

Homogenous 2nd Degree Equation

Every homogenous second degree equation

$$ax^2 + 2hxy + by^2 = 0$$

represents straight lines through the origin.

Consider the equations are $y = m_1x$ and $y = m_2x$

$$\Rightarrow m_1x - y = 0 \quad \text{and} \quad m_2x - y = 0$$

Taking product

$$(m_1x - y)(m_2x - y) = 0$$

$$\Rightarrow m_1m_2x^2 - m_1xy - m_2xy + y^2 = 0$$

$$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0 \dots\dots\dots (i)$$

Also we have

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \quad \div \text{ing by } b$$

$$\Rightarrow \frac{a}{b}x^2 - \left(-\frac{2h}{b}\right)xy + y^2 = 0$$

Comparing it with (i), we have

$$\boxed{m_1m_2 = \frac{a}{b}} \quad \text{and} \quad \boxed{m_1 + m_2 = -\frac{2h}{b}}$$

Let θ be the angles between the lines then

$$\begin{aligned} \tan \theta &= \frac{m_1 - m_2}{1 + m_1m_2} \\ &= \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1m_2} = \frac{\sqrt{m_1^2 + m_2^2 - 2m_1m_2}}{1 + m_1m_2} \\ &= \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 - 4m_1m_2}}{1 + m_1m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} \\ &= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{b+a}{b}} \\ &\Rightarrow \tan \theta = \frac{\sqrt{4(h^2 - ab)}}{b+a} \Rightarrow \boxed{\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}} \end{aligned}$$

Find the lines represented by each of the following and also find measure of the angle between them (problem 1-6)

Question # 1

$$10x^2 - 23xy - 5y^2 = 0$$

Solution

$$\begin{aligned} 10x^2 - 23xy - 5y^2 &= 0 \dots\dots\dots (i) \\ \Rightarrow 10x^2 - 25xy + 2xy - 5y^2 &= 0 \\ \Rightarrow 5x(2x - 5y) + y(2x - 5y) &= 0 \Rightarrow (2x - 5y)(5x + y) = 0 \\ \Rightarrow 2x - 5y = 0 \quad \text{and} \quad 5x + y = 0 \\ &\text{are the required lines.} \end{aligned}$$

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\text{So } a=10 \quad , \quad 2h=-23 \Rightarrow h=-\frac{23}{2} \quad , \quad b=-5$$

Let θ be angle between lines then

$$\begin{aligned} \tan \theta &= \frac{2\sqrt{h^2 - ab}}{a + b} \\ &= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10 - 5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5} \\ &= \frac{2\sqrt{729/4}}{5} = \frac{2(27/2)}{5} = \frac{27}{5} \\ \Rightarrow \theta &= \tan^{-1}\left(\frac{27}{5}\right) = 79^\circ 31' \end{aligned}$$

Hence acute angle between the lines = $79^\circ 31'$

Question # 2

$$3x^2 + 7xy + 2y^2 = 0$$

Solution *Do yourself as above*

Question # 3

$$9x^2 + 24xy + 16y^2 = 0$$

Solution *Do yourself as above*

Question # 4

$$2x^2 + 3xy - 5y^2 = 0$$

$$\begin{aligned} \text{Solution } 2x^2 + 3xy - 5y^2 &= 0 \dots\dots\dots (i) \\ \Rightarrow 2x^2 + 5xy - 2xy - 5y^2 &= 0 \\ \Rightarrow x(2x + 5y) - y(2x + 5y) &= 0 \\ \Rightarrow (2x + 5y)(x - y) &= 0 \\ \Rightarrow 2x + 5y = 0 \quad \text{and} \quad x - y &= 0 \end{aligned}$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow a=2, \quad 2h=3 \Rightarrow h=\frac{3}{2}, \quad b=-5$$

Let θ be angle between lines then

$$\begin{aligned} \tan \theta &= \frac{2\sqrt{h^2 - ab}}{a+b} \\ &= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5} = \frac{2\sqrt{\frac{9}{4} + 10}}{-3} \\ &= -\frac{2\sqrt{49/4}}{3} = -\frac{2(7/2)}{3} = -\frac{7}{3} \end{aligned}$$

$$\Rightarrow -\tan \theta = \frac{7}{3}$$

$$\Rightarrow \tan(180 - \theta) = \frac{7}{3} \quad \because \tan(180 - \theta) = -\tan \theta$$

$$\Rightarrow 180 - \theta = \tan^{-1}\left(\frac{7}{3}\right) \Rightarrow 180 - \theta = 66^\circ 48'$$

$$\Rightarrow \theta = 180 - 66^\circ 48' = 113^\circ 12'$$

$$\text{Hence acute angle between the lines} = 180 - 113^\circ 12' = 66^\circ 48'$$

Question # 5

$$6x^2 - 19xy + 15y^2 = 0$$

Solution

Do yourself as above

Question # 6

$$x^2 - 2xy \sec \alpha + y^2 = 0$$

Solution

$$x^2 - 2xy \sec \alpha + y^2 = 0 \dots\dots\dots (i)$$

÷ing by y^2

$$\begin{aligned} \frac{x^2}{y^2} - \frac{2xy \sec \alpha}{y^2} + \frac{y^2}{y^2} &= 0 \\ \Rightarrow \left(\frac{x}{y}\right)^2 - 2 \sec \alpha \left(\frac{x}{y}\right) + 1 &= 0 \end{aligned}$$

This is quadric equation in $\frac{x}{y}$ with $a=1$, $b=-2 \sec \alpha$, $c=1$

$$\frac{x}{y} = \frac{2 \sec \alpha \pm \sqrt{(-2 \sec \alpha)^2 - 4(1)(1)}}{2(1)}$$

$$\begin{aligned}
&= \frac{2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2(1)} = \frac{2\sec\alpha \pm \sqrt{4(\sec^2\alpha - 1)}}{2} \\
&= \frac{2\sec\alpha \pm \sqrt{4\tan^2\alpha}}{2} \quad \because 1 + \tan^2\alpha = \sec^2\alpha \\
&= \frac{2\sec\alpha \pm 2\tan\alpha}{2} \\
\Rightarrow \frac{x}{y} &= \sec\alpha \pm \tan\alpha \\
&= \frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha} = \frac{1 \pm \sin\alpha}{\cos\alpha} \\
\Rightarrow \frac{x}{y} &= \frac{1 + \sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{x}{y} = \frac{1 - \sin\alpha}{\cos\alpha} \\
\Rightarrow x\cos\alpha &= (1 + \sin\alpha)y \quad \text{and} \quad x\cos\alpha = (1 - \sin\alpha)y \\
\Rightarrow x\cos\alpha - (1 + \sin\alpha)y &= 0 \quad \text{and} \quad x\cos\alpha - (1 - \sin\alpha)y = 0
\end{aligned}$$

These are required equations of lines.

Now comparing (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$a=1, \quad 2h=-2\sec\alpha \Rightarrow h=-\sec\alpha, \quad b=1$$

If θ is angle between lines then

$$\begin{aligned}
\tan\theta &= \frac{2\sqrt{h^2 - ab}}{a+b} \\
&= \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1+1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha} \\
\Rightarrow \tan\theta &= \tan\alpha \Rightarrow \theta = \alpha.
\end{aligned}$$

Note:

If one wish to get the answer similar to given at the end of textbook, then follow the solution as follows after getting:

$$x\cos\alpha - (1 + \sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (1 - \sin\alpha)y = 0$$

Multiplying equation at left with $\frac{1 - \sin\alpha}{\cos\alpha}$ and equation at right with $\frac{1 + \sin\alpha}{\cos\alpha}$ to get

$$\begin{aligned}
x(1 - \sin\alpha) - \frac{(1 - \sin^2\alpha)}{\cos\alpha}y &= 0 \quad \text{and} \quad x(1 + \sin\alpha) - \frac{(1 - \sin^2\alpha)}{\cos\alpha}y = 0 \\
\Rightarrow x(1 - \sin\alpha) - y\cos\alpha &= 0 \quad \text{and} \quad x(1 + \sin\alpha) - y\cos\alpha = 0
\end{aligned}$$

Question # 7

Find a joint equation of the lines through the origin and perpendicular to the lines:

$$x^2 - 2xy\tan\alpha - y^2 = 0$$

Solution Given: $x^2 - 2xy\tan\alpha - y^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$\begin{aligned}
 m_1 + m_2 &= -\frac{2h}{b} \\
 &= -\frac{-2 \tan \alpha}{-1} \\
 \Rightarrow m_1 + m_2 &= -2 \tan \alpha \\
 \& \quad m_1 m_2 &= \frac{a}{b} = \frac{1}{-1} \Rightarrow m_1 m_2 = -1
 \end{aligned}
 \quad \left| \quad \begin{aligned}
 a &= 1, \\
 2h &= -2 \tan \alpha \\
 \Rightarrow h &= -\tan \alpha \\
 b &= -1
 \end{aligned}
 \right.$$

Now slopes of lines \perp ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$\begin{aligned}
 y &= -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \quad (\text{Passing through origin}) \\
 \Rightarrow m_1 y &= -x \quad \& \quad m_2 y = -x \\
 \Rightarrow x + m_1 y &= 0 \quad \& \quad x + m_2 y = 0
 \end{aligned}$$

Their joint equation:

$$\begin{aligned}
 (x + m_1 y)(x + m_2 y) &= 0 \\
 \Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 &= 0 \\
 \Rightarrow x^2 + (-2 \tan \alpha)xy + (-1)y^2 &= 0 \\
 \Rightarrow x^2 - 2xy \tan \alpha - y^2 &= 0
 \end{aligned}$$

Question # 8

Find a joint equation of the lines through the origin and perpendicular to the lines:

$$ax^2 + 2hxy + by^2 = 0$$

Solution

Given: $ax^2 + 2hxy + by^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1 m_2 = \frac{a}{b}$$

Now slopes of lines \perp ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$\begin{aligned}
 y &= -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \quad (\text{Passing through origin}) \\
 \Rightarrow m_1 y &= -x \quad \& \quad m_2 y = -x \\
 \Rightarrow x + m_1 y &= 0 \quad \& \quad x + m_2 y = 0
 \end{aligned}$$

Their joint equation:

$$\begin{aligned}
 (x + m_1 y)(x + m_2 y) &= 0 \\
 \Rightarrow x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 &= 0 \\
 \Rightarrow x^2 + \left(-\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)y^2 &= 0 \\
 \Rightarrow bx^2 - 2hxy + ay^2 &= 0
 \end{aligned}$$

Question # 9

Find the area of the region bounded by:

$$10x^2 - xy - 21y^2 = 0 \quad \text{and} \quad x + y + 1 = 0$$

Solution $10x^2 - xy - 21y^2 = 0 \quad , \quad x + y + 1 = 0$

$$\Rightarrow 10x^2 - 15xy + 14xy - 21y^2 = 0$$

$$\Rightarrow 5x(2x - 3y) + 7y(2x - 3y) = 0$$

$$\Rightarrow (2x - 3y)(5x + 7y) = 0$$

$$\Rightarrow 2x - 3y = 0 \quad \text{or} \quad 5x + 7y = 0$$

So we have equation of lines

$$l_1: 2x - 3y = 0 \dots\dots\dots (i)$$

$$l_2: 5x + 7y = 0 \dots\dots\dots (ii)$$

$$l_3: x + y + 1 = 0 \dots\dots\dots (iii)$$

Now do yourself as Q # 14 (Ex. 4.4)
