

Circle

The set of all point in the plane that are equally distant from a fixed point is called a *circle*.

The fixed point is called *centre* of the circle and the distance from the centre of the circle to any point on the circle is called the *radius* of circle.

Equation of Circle

Let r be radius and $C(h, k)$ be centre of circle. Let $P(x, y)$ be any point on circle then

$$|PC| = r$$

$$\Rightarrow \sqrt{(x-h)^2 + (y-k)^2} = r$$

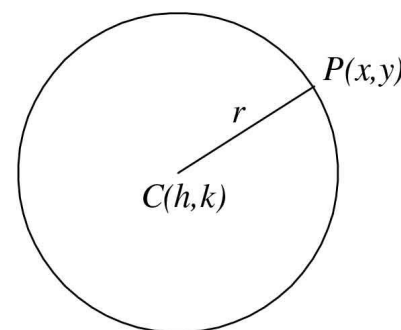
On squaring

$$(x-h)^2 + (y-k)^2 = r^2$$

This is equation of circle in standard form.

If centre of circle is at origin i.e. $C(h, k) = C(0, 0)$ then equation of circle becomes

$$x^2 + y^2 = r^2$$



Equation of circle with end points of diameter

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be end points of diameter.

Let $P(x, y)$ be any point on circle then

$$m\angle APB = 90^\circ$$

(Note: An angle in a semi circle is a right angle – see Theorem 4 at page 270)

Thus the line AP and BP are \perp ar to each other and we have

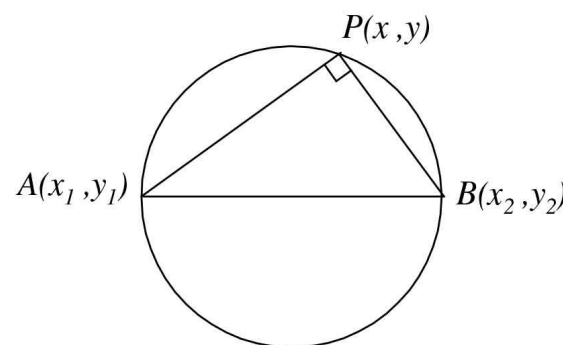
$$(\text{Slope of } AP)(\text{Slope of } BP) = -1$$

$$\Rightarrow \left(\frac{y - y_1}{x - x_1} \right) \left(\frac{y - y_2}{x - x_2} \right) = -1$$

$$\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

is the required equation of circle with end points of diameter $A(x_1, y_1)$ & $B(x_2, y_2)$.



General form of an equation of a circle

The equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

represents a circle.

$$\Rightarrow x^2 + 2gx + g^2 + y^2 + 2fy + f^2 + c = g^2 + f^2$$

$$\Rightarrow (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

$$\Rightarrow (x - (-g))^2 + (y - (-f))^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

This is equation of circle in standard form with

$$\text{centre at } (-g, -f) \text{ and radius } = \sqrt{g^2 + f^2 - c}$$

Question # 1

In each of the following, find an equation of the circle with

- (a) centre $(5, -2)$ and radius 4 (b) centre $(\sqrt{2}, -3\sqrt{3})$ and radius $2\sqrt{2}$
 (c) ends of a diameter at $(-3, 2)$ and $(5, -6)$.

Solution

- (a) Given: centre $C(h, k) = (5, -2)$, radius $= r = 4$

Equation of circle:

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ \Rightarrow (x-5)^2 + (y+2)^2 &= (4)^2 \\ \Rightarrow x^2 - 10x + 25 + y^2 + 4y + 4 &= 16 \\ \Rightarrow x^2 + y^2 - 10x + 4y + 25 + 4 - 16 &= 0 \\ \Rightarrow x^2 + y^2 - 10x + 4y + 13 &= 0 \end{aligned}$$

- (b) Given: centre $C(h, k) = (\sqrt{2}, -3\sqrt{3})$, radius $= r = 2\sqrt{2}$

Equation of circle:

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ \Rightarrow (x-\sqrt{2})^2 + (y+3\sqrt{3})^2 &= (2\sqrt{2})^2 \\ \Rightarrow x^2 - 2\sqrt{2}x + 2 + y^2 + 6\sqrt{3}y + 27 &= 8 \\ \Rightarrow x^2 + y^2 - 2\sqrt{2}x + 6\sqrt{3}y + 2 + 27 - 8 &= 0 \\ \Rightarrow x^2 + y^2 - 2\sqrt{2}x + 6\sqrt{3}y + 21 &= 0 \end{aligned}$$

- (c) Given end points of diameter:

$$A(x_1, y_1) = (-3, 2) \text{ , } B(x_2, y_2) = (5, -6)$$

Equation of circle with ends of diameter is

$$\begin{aligned} (x-x_1)(x-x_2) + (y-y_1)(y-y_2) &= 0 \\ \Rightarrow (x-(-3))(x-5) + (y-2)(y-(-6)) &= 0 \\ \Rightarrow (x+3)(x-5) + (y-2)(y+6) &= 0 \\ \Rightarrow x^2 + 3x - 4x - 15 + y^2 - 2y + 6y - 12 &= 0 \\ \Rightarrow x^2 + y^2 - 2x + 4y - 27 &= 0 \end{aligned}$$

Question # 2

Find the centre and radius of the circle with the given equation

- (a) $x^2 + y^2 + 12x - 10y = 0$ (b) $5x^2 + 5y^2 + 14x + 12y - 10 = 0$

$$(c) \ x^2 + y^2 - 6x + 4y + 13 = 0 \quad (d) \ 4x^2 + 4y^2 - 8x + 12y - 25 = 0$$

Solution

$$(a) \ x^2 + y^2 + 12x - 10y = 0$$

$$\text{Here } 2g = 12, \quad 2f = -10, \quad c = 0$$

$$\Rightarrow g = 6, \quad f = -5$$

$$\text{So centre} = (-g, -f) = (-6, 5)$$

$$\begin{aligned} \text{Radius} &= \sqrt{g^2 + f^2 - c} = \sqrt{(6)^2 + (-5)^2 - 0} \\ &= \sqrt{36 + 25} = \sqrt{61} \end{aligned}$$

$$(b) \quad 5x^2 + 5y^2 + 14x + 12y - 10 = 0$$

$$\Rightarrow x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0 \quad \div \text{ing by } 5$$

$$\text{Here } 2g = \frac{14}{5}, \quad 2f = \frac{12}{5}, \quad c = -2$$

$$\Rightarrow g = \frac{7}{5}, \quad f = \frac{6}{5}$$

$$\text{Centre} = (-g, -f) = \left(-\frac{7}{5}, -\frac{6}{5}\right)$$

$$\begin{aligned} \text{Radius} &= \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{6}{5}\right)^2 - (-2)} \\ &= \sqrt{\frac{49}{25} + \frac{36}{25} + 2} = \sqrt{\frac{27}{5}} = 3\sqrt{\frac{3}{5}} \end{aligned}$$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 3

Written an equation of the circle that passes through the given points

$$(a) \ A(4, 5), \ B(-4, -3), \ C(8, -3)$$

$$(b) \ A(-7, 7), \ B(5, -1), \ C(10, 0)$$

$$(c) \ A(a, 0), \ B(0, b), \ C(0, 0) \quad (d) \ A(5, 6), \ B(-3, 2), \ C(3, -4)$$

Solution

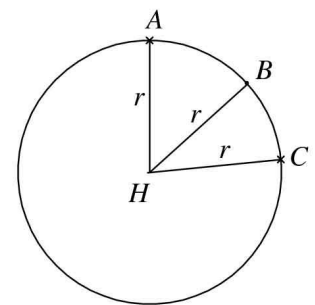
$$\text{Given: } A(4, 5), \ B(-4, -3), \ C(8, -3)$$

Let $H(h, k)$ be centre and r be radius of circle, then

$$|\overline{AH}| = |\overline{BH}| = |\overline{CH}| = r$$

$$\Rightarrow |\overline{AH}|^2 = |\overline{BH}|^2 = |\overline{CH}|^2 = r^2$$

$$\Rightarrow (h-4)^2 + (k-5)^2 = (h+4)^2 + (k+3)^2 = (h-8)^2 + (k+3)^2 = r^2 \dots\dots (i)$$



From eq. (i)

$$\begin{aligned}
 (h-4)^2 + (k-5)^2 &= (h+4)^2 + (k+3)^2 \\
 \Rightarrow h^2 - 8h + 16 + k^2 - 10k + 25 &= h^2 + 8h + 16 + k^2 + 6k + 9 \\
 \Rightarrow h^2 - 8h + 16 + k^2 - 10k + 25 - h^2 - 8h - 16 - k^2 - 6k - 9 &= 0 \\
 \Rightarrow -16h - 16k + 16 &= 0 \quad \Rightarrow h + k - 1 = 0 \dots\dots\dots (ii)
 \end{aligned}$$

Again from (i)

$$\begin{aligned}
 (h+4)^2 + (k+3)^2 &= (h-8)^2 + (k+3)^2 \\
 \Rightarrow (h+4)^2 &= (h-8)^2 \Rightarrow h^2 + 8h + 16 = h^2 - 16h + 64 \\
 \Rightarrow h^2 + 8h + 16 - h^2 + 16h - 64 &= 0 \\
 \Rightarrow 24h - 48 &= 0 \quad \Rightarrow 24h = 48 \quad \Rightarrow \boxed{h = 2}
 \end{aligned}$$

Putting value of h in (ii)

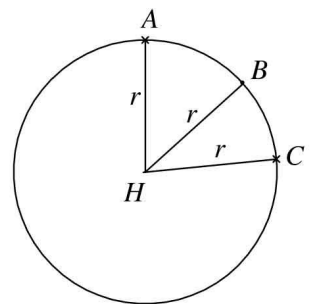
$$2 + k - 1 = 0 \quad \Rightarrow k + 1 = 0 \quad \Rightarrow \boxed{k = -1}$$

Again from (i)

$$\begin{aligned}
 r^2 &= (h-4)^2 + (k-5)^2 \\
 &= (2-4)^2 + (-1-5)^2 \quad \because h=2, k=-1 \\
 &= (-2)^2 + (-6)^2 = 4 + 36 = 40 \quad \Rightarrow r = \sqrt{40}
 \end{aligned}$$

Now equation of circle with centre at $H(2, -1)$ & $r = \sqrt{40}$

$$\begin{aligned}
 (x-2)^2 + (y+1)^2 &= (\sqrt{40})^2 \\
 \Rightarrow x^2 - 4x + 4 + y^2 + 2y + 1 &= 40 \Rightarrow x^2 + y^2 - 4x + 2y + 4 + 1 - 40 = 0 \\
 \Rightarrow x^2 + y^2 - 4x + 2y - 35 &= 0 \quad \text{Ans.}
 \end{aligned}$$



(b) Given: $A(-7, 7)$, $B(5, -1)$, $C(10, 0)$

Let $H(h, k)$ be centre and r be radius of circle, then

$$\begin{aligned}
 |\overline{AH}| &= |\overline{BH}| = |\overline{CH}| = r \\
 \Rightarrow |\overline{AH}|^2 &= |\overline{BH}|^2 = |\overline{CH}|^2 = r^2
 \end{aligned}$$

$$\Rightarrow (h+7)^2 + (k-7)^2 = (h-5)^2 + (k+1)^2 = (h-10)^2 + (k-0)^2 = r^2 \dots\dots\dots (i)$$

From equation (i) we have

$$\begin{aligned}
 (h+7)^2 + (k-7)^2 &= (h-10)^2 + (k-0)^2 \\
 \Rightarrow h^2 + 14h + 49 + k^2 - 14k + 49 &= h^2 - 20h + 100 + k^2 \\
 \Rightarrow h^2 + 14h + 49 + k^2 - 14k + 49 - h^2 + 20h - 100 - k^2 &= 0 \\
 \Rightarrow 34h - 14k - 2 &= 0 \quad \Rightarrow 17h - 7k - 1 = 0 \dots\dots\dots (ii)
 \end{aligned}$$

Again from (i)

$$\begin{aligned}
 (h-5)^2 + (k+1)^2 &= (h-10)^2 + (k-0)^2 \\
 \Rightarrow h^2 - 10h + 25 + k^2 + 2k + 1 &= h^2 - 20h + 100 + k^2
 \end{aligned}$$

$$\Rightarrow h^2 - 10h + 25 + k^2 + 2k + 1 - h^2 + 20h - 100 - k^2 = 0$$

$$\Rightarrow 10h + 2k - 74 = 0$$

$$\Rightarrow 5h + k - 37 = 0 \dots\dots\dots (iii)$$

Multiplying eq. (iii) by 7 and subtracting from (ii)

$$17h - 7k - 1 = 0$$

$$35h + 7k - 259 = 0$$

$$\hline 52h \quad - 260 = 0$$

$$\Rightarrow 52h = 260 \Rightarrow \boxed{h = 5}$$

Putting value of h in eq. (iii)

$$5(5) + k - 37 = 0 \Rightarrow 25 + k - 37 = 0 \Rightarrow k - 12 = 0 \Rightarrow \boxed{k = 12}$$

Again from eq. (i), we have

$$r^2 = (h+7)^2 + (k-7)^2$$

$$= (5+7)^2 + (12-7)^2 = (12)^2 + (5)^2 = 144 + 25 = 169$$

$$\Rightarrow r = 13$$

Now equation of circle with centre $(5,12)$ and radius 13:

$$(x-5)^2 + (y-12)^2 = (13)^2$$

$$\Rightarrow x^2 - 10x + 25 + y^2 - 24y + 144 = 169$$

$$\Rightarrow x^2 + y^2 - 10x - 24y + 25 + 144 - 169 = 0$$

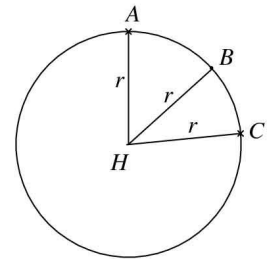
$$\Rightarrow x^2 + y^2 - 10x - 24y = 0$$

(c) Given: $A(a,0)$, $B(0,b)$, $C(0,0)$

Let $H(h,k)$ be centre and r be radius of circle, then

$$|\overline{AH}| = |\overline{BH}| = |\overline{CH}| = r$$

$$\Rightarrow |\overline{AH}|^2 = |\overline{BH}|^2 = |\overline{CH}|^2 = r^2$$



$$\Rightarrow (h-a)^2 + (k-0)^2 = (h-0)^2 + (k-b)^2 = (h-0)^2 + (k-0)^2 = r^2$$

$$\Rightarrow (h-a)^2 + k^2 = h^2 + (k-b)^2 = h^2 + k^2 = r^2 \dots\dots\dots (i)$$

From equation (i)

$$(h-a)^2 + k^2 = h^2 + k^2 \Rightarrow h^2 - 2ha + a^2 + k^2 = h^2 + k^2$$

$$\Rightarrow -2ha + a^2 = 0 \Rightarrow -2ha = -a^2 \Rightarrow h = \frac{a^2}{2a} \Rightarrow \boxed{h = \frac{a}{2}}$$

Again from equation (i)

$$h^2 + (k-b)^2 = h^2 + k^2$$

$$\Rightarrow h^2 + k^2 - 2bk + b^2 = h^2 + k^2 \Rightarrow -2bk + b^2 = 0$$

$$\Rightarrow 2bk = b^2 \Rightarrow k = \frac{b^2}{2b} \Rightarrow \boxed{k = \frac{b}{2}}$$

Again from equation (i)

$$\begin{aligned} r^2 &= h^2 + k^2 \\ &= \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} \Rightarrow r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \end{aligned}$$

Now equation of circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{\sqrt{a^2 + b^2}}{2}$

$$\begin{aligned} \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 &= \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2 \\ \Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} &= \frac{a^2}{4} + \frac{b^2}{4} \\ \Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4} &= 0 \Rightarrow x^2 + y^2 - ax - by = 0 \end{aligned}$$

Alternative Method

Given point on circle: $A(a,0)$, $B(0,b)$, $C(0,0)$

Consider an equation of circle in standard form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots (i)$$

Since $A(a,0)$ lies on circle, therefore

$$\begin{aligned} (a)^2 + (0)^2 + 2g(a) + 2f(0) + c &= 0 \\ \Rightarrow a^2 + 2ga + c &= 0 \dots\dots\dots (ii) \end{aligned}$$

Also $B(0,b)$ lies on the circle, then

$$\begin{aligned} (0)^2 + (b)^2 + 2g(0) + 2f(b) + c &= 0 \\ \Rightarrow b^2 + 2fb + c &= 0 \dots\dots\dots (iii) \end{aligned}$$

Also $C(0,0)$ lies on the circle, therefore

$$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0 \Rightarrow c = 0$$

Putting value of c in (ii)

$$a^2 + 2ga + 0 = 0 \Rightarrow 2ga = -a^2 \Rightarrow g = -\frac{a^2}{2a} \Rightarrow g = -\frac{a}{2}$$

Putting value of c in (iii)

$$\begin{aligned} b^2 + 2fb + 0 &= 0 \Rightarrow 2fb = -b^2 \\ \Rightarrow f &= -\frac{b^2}{2b} \Rightarrow f = -\frac{b}{2} \end{aligned}$$

Putting value of g , f and c in (i)

$$\begin{aligned} x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 &= 0 \\ \Rightarrow x^2 + y^2 - ax - by &= 0 \end{aligned}$$

Question # 4

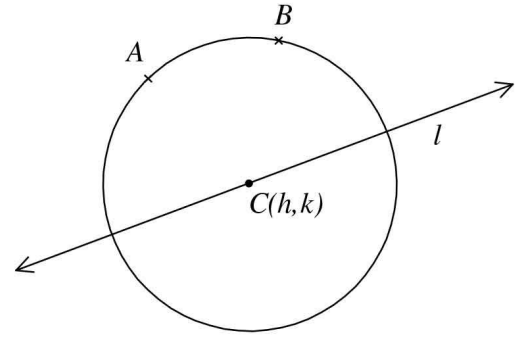
In each of the following, find an equation of the circle passes through

(a) $A(3,-1)$, $B(0,1)$ and have centre at $4x - 3y - 3 = 0$

(b) $A(-3,1)$ with radius 2 and centre at $2x - 3y + 3 = 0$

(c) $A(5,1)$ and tangent to the line $2x - y - 10 = 0$ at $B(3,-4)$

(d) $A(1,4)$, $B(-1,8)$ and tangent to the line $x + 3y - 3 = 0$

**Solution**

(a) Given: $A(3,-1)$, $B(0,1)$
 $l: 4x - 3y - 3 = 0$

Let $C(h,k)$ be centre and r be radius of circle

$\therefore A \text{ \& } B$ lies on circle

$$\therefore |\overline{CA}| = |\overline{CB}| = r$$

$$\Rightarrow \sqrt{(h-3)^2 + (k+1)^2} = \sqrt{(h-0)^2 + (k-1)^2} = r \dots\dots\dots (i)$$

$$\Rightarrow (h-3)^2 + (k+1)^2 = h^2 + (k-1)^2 \quad \text{on squaring}$$

$$\Rightarrow h^2 - 6h + 9 + k^2 + 2k + 1 = h^2 + k^2 - 2k + 1$$

$$\Rightarrow h^2 - 6h + 9 + k^2 + 2k + 1 - h^2 - k^2 + 2k - 1 = 0$$

$$\Rightarrow -6h + 4k + 9 = 0 \dots\dots\dots (ii)$$

Now since $C(h,k)$ lies on given equation l

$$\therefore 4h - 3k - 3 = 0 \dots\dots\dots (iii)$$

×ing equation (ii) by 3 & (iii) by 4 then adding

$$-18h + 12k + 27 = 0$$

$$16h - 12k - 12 = 0$$

$$\hline -2h \quad +15 = 0$$

$$\Rightarrow 2h = 15 \quad \Rightarrow \boxed{h = \frac{15}{2}}$$

Putting in (iii)

$$4\left(\frac{15}{2}\right) - 3k - 3 = 0 \quad \Rightarrow 30 - 3k - 3 = 0$$

$$\Rightarrow -3k + 27 = 0 \Rightarrow 3k = 27 \Rightarrow \boxed{k = 9}$$

Now from eq. (i)

$$\begin{aligned} r &= \sqrt{h^2 + (k-1)^2} \\ &= \sqrt{\left(\frac{15}{2}\right)^2 + (9-1)^2} = \sqrt{\frac{225}{4} + 64} = \sqrt{\frac{448}{4}} \end{aligned}$$

Now equation of circle with centre at $C(h,k) = \left(\frac{15}{2}, 9\right)$ and radius $\sqrt{\frac{481}{4}}$

$$\begin{aligned}\left(x - \frac{15}{2}\right)^2 + (y - 9)^2 &= \left(\sqrt{\frac{481}{4}}\right)^2 \\ \Rightarrow x^2 - 15x + \frac{225}{4} + y^2 - 18y + 81 - \frac{481}{4} &= 0 \\ \Rightarrow x^2 + y^2 - 15x - 18y + 17 &= 0\end{aligned}$$

(b) Given: $A(-3,1)$ lies on circle, radius = $r = 2$

$$l: 2x - 3y + 3 = 0$$

Let $C(h,k)$ be centre of circle.

Since $A(-3,1)$ lies on circle

$$\therefore r = |AC|$$

$$\Rightarrow 2 = \sqrt{(h+3)^2 + (k-1)^2}$$

$$\Rightarrow 4 = (h+3)^2 + (k-1)^2$$

$$\Rightarrow 4 = h^2 + 6h + 9 + k^2 - 2k + 1 \Rightarrow h^2 + 6h + 9 + k^2 - 2k + 1 - 4 = 0$$

$$\Rightarrow h^2 + 6h + 9 + k^2 - 2k - 3 = 0 \dots\dots\dots (i)$$

Since centre $C(h,k)$ lies on l

$$\therefore 2h - 3k + 3 = 0$$

$$\Rightarrow 2h = 3k - 3 \Rightarrow h = \frac{3k-3}{2} \dots\dots\dots (ii)$$

Putting value of h in (i)

$$\left(\frac{3k-3}{2}\right)^2 + k^2 + 6\left(\frac{3k-3}{2}\right) - 2k + 6 = 0$$

$$\Rightarrow \frac{9k^2 - 18k + 9}{4} + k^2 + 9k - 9 - 2k + 6 = 0$$

$$\Rightarrow 9k^2 - 18k + 9 + 4k^2 + 36k - 36 - 8k + 24 = 0 \quad \times \text{ing by } 4$$

$$\Rightarrow 13k^2 + 10k - 3 = 0 \Rightarrow 13k^2 + 13k - 3k - 3 = 0$$

$$\Rightarrow 13k(k+1) - 3(k+1) = 0 \Rightarrow (k+1)(13k-3) = 0$$

$$\Rightarrow k = -1 \quad \text{or} \quad k = \frac{3}{13},$$

Putting value of k in (ii)

$$h = \frac{3(-1)-3}{2}$$

$$= \frac{-6}{2}$$

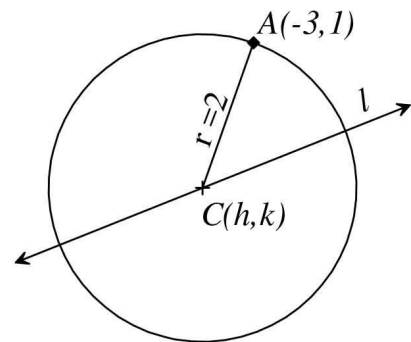
$$= -3$$

$$\Rightarrow (-3, -1) \text{ is centre of circle}$$

$$h = \frac{3\left(\frac{3}{13}\right)-3}{2}$$

$$= \frac{\frac{9}{13}-3}{2} = \frac{-\frac{30}{13}}{2} = \frac{-15}{13}$$

$$\Rightarrow \left(-\frac{15}{13}, \frac{3}{13}\right) \text{ is centre of circle.}$$



Now equation of circle with centre at $(-3,1)$ and radius 2

$$(x+3)^2 + (y+1)^2 = (2)^2 \Rightarrow (x+3)^2 + (y+1)^2 = 4$$

Now equation of circle with centre at $\left(-\frac{15}{13}, \frac{3}{13}\right)$ and radius 2

$$\begin{aligned} \left(x + \frac{15}{13}\right)^2 + \left(y - \frac{3}{13}\right)^2 &= (2)^2 \\ \Rightarrow \left(x + \frac{15}{13}\right)^2 + \left(y - \frac{3}{13}\right)^2 &= 4 \end{aligned}$$

(c) Given: $A(5,1)$ and $l: 2x - y - 10 = 0$ is tangent at $B(3,-4)$

Let $C(h,k)$ be centre and r be radius of circle.

$\because A(5,1)$ and $B(3,-4)$ lies on circle

$$\therefore |AC| = |BC| = r$$

$$\Rightarrow \sqrt{(h-5)^2 + (k-1)^2} = \sqrt{(h-3)^2 + (k+4)^2} = r \dots (i)$$

$$\Rightarrow (h-5)^2 + (k-1)^2 = (h-3)^2 + (k+4)^2 \quad \text{On squaring}$$

$$\Rightarrow h^2 - 10h + 25 + k^2 - 2k + 1 = h^2 - 6h + 9 + k^2 + 8k + 16$$

$$\Rightarrow h^2 - 10h + 25 + k^2 - 2k + 1 - h^2 + 6h - 9 - k^2 - 8k - 16 = 0$$

$$\Rightarrow -4h - 10k + 1 = 0 \dots (ii)$$

$$\text{Now slope of tangent } l = m_1 = -\frac{a}{b} = -\frac{2}{-1} = 2$$

$$\text{And slope of radial segment } \overline{CB} = m_2 = \frac{k+4}{h-3}$$

Since radial segment is perpendicular to tangent therefore

$$m_1 m_2 = -1$$

$$\Rightarrow 2\left(\frac{k+4}{h-3}\right) = -1 \Rightarrow 2k+8 = -h+3$$

$$\Rightarrow h-3+2k+8 = 0$$

$$\Rightarrow h+2k+5 = 0 \dots (iii)$$

Multiplying eq. (iii) by 4 and adding in (ii)

$$4h + 8k + 20 = 0$$

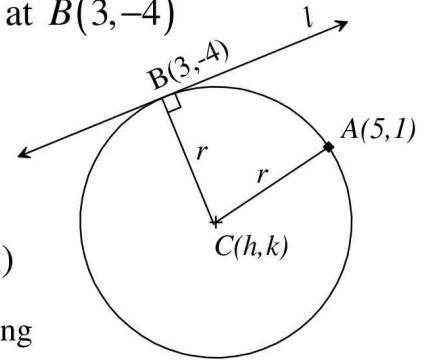
$$-4h - 10k + 1 = 0$$

$$\hline -2k + 21 = 0$$

$$\Rightarrow 2k = 21 \Rightarrow \boxed{k = \frac{21}{2}}$$

Putting value of k in (iii)

$$h + 2\left(\frac{21}{2}\right) + 5 = 0 \Rightarrow h + 21 + 5 = 0$$



$$\Rightarrow h + 26 = 0 \Rightarrow \boxed{h = -26}$$

Now from eq. (i)

$$\begin{aligned} r &= \sqrt{(h-3)^2 + (k+4)^2} \\ &= \sqrt{(-26-3)^2 + \left(\frac{21}{2} + 4\right)^2} = \sqrt{(-29)^2 + \left(\frac{29}{2}\right)^2} \\ &= \sqrt{841 + \frac{841}{4}} = \sqrt{\frac{4205}{4}} \end{aligned}$$

Now equation of circle with centre at $\left(-26, \frac{21}{2}\right)$ and radius $\sqrt{\frac{4205}{4}}$

$$\begin{aligned} (x+26)^2 + \left(y - \frac{21}{2}\right)^2 &= \left(\sqrt{\frac{4205}{4}}\right)^2 \\ \Rightarrow x^2 + 52x + 676 + y^2 - 21y + \frac{441}{4} - \frac{4205}{4} &= 0 \\ \Rightarrow x^2 + y^2 + 52x - 21y - 265 &= 0 \end{aligned}$$

(d) Given; $A(1,4)$, $B(-1,8)$

$$l: x + 3y - 3 = 0$$

Let $C(h,k)$ be centre and r be radius of circle then

$$\begin{aligned} |AC| &= |BC| = r \\ \Rightarrow \sqrt{(h-1)^2 + (k-4)^2} &= \sqrt{(h+1)^2 + (k-8)^2} = r \dots\dots\dots (i) \end{aligned}$$

Also l is tangent to circle

\therefore radius of circle = \perp ar distance of $C(h,k)$ form l

$$\begin{aligned} \Rightarrow r &= \frac{|h+3k-3|}{\sqrt{(1)^2 + (3)^2}} \\ \Rightarrow r &= \frac{|h+3k-3|}{\sqrt{10}} \dots\dots\dots (ii) \end{aligned}$$

Now from (i)

$$\sqrt{(h-1)^2 + (k-4)^2} = \sqrt{(h+1)^2 + (k-8)^2}$$

On squaring

$$\begin{aligned} (h-1)^2 + (k-4)^2 &= (h+1)^2 + (k-8)^2 \\ \Rightarrow h^2 - 2h + 1 + k^2 - 8k + 16 &= h^2 + 2h + 1 + k^2 - 16k + 64 \\ \Rightarrow h^2 - 2h + 1 + k^2 - 8k + 16 - h^2 - 2h - 1 - k^2 + 16k - 64 &= 0 \\ \Rightarrow -4h + 8k - 48 &= 0 \\ \Rightarrow h - 2k + 12 &= 0 \dots\dots\dots (iii) \end{aligned}$$

Now from (i) & (ii)

$$\sqrt{(h-1)^2 + (k-4)^2} = \frac{|h+3k-3|}{\sqrt{10}}$$

On squaring

$$(h-1)^2 + (k-4)^2 = \frac{|h+3k-3|^2}{10}$$

$$\Rightarrow 10[(h-1)^2 + (k-4)^2] = h^2 + 9k^2 + 9 + 6hk - 18k - 6h$$

$$\Rightarrow 10[h^2 - 2h + 1 + k^2 - 8k + 16] = h^2 + 9k^2 + 9 + 6hk - 18k - 6h$$

$$\Rightarrow 10h^2 - 20h + 10 + 10k^2 - 80k + 160 - h^2 - 9k^2 - 9 - 6hk + 18k + 6h = 0$$

$$\Rightarrow 9h^2 + k^2 - 14h - 62k - 6hk + 161 = 0 \dots\dots\dots \text{(iv)}$$

From (iii)

$$h = 2k - 12 \dots\dots\dots \text{(v)}$$

Putting in (iv)

$$9(2k-12)^2 + k^2 - 14(2k-12) - 62k - 6(2k-12)k + 161 = 0$$

$$\Rightarrow 9(4k^2 - 48k + 144) + k^2 - 28k + 168 - 62k - 12k^2 + 72k + 161 = 0$$

$$\Rightarrow 36k^2 - 432k + 1296 + k^2 - 28k + 168 - 62k - 12k^2 + 72k + 161 = 0$$

$$\Rightarrow 25k^2 - 450k + 1625 = 0$$

$$\Rightarrow k^2 - 18k + 65 = 0 \quad \div \text{ing by 25}$$

$$\Rightarrow k^2 - 13k - 5k + 65 = 0 \quad \Rightarrow k(k-13) - 5(k-13) = 0$$

$$\Rightarrow (k-13)(k-5) = 0$$

$$\Rightarrow k=13 \quad \text{or} \quad k=5$$

Putting in eq. (v)

$$h = 2(13) - 12$$

$$= 26 - 12 = 14$$

Now from (i)

$$r = \sqrt{(h-1)^2 + (k-4)^2}$$

$$\Rightarrow r = \sqrt{(14-1)^2 + (13-4)^2}$$

$$= \sqrt{(13)^2 + (9)^2} = \sqrt{169 + 81}$$

$$= \sqrt{250}$$

Now eq. of circle with centre (14,13)

and radius $\sqrt{170}$

$$(x-14)^2 + (y-13)^2 = (\sqrt{250})^2$$

$$\Rightarrow (x-14)^2 + (y-13)^2 = 250$$

Putting in (v)

$$h = 2(5) - 12$$

$$= 10 - 12 = -2$$

Now from (i)

$$r = \sqrt{(h-1)^2 + (k-4)^2}$$

$$= \sqrt{(2-1)^2 + (5-4)^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

Now eq. of circle with centre (2,5)

and radius $\sqrt{2}$

$$(x-2)^2 + (y-5)^2 = (\sqrt{2})^2$$

$$\Rightarrow (x-2)^2 + (y-5)^2 = 2$$

Question # 5

Find an equation of a circle of radius a and lying in the second quadrant such that it is tangent to both the axes.

Solution

Radius of circle = $r = a$

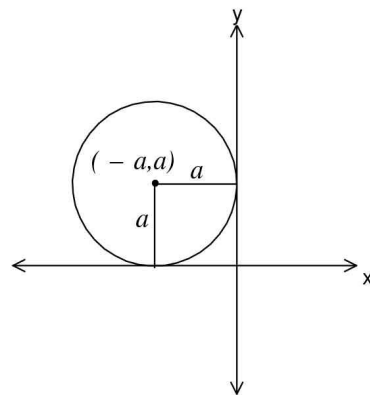
\therefore circle lies in second quadrant and touching both the axis therefore centre of circle is $(-a, a)$

So equation of circle

$$(x - (-a))^2 + (y - a)^2 = (a)^2$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2ay + a^2 - a^2 = 0$$

$$\Rightarrow x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

**Question # 6**

Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangents to the circle

$$x^2 + y^2 + 6x - 4y = 0$$

Solution

Suppose

$$l_1: 3x - 2y = 0$$

$$l_2: 2x + 3y - 13 = 0$$

$$S: x^2 + y^2 + 6x - 4y = 0$$

From S

$$2g = 6, \quad 2f = -4, \quad c = 0$$

$$\Rightarrow g = 3, \quad f = -2,$$

$$\text{Centre } C(-g, -f) = C(-3, 2)$$

$$\begin{aligned} \text{Radius } r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(3)^2 + (-2)^2 - 0} \\ &= \sqrt{9 + 4} = \sqrt{13} \end{aligned}$$

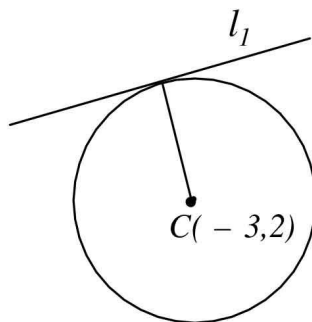
Now to check l_1 is tangent to circle, we find

$$\begin{aligned} \perp \text{ ar distance of } l_1 \text{ from centre} &= \frac{|3(-3) - 2(2) + 0|}{\sqrt{(3)^2 + (-2)^2}} \\ &= \frac{|-9 - 4|}{\sqrt{9 + 4}} = \frac{|-13|}{\sqrt{13}} = \frac{13}{\sqrt{13}} \\ &= \sqrt{13} = \text{radius of circle} \end{aligned}$$

$\Rightarrow l_1$ is tangent to given circle.

Now to check l_2 is tangent to circle, let

$$\perp \text{ ar distance of } l_2 \text{ from centre} = \frac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$$



$$\begin{aligned}
 &= \frac{|-6+6-13|}{\sqrt{4+9}} = \frac{|-13|}{\sqrt{13}} \\
 &= \frac{13}{\sqrt{13}} = \sqrt{13} = \text{Radius of circle}
 \end{aligned}$$

$\Rightarrow l_2$ is also tangent to given circle.

Circles touching each other externally or internally

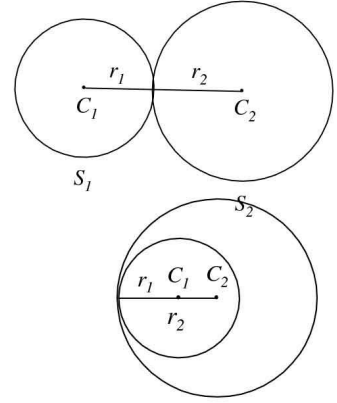
Let C_1 be centre and r_1 be radius of circle S_1 and C_2 be centre and r_2 be radius of circle S_2 .

Then they touch each other externally if

$$|C_1 C_2| = r_1 + r_2$$

And they touch each other internally if

$$|C_1 C_2| = |r_2 - r_1|$$



Question # 7

Show that the circles

$x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.

Solution

Let $S_1: x^2 + y^2 + 2x - 2y - 7 = 0$

$S_2: x^2 + y^2 - 6x + 4y + 9 = 0$

For S_1 :

$$\begin{aligned}
 2g &= 2, \quad 2f = -2, \quad c = -7 \\
 \Rightarrow g &= 1, \quad f = -1,
 \end{aligned}$$

Let C_1 be centre and r_1 be radius of circle S_1 , then

$$C_1(-g, -f) = C_1(-1, 1)$$

$$\begin{aligned}
 \text{Radius} = r_1 &= \sqrt{g^2 + f^2 - c} \\
 &= \sqrt{(1)^2 + (-1)^2 - (-7)} = \sqrt{1+1+7} = \sqrt{9} = 3
 \end{aligned}$$

For S_2 :

$$\begin{aligned}
 2g &= -6, \quad 2f = 4, \quad c = 9 \\
 \Rightarrow g &= -3, \quad f = 2
 \end{aligned}$$

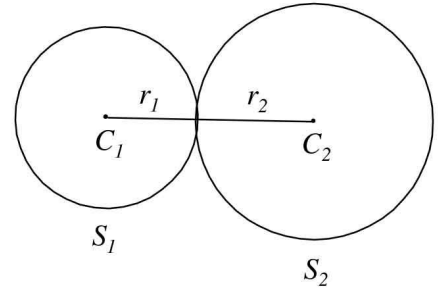
Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g, -f) = C_2(3, -2)$$

$$\begin{aligned}
 \text{Radius} = r_2 &= \sqrt{g^2 + f^2 - c} \\
 &= \sqrt{(-3)^2 + (2)^2 - 9} = \sqrt{9+4-9} = \sqrt{4} = 2
 \end{aligned}$$

Now circles touch each other externally if

$$\begin{aligned}
 |C_1 C_2| &= r_1 + r_2 \\
 \Rightarrow \sqrt{(3+1)^2 + (-2-1)^2} &= 3+2
 \end{aligned}$$



$$\Rightarrow \sqrt{16+9} = 5 \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

Hence both circles touch each other externally.

Question # 8

Show that the circles

$$x^2 + y^2 + 2x - 8 = 0 \text{ and } x^2 + y^2 - 6x + 6y - 46 = 0 \text{ touches internally.}$$

Solution

Suppose $S_1: x^2 + y^2 + 2x - 8 = 0$

$$S_2: x^2 + y^2 - 6x + 6y - 46 = 0$$

For $S_1: 2g = 2, \quad 2f = 0, \quad c = -8$

$$\Rightarrow g = 1, \quad f = 0$$

Let C_1 be centre and r_1 be radius of circle S_1 then

$$C_1(-g, -f) = C(-1, 0)$$

$$\text{Radius} = r_1 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(1)^2 + (0)^2 + 8} = \sqrt{9} = 3$$

For $S_2: 2g = -6, \quad 2f = 6, \quad c = -46$

$$\Rightarrow g = -3, \quad f = 3$$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g, -f) = C_2(3, -3)$$

$$\text{Radius} = r_2 = \sqrt{g^2 + f^2 - c}$$

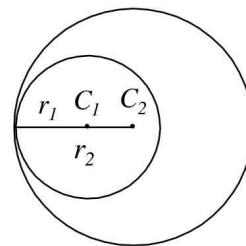
$$= \sqrt{(3)^2 + (-3)^2 - (-46)} = \sqrt{9 + 9 + 46} = \sqrt{64} = 8$$

Now circles touch each other internally if

$$|C_1C_2| = |r_2 - r_1| \Rightarrow \sqrt{(3+1)^2 + (-3-0)^2} = |8-3|$$

$$\Rightarrow \sqrt{16+9} = |5| \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

Hence circles are touching each other internally.



Question # 9

Find an equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at $A(1, -3)$.

Solution

Given: Radius $r = 2$,

Tangent: $x - y - 4 = 0$ at $A(1, -3)$

Suppose $C(h, k)$ be the centre then

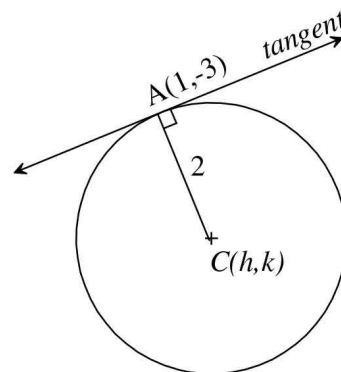
$$|AC| = 2$$

$$\Rightarrow \sqrt{(h-1)^2 + (k+3)^2} = 2$$

On squaring

$$(h-1)^2 + (k+3)^2 = 4$$

$$\Rightarrow h^2 - 2h + 1 + k^2 + 6k + 9 - 4 = 0 \Rightarrow h^2 + k^2 - 2h + 6k + 6 = 0 \dots\dots\dots (i)$$



$$\text{Now slope of radial line } AC = \frac{k+3}{h-1}$$

$$\text{Slope of line tangent} = -\frac{1}{-1} = 1$$

Since radial line is \perp ar to tangent, therefore

$$(\text{Slope of radial line})(\text{Slope of tangent}) = -1$$

$$\Rightarrow \left(\frac{k+3}{h-1}\right)(1) = -1$$

$$\Rightarrow k+3 = -(h-1) \Rightarrow k = -h+1-3 \Rightarrow k = -h-2 \dots\dots\dots (ii)$$

$$\text{Putting in (i)} \quad h^2 + (-h-2)^2 - 2h + 6(-h-2) + 6 = 0$$

$$\Rightarrow h^2 + h^2 + 4h + 4 - 2h - 6h - 12 + 6 = 0 \Rightarrow 2h^2 - 4h - 2 = 0 \Rightarrow h^2 - 2h - 1 = 0$$

$$\begin{aligned} \Rightarrow h &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \end{aligned}$$

Putting $h = 1 + \sqrt{2}$ in (ii)

$$k = -1 - \sqrt{2} - 2$$

$$\Rightarrow k = -3 - \sqrt{2}$$

Now equation of circle with

centre $(1 + \sqrt{2}, -3 - \sqrt{2})$ and radius 2.

$$(x - (1 + \sqrt{2}))^2 - (y - (-3 - \sqrt{2}))^2 = (2)^2$$

$$\Rightarrow (x - 1 - \sqrt{2})^2 - (y + 3 + \sqrt{2})^2 = 4$$

Putting $h = 1 - \sqrt{2}$ in (ii)

$$k = -1 + \sqrt{2} - 2 \Rightarrow k = -3 + \sqrt{2}$$

Now equation of circle with

centre $(1 - \sqrt{2}, -3 + \sqrt{2})$ and radius 2.

$$(x - (1 - \sqrt{2}))^2 - (y - (-3 + \sqrt{2}))^2 = (2)^2$$

$$\Rightarrow (x - 1 + \sqrt{2})^2 - (y + 3 - \sqrt{2})^2 = 4$$