} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-x} \quad \Rightarrow y = ce^{-x}$$

Question # 3

$$ydx + xdy = 0$$

Solution

$$ydx + xdy = 0 \quad \Rightarrow ydx = -xdy$$
$$\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

On integrating

$$\ln x = -\ln y + \ln c$$
$$\Rightarrow \ln x = \ln\frac{c}{y}$$
$$\Rightarrow x = \frac{c}{y} \quad \Rightarrow xy = c$$

Question # 4

$$\frac{dy}{dx} = \frac{1-x}{y}$$

Solution *Do yourself*

Question # 5

$$\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$$

Solution

$$\frac{dy}{dx} = \frac{y}{x^2} \quad \Rightarrow \frac{dy}{y} = x^{-2}dx$$

Integrating

$$\int\frac{dy}{y} = \int x^{-2}dx$$
$$\Rightarrow \ln y = \frac{x^{-2+1}}{-2+1} + \ln c$$
$$\Rightarrow \ln y = \frac{x^{-1}}{-1} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x}\ln e + \ln c$$
$$= \ln e^{-\frac{1}{x}} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-\frac{1}{x}} \quad \Rightarrow y = ce^{-\frac{1}{x}}$$

Question # 6

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$

Solution

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$
$$\Rightarrow \sin y\, dy = \frac{dx}{\operatorname{cosec} x}$$
$$\Rightarrow \sin y\, dy = \sin x\, dx$$

Integrating

$$\int \sin y\, dy = \int \sin x\, dx$$
$$\Rightarrow -\cos y = -\cos x - c$$
$$\Rightarrow \cos y = \cos x + c$$

Question # 7

$$xdy + y(x-1)dx = 0$$

Solution

$$xdy + y(x-1)dx = 0$$
$$\Rightarrow xdy = -y(x-1)dx$$
$$\Rightarrow \frac{dy}{y} = -\frac{x-1}{x}dx$$
$$\Rightarrow \frac{dy}{y} = -\left(\frac{x}{x} - \frac{1}{x}\right)dx$$
$$\Rightarrow \frac{dy}{y} = -\left(1 - \frac{1}{x}\right)dx$$

On integrating

$$\int\frac{dy}{y} = -\int\left(1 - \frac{1}{x}\right)dx$$
$$\Rightarrow \ln y = -x + \ln x + \ln c$$
$$= -x\ln e + \ln x + \ln c$$
$$= \ln e^{-x} + \ln x + \ln c$$
$$\Rightarrow \ln y = \ln cxe^{-x} \Rightarrow y = cxe^{-x}$$

Question # 8

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}, (x, y > 0)$$

Solution

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$
$$\Rightarrow \frac{x^2+1}{x}dx = \frac{y+1}{y}dy$$

On integrating

$$\int\frac{x^2+1}{x}dx = \int\frac{y+1}{y}dy$$
$$\Rightarrow \int\left(\frac{x^2}{x} + \frac{1}{x}\right)dx = \int\left(\frac{y}{y} + \frac{1}{y}\right)dy$$

$$\Rightarrow \int\left(x+\frac{1}{x}\right)dx = \int\left(1+\frac{1}{y}\right)dy$$

$$\Rightarrow \int x\,dx+\int\frac{1}{x}dx = \int dy+\int\frac{1}{y}dy$$

$$\Rightarrow \frac{x^2}{2}+\ln x = y+\ln y-\ln c$$

$$\Rightarrow \frac{x^2}{2}\ln e+\ln x+\ln c = y\ln e+\ln y$$

$$\Rightarrow \ln e^{\frac{x^2}{2}}+\ln x+\ln c = \ln e^y+\ln y$$

$$\Rightarrow \ln cxe^{\frac{x^2}{2}} = \ln ye^y$$

$$\Rightarrow cxe^{\frac{x^2}{2}} = ye^y \quad \text{i.e. } ye^y = cxe^{\frac{x^2}{2}}$$

Question # 9

$$\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\left(1+y^2\right)$$

Solution *Do yourself*

Question # 10

$$2x^2y\frac{dy}{dx}=x^2-1$$

Solution *Do yourself*

Question # 11

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

Solution

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

$$\Rightarrow \frac{dy}{dx} = x-\frac{2xy}{2y+1}$$

$$= x\left(1-\frac{2y}{2y+1}\right)$$

$$= x\left(\frac{2y+1-2y}{2y+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = x\left(\frac{1}{2y+1}\right) \Rightarrow (2y+1)dy = x\,dx$$

Now do yourself

Question # 12

$$\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2 = 0$$

Solution

$$\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2 = 0$$

$$\Rightarrow \left(x^2-yx^2\right)\frac{dy}{dx} = -y^2-xy^2$$

$$\Rightarrow x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$\Rightarrow \frac{1-y}{y^2}dy = -\frac{1+x}{x^2}dx$$

Now do yourself

Question # 13

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

Solution

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

$$\Rightarrow \sec^2 x\tan y\,dx = -\sec^2 y\tan x\,dy$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$$

On integrating

$$\int\frac{\sec^2 x}{\tan x}dx = -\int\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \int\frac{\frac{d}{dx}(\tan x)}{\tan x}dx = -\int\frac{\frac{d}{dy}(\tan y)}{\tan y}dy$$

$$\Rightarrow \ln\tan x = -\ln\tan y+\ln c$$

$$\Rightarrow \ln\tan x+\ln\tan y = \ln c$$

$$\Rightarrow \ln(\tan x\tan y) = \ln c$$

$$\Rightarrow \tan x\tan y = c$$

Question # 14

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

Solution

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

$$\Rightarrow y-x\frac{dy}{dx} = 2y^2+2\frac{dy}{dx}$$

$$\Rightarrow y-2y^2 = 2\frac{dy}{dx}+x\frac{dy}{dx}$$

$$\Rightarrow y(1-2y) = (2+x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{2+x} = \frac{dy}{y(1-2y)}$$

On integrating

$$\int\frac{dx}{2+x} = \int\frac{dy}{y(1-2y)} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now consider

$$\frac{1}{y(1-2y)} = \frac{A}{y}+\frac{B}{1-2y}$$

$\Rightarrow\ 1 = A(1-2y)+By$ .... (ii)

Put $y=0$ in (ii)

$$1 = A(1-2(0))+0 \Rightarrow A=1$$

Put $1-2y=0 \Rightarrow 2y=1 \Rightarrow y=\frac{1}{2}$ in (ii)

$$1 = 0+B\left(\frac{1}{2}\right) \Rightarrow B=2$$

So $\dfrac{1}{y(1-2y)} = \dfrac{1}{y}+\dfrac{2}{1-2y}$

Using in (i) $\leftarrow *$

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dy$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{-2}{1-2y}dy$$

$$\Rightarrow \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(1-2y)}{1-2y}dy$$

$$\Rightarrow \ln(2+x) = \ln y-\ln(1-2y)-\ln c$$

$$\Rightarrow \ln(2+x)+\ln c = \ln y-\ln(1-2y)$$

$$\Rightarrow \ln c(2+x) = \ln\frac{y}{(1-2y)}$$

$$\Rightarrow c(2+x) = \frac{y}{(1-2y)}$$

$$\Rightarrow y = c(2+x)(1-2y)$$

Alternative $(\leftarrow *)$

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dx$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{2}{2y-1}dy$$

$$\Rightarrow \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(2y-1)}{2y-1}dy$$

$$\Rightarrow \ln(2+x) = \ln y-\ln(2y-1)-\ln c$$

$$\Rightarrow \ln(2+x)+\ln c = \ln y-\ln(2y-1)$$

$$\Rightarrow \ln c(2+x) = \ln\frac{y}{(2y-1)}$$

$$\Rightarrow c(2+x) = \frac{y}{(2y-1)}$$

i.e. $\dfrac{y}{(2y-1)} = c(2+x)$

Review

- $\int\tan x\,dx = \ln|\sec x| = -\ln|\cos x|$
- $\int\cot x\,dx = \ln|\sin x| = -\ln|\csc x|$
- $\int\sec x\,dx = \ln|\sec x+\tan x|$
- $\int\csc x\,dx = \ln|\csc x-\cot x|$

Question # 15

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

Solution

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

$$\Rightarrow \cos x\tan y\frac{dy}{dx} = -1$$

$$\Rightarrow \tan y\,dy = -\frac{1}{\cos x}dx$$

$$\Rightarrow \tan y\,dy = -\sec x\,dx$$

On integrating

$$\int\tan y\,dy = -\int\sec x\,dx$$

$$\Rightarrow -\ln|\cos y| = -\ln|\sec x+\tan x|-\ln c$$

$$\Rightarrow \ln|\cos y| = +\ln|\sec x+\tan x|+\ln c$$

$$\Rightarrow \ln|\cos y| = \ln|c(\sec x+\tan x)|$$

$$\Rightarrow \cos y = c(\sec x+\tan x)$$

Question # 16

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

Solution

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

$$\Rightarrow y-x\frac{dy}{dx} = 3+3x\frac{dy}{dx}$$

$$\Rightarrow y-3 = 3x\frac{dy}{dx}+x\frac{dy}{dx}$$

$$= (3x+x)\frac{dy}{dx}$$

$$\Rightarrow y-3 = 4x\frac{dy}{dx} \quad\Rightarrow \frac{dx}{x} = 4\frac{dy}{y-3}$$

Now do yourself

Question # 17

$$\sec x + \tan y \frac{dy}{dx} = 0$$

Solution

$$\sec x + \tan y \frac{dy}{dx} = 0$$

$$\Rightarrow \tan y \frac{dy}{dx} = -\sec x$$

$$\Rightarrow \tan y\, dy = -\sec x\, dx$$

Now do yourself as Question # 15

Question # 18

$$\left(e^x + e^{-x}\right)\frac{dy}{dx} = e^x - e^{-x}$$

Solution

$$\left(e^x + e^{-x}\right)\frac{dy}{dx} = e^x - e^{-x}$$

$$\Rightarrow dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

On integrating

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\Rightarrow y = \int \frac{\frac{d}{dx}\left(e^x + e^{-x}\right)}{e^x + e^{-x}} dx$$

$$\Rightarrow y = \ln\left(e^x + e^{-x}\right) + c$$

Question # 19

Find the general solution of the equation $\frac{dy}{dx} - x = xy^2$. Also find the perpendicular solution if $y = 1$ when $x = 0$.

Solution

$$\frac{dy}{dx} - x = xy^2 \quad \Rightarrow \frac{dy}{dx} = x + xy^2$$

$$\Rightarrow \frac{dy}{dx} = x\left(1 + y^2\right) \quad \Rightarrow \frac{dy}{1 + y^2} = x\,dx$$

$$\Rightarrow \int \frac{dy}{1 + y^2} = \int x\,dx$$

$$\Rightarrow Tan^{-1} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = Tan\left(\frac{x^2}{2} + c\right)$$

Question # 20

Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x = 4$ when $t = 0$

Solution

$$\frac{dx}{dt} = 2x \quad \Rightarrow \frac{dx}{x} = 2dt$$

$$\Rightarrow \int \frac{dx}{x} = 2\int dt$$

$$\Rightarrow \ln x = 2t + \ln c$$

$$= \ln e^{2t} + \ln c \qquad \because \ln e^x = x$$

$$\Rightarrow \ln x = \ln ce^{2t}$$

$$\Rightarrow x = ce^{2t} \quad \ldots\ldots \text{(i)}$$

When $t = 0$ then $x = 4$, putting in (i)

$$4 = ce^{2(0)} \quad \Rightarrow 4 = c\,e^0$$

$$\Rightarrow 4 = c(1) \quad \Rightarrow c = 4$$

Putting in (i)

$$\Rightarrow x = 4e^{2t}$$

Question # 21

Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the perpendicular solution if $s = 4e$, when $t = 0$

Solution

$$\frac{ds}{dt} + 2st = 0$$

$$\Rightarrow \frac{ds}{dt} = -2st \quad \Rightarrow \frac{ds}{s} = -2t\,dt$$

On integrating

$$\int \frac{ds}{s} = -2\int t\,dt$$

$$\Rightarrow \ln s = -2\frac{t^2}{2} + \ln c$$

$$= -t^2 + \ln c$$

$$= \ln e^{-t^2} + \ln c \qquad \because \ln e^x = x$$

$$\Rightarrow \ln s = \ln ce^{-t^2}$$

$$\Rightarrow s = ce^{-t^2} \quad \ldots\ldots \text{(i)}$$

When $t = 0$ then $s = 4e$, using in (i)

$$4e = ce^{-(0)^2} \quad \Rightarrow 4e = c(1)$$

$$\Rightarrow c = 4e$$

Putting in (i)

$$s = 4e \cdot e^{-t^2}$$

$$\Rightarrow s = 4e^{1-t^2}$$

Question # 22
In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution

Number of bacteria initially = 200
No. of bacteria after two hours = 2(200)
= 400
No. of bacteria after four hours = 2(400)
= 800 *Ans.*

Question # 23
A ball is thrown vertically upward with a velocity of $2450 cm/\sec$. Neglecting air resistance, find
(i) velocity of ball at any time t
(ii) distance travelled in any time t
(iii) maximum height attained by the ball.

Solution

i) When a body is projected upward its acceleration is $-g$. (where $g = 980$ cm/sec^2)

i.e. acceleration = $\dfrac{dv}{dt} = -g$,
where v is velocity of ball.

$$\Rightarrow \frac{dv}{dt} = -980$$
$$\Rightarrow dv = -980\,dt$$

On integrating

$$\int dv = -980\int dt$$
$$\Rightarrow v = -980t + c_1 \quad \text{........ (i)}$$

Initially, when $t = 0$ then $v = 2450$ cm/sec

$$2450 = -980(0) + c_1$$
$$\Rightarrow c_1 = 2450$$

Putting in (i)

$$\boxed{v = -980t + 2450}$$

ii) Since velocity $= v = \dfrac{dx}{dt}$
where x is height of ball.

$$\Rightarrow \frac{dx}{dt} = -980t + 2450$$
$$\Rightarrow dx = (-980t + 2450)\,dt$$

Integrating

$$\int dx = \int(-980t + 2450)\,dt$$
$$\Rightarrow x = -980\frac{t^2}{2} + 2450t + c_2$$
$$\Rightarrow x = -490t^2 + 2450t + c_2 \quad \text{....... (ii)}$$

Initially, when $t = 0$ then $x = 0$

$$0 = -490\,(0) + 2450(0) + c_2$$
$$\Rightarrow c_2 = 0$$

Putting value of c_2 in (ii)

$$\Rightarrow x = -490t^2 + 2450t + 0$$
$$\Rightarrow \boxed{x = 2450t - 490t^2}$$

iii)

$$\because v = -980t + 2450$$

When body is at max. height then $v = 0$

$$\Rightarrow -980t + 2450 = 0$$
$$\Rightarrow 980t = 2450 \quad \Rightarrow t = \frac{2450}{980}$$
$$\Rightarrow t = 2.5 \text{ sec}$$

Since $x = 2450t - 4980t^2$

When $t = 2.5$ sec

$$x = 2450(2.5) - 490(2.5)^2$$
$$= 6125 - 3062.5$$
$$= 3062.5$$

Hence ball attains max. height of 3062.5 cm.
