Question # 1
Prove that normal lines of a circle pass through the centre of the circle.

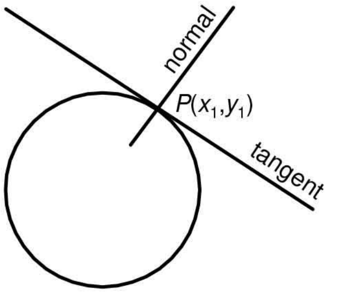
Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$

Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad 2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at $(x_1, y_1) = m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}.$



Since normal is $\perp$ar to tangent therefore

Slope of normal at $(x_1, y_1) = -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}.$

Now equation of normal at (x_1, y_1) having slope $\frac{y_1}{x_1}$

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow \quad x_1 y - x_1 y_1 = y_1 x - y_1 x_1$$

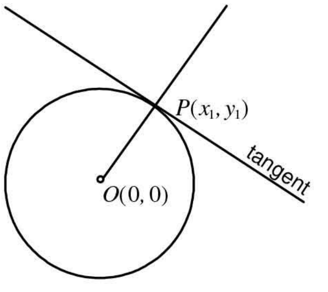
$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Clearly centre of circle $(0,0)$ satisfies (i), hence normal lines of the circles passing through the centre of the circle.

Question # 2
Prove that the straight line drawn from the centre of a circle perpendicular to a tangent passes through the point of tangency.

Solution Consider a circle with centre at origin and radius r.

$$x^2 + y^2 = r^2.$$



Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Slope of tangent at $(x_1, y_1) = m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}.$

Slope of line $\perp$ar to tangent $= -\frac{1}{m} = -\frac{1}{-x_1/y_1} = \frac{y_1}{x_1}.$

Now equation of line perpendicular to tangent passing through centre $(0,0)$

$$y - 0 = \frac{y_1}{x_1}(x - 0)$$

$$\Rightarrow \quad x_1 y = y_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Clearly the point of tangency (x_1, y_1) satisfy (i), hence the straight line drawn from the centre of circle perpendicular to a tangent passes through the point of tangency.

Question # 3

Prove that the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

Solution Let OAB be a right triangle with $|OA| = a$, $|OB| = b$.

Then the coordinates of A and B are $(a,0)$ and $(0,b)$ respectively.

Let C be the mid-point of hypotenuse AB. Then

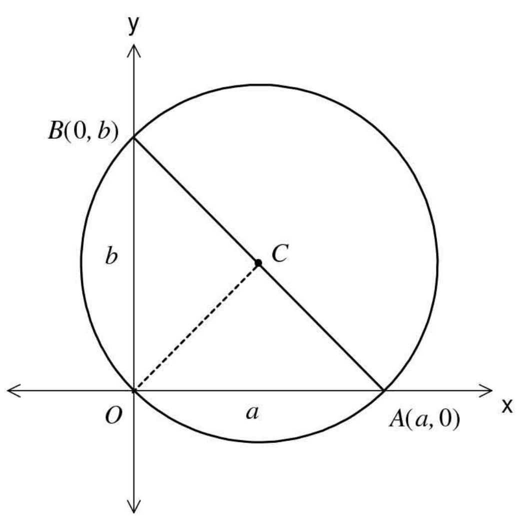
$$\text{coordinate of } C = \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right).$$

Now

$$|CA| = \sqrt{\left(\frac{a}{2}-a\right)^2 + \left(\frac{b}{2}-0\right)^2}$$

$$= \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}}$$



$$|CB| = \sqrt{\left(\frac{a}{2}-0\right)^2 + \left(\frac{b}{2}-b\right)^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}}$$

$$|CO| = \sqrt{\left(0-\frac{a}{2}\right)^2 + \left(0-\frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}}.$$

Since $|CA| = |CB| = |CO|$, therefore C is the centre of the circumcircle.

Hence the mid-point of the hypotenuse of a right triangle is the circumcentre of the triangle.

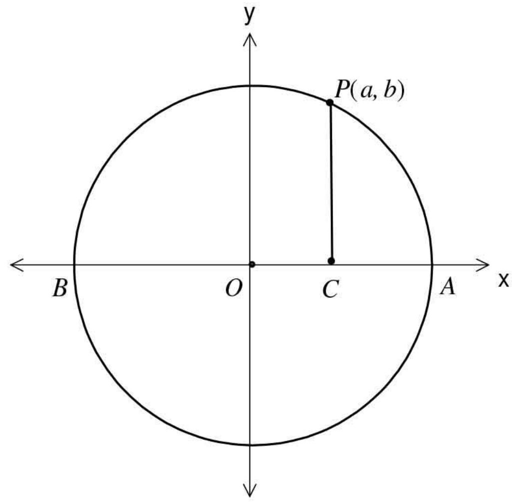
Mean proportional

Let a,b and c be three numbers. The number b is said to be *mean proportional* between a and b if a,b,c are in geometric means or

$$b^2 = ac \qquad \text{or} \qquad \frac{b}{a} = \frac{a}{c}.$$

Question # 4

Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.



Solution Consider a circle of radius r and centre $(0,0)$, then equation of circle

$$x^2 + y^2 = r^2$$

Let A and B are end-points of diameter of circle along x-axis, then coordinate of A and B are $(-r,0)$ and $(0,r)$ respectively.

Also let $P(a,b)$ be any point on circle and $\perp$ ar from P cuts diameter at C. Then coordinate of C are $(a,0)$.

Since $P(a,b)$ lies on a circle, therefore

$$a^2 + b^2 = r^2 \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Now

$$|AC| = \sqrt{(r+a)^2 - (0-0)^2} = r + a.$$

$$|CB| = \sqrt{(r-a)^2 - (0-0)^2} = r - a.$$

$$|PC| = \sqrt{(a-a)^2 + (b-0)^2} = \sqrt{0 + b^2} = b.$$

Now

$$\begin{aligned} |AC| \cdot |CB| &= (r+a)(r-a) \\ &= r^2 - a^2 \\ &= a^2 + b^2 - a^2 \qquad \text{from (i)} \\ &= b^2 = |PC|^2 \end{aligned}$$

$$\Rightarrow |AC| \cdot |CB| = |PC| \cdot |PC| \qquad \Rightarrow \frac{|AC|}{|PC|} = \frac{|PC|}{|CB|}$$

$\Rightarrow |PC|$ is a mean proportional to $|AC|$ and $|CB|$.