Question # 1

Find the focus, vertex and directix of the parabola sketch its graph.

(i) $y^2 = 8x$ (ii) $x^2 = -16y$

(iii) $x^2 = 5y$ (iv) $y^2 = -12x$

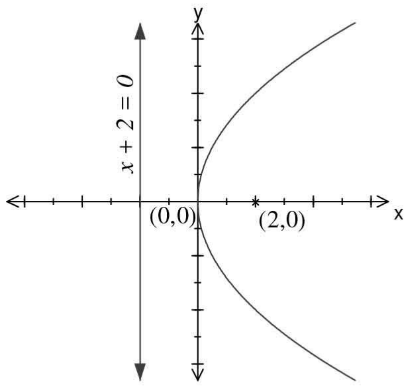
(v) $x^2 = 4(y-1)$

(vi) $y^2 = -8(x-3)$

(vii) $(x-1)^2 = 8(y+2)$ (viii) $y = 6x^2 - 1$

(ix) $x+8-y^2+2y = 0$

(x) $x^2-4x-8y+4 = 0$



Solution

(i) $y^2 = 8x$

Here $4a = 8 \Rightarrow a = 2$

Vertex: $O(0,0)$

The axis of parabola is along x-axis and opening of parabola is to the right side.

Focus: $(a,0) = (2,0)$

Directrix: $x+a = 0$

$\Rightarrow x+2 = 0$

(ii) $x^2 = -16y$

Here $4a = 16 \Rightarrow a = 4$

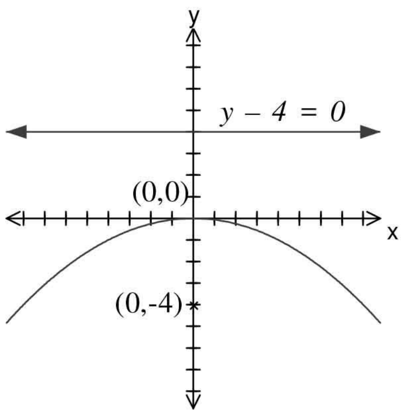
Vertex: $O(0,0)$

The axis of parabola is along $y-$axis and opening of the parabola is downward.

So Focus: $F(0,-a) = F(0,-4)$

Directrix: $y-a = 0$

$\Rightarrow y-4 = 0$



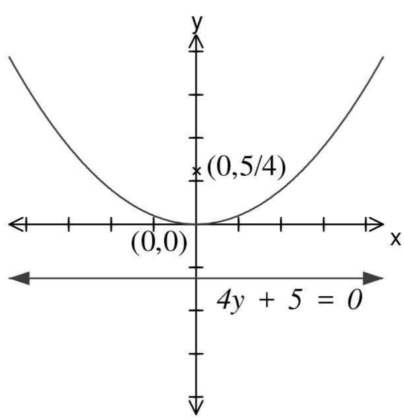
(ii) $x^2 = 5y$

Here $4a = 5 \Rightarrow a = \dfrac{5}{4}$

And vertex: $O(0,0)$

The axis of the parabola is along y-axis and opening of the parabola is upward.

Focus: $F(0,a) = F\left(0,\dfrac{5}{4}\right)$



Directrix: $y+a=0 \Rightarrow y+\frac{5}{4}=0$

$\Rightarrow 4y+5=0$

(iv) $y^2=-12x$

Here $4a=12 \Rightarrow a=3$

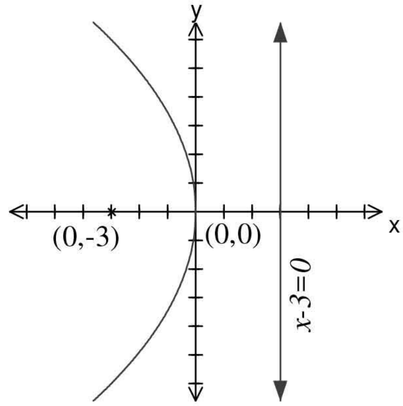
And vertex: $O(0,0)$

The axis of the parabola is along x-axis and opening of the parabola is to the left side.

Focus: $F(-a,0)=(-3,0)$

Directrix: $x-a=0$

$\Rightarrow x-3=0$



(v) $x^2=4(y-1)$ ……….. (i)

Put $X=x$, $Y=y-1$

$\Rightarrow X^2=4Y$ ………… (ii)

Here $4a=4 \Rightarrow a=1$

And vertex of parabola (ii) is $O(0,0)$ with axis of parabola is along $Y-$axis open upward.

$\because$ Vertex: $O(0,0)$

$\Rightarrow X=0$, $Y=0$

$\Rightarrow x=0$, $y-1=0 \Rightarrow y=1$

$\Rightarrow (0,1)$ is vertex of parabola (i)

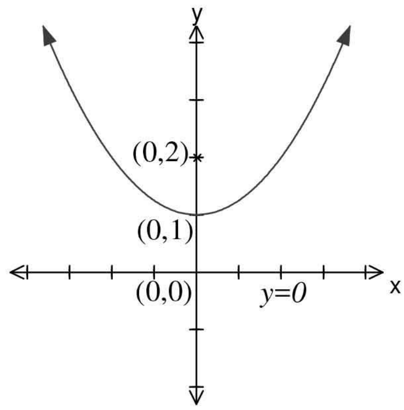
Now focus: $F(0,a)=F(0,1)$

$\Rightarrow X=0$, $Y=1$

$\Rightarrow x=0$, $y-1=1$

$y=1+1 \Rightarrow y=2$

$\Rightarrow (0,2)$ is focus of parabola (i)



Directrix of parabola (ii) is

$Y+a=0 \Rightarrow Y+1=0$

$\Rightarrow y-1+1=0$

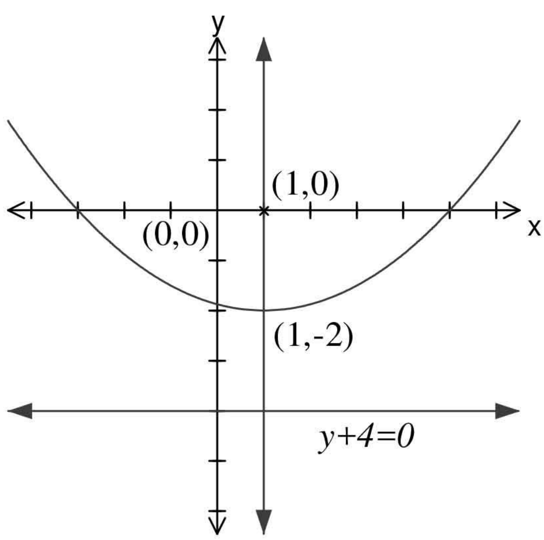
$\Rightarrow y=0$ is directrix of parabola (i)

(vi) *Do yourself*

(vii) $(x-1)^2=8(y+2)$ …………. (i)

Put $X=x-1$, $Y=y+2$ in (i)

$X^2=8Y$ …………. (ii)



Here $4a = 8 \Rightarrow a = 2$

Axis of parabola is along Y-axis open upward with vertex of $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 0$

$\Rightarrow x = 2 \quad , \quad y = -2$

$\Rightarrow (1,-2)$ is vertex of parabola (i)

Focus of (ii) is $(0,a) = (0,2)$

$\Rightarrow X = 0 \quad , \quad Y = 2$

$\Rightarrow x-1 = 0 \quad , \quad y+2 = 2$

$\Rightarrow x = 1 \quad , \quad y = 0$

$\Rightarrow (1,0)$ is the focus of given parabola (i)

Directrix of (ii)

$Y+a = 0 \quad \Rightarrow \quad Y+2=0$

$\Rightarrow y+2+2 = 0 \quad \Rightarrow \quad y+4 = 0$ is directrix of given parabola.

(viii) $y = 6x^2 - 1$

$\Rightarrow 6x^2 = y+1 \quad \Rightarrow \quad x^2 = \frac{1}{6}(y+1)$

Now try yourself

(ix) $x+8-y^2+2y = 0$

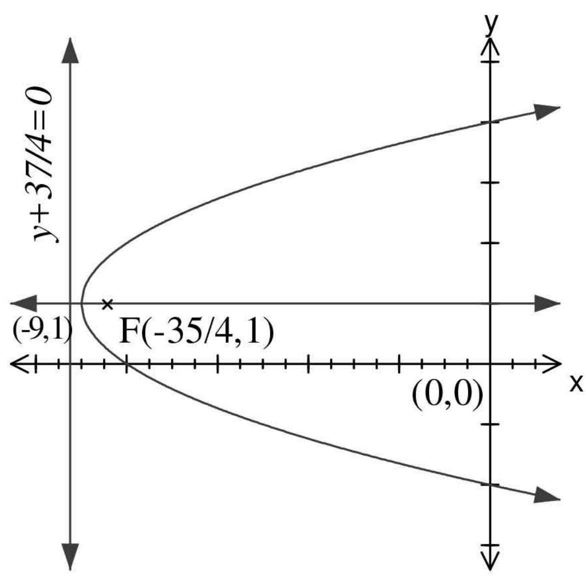
$\Rightarrow y^2 - 2y = x+8$

$\Rightarrow y^2 - 2y + 1 = x+8+1$

$\Rightarrow (y-1)^2 = x+9$

Put $X = x+9 \quad , \quad Y = y-1$

$Y^2 = X$



Here $4a = 1 \Rightarrow a = \frac{1}{4}$

The axis of parabola is along x-axis and it is opening to the right side.

Vertex of parabola (ii) is $(0,0)$

$\Rightarrow X = 0 \quad , \quad Y = 0$

$\Rightarrow x+9 = 0 \quad , \quad y-1 = 0$

$\Rightarrow x = -9 \quad , \quad y = 1$

$\Rightarrow (-9,1)$ is vertex of the parabola (i)

Focus: $(a,0) = \left(\frac{1}{4},0\right)$

$$X = \frac{1}{4} \quad , \quad Y = 0$$

$$\Rightarrow \quad x+9 = \frac{1}{4} \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = \frac{1}{4}-9 \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = -\frac{35}{4} \quad , \quad y = 1$$

$\Rightarrow \left(-\frac{35}{4},1\right)$ is focus of parabola (i)

Directrix of parabola (ii)

$$X + a = 0$$

$$\Rightarrow \quad X + \frac{1}{4} = 0$$

$$\Rightarrow \quad x+9+\frac{1}{4} = 0$$

$\Rightarrow \quad x+\frac{37}{4} = 0$ is directrix of parabola (i)

(x) *Do yourself as above*

Question # 2

Write an equation of the parabola with given elements.

(i) Focus $(-3,1)$; directrix $x=3$
(ii) Focus $(2,5)$; directrix $y=1$
(iii) Focus $(-3,1)$; directrix $x-2y-3=0$
(iv) Focus $(1,2)$; vertex $(3,2)$
(v) Focus $(-1,0)$; vertex $(-1,2)$
(vi)Directrix $x=-2$,Focus $(2,2)$
(vii)Directrix $y=3$; vertex $(2,2)$
(viii) Directrix $y=1$, length of latusrectum is 8. Opens downward.
(ix)Axis $y=0$, through $(2,1)$ and $(11,-2)$
(x)Axis parallel to y-axis, the points $(0,3),(3,4)$ *and* $(4,11)$ lie on the graph.

Solution

(i) Focus: $F(-3,1)$

Directrix: $x = 3$ i.e. $x-3 = 0$

Let $P(x,y)$ be any point on parabola then by definition

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-3|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \sqrt{x^2+6x+9+y^2-2y+1} = |x-3|$$

On squaring

$$\Rightarrow x^2+6x+9+y^2-2y+1 = x^2-6x+9$$

$\Rightarrow\ x^2+6x+9+y^2-2y+1-x^2+6x-9 = 0$

$\Rightarrow\ y^2+12x-2y+1 = 0$

is required equation of parabola.

(ii) *Do yourself as above.*

(iii) Focus: $F(-3,1)$

Directrix: $x-2y-3 = 0$

Let $P(x,y)$ be any point on parabola, then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix.

$$\Rightarrow\ \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}$$

$$\Rightarrow\ \sqrt{x^2+6x+9+y^2-2y+1} = \frac{|x-2y-3|}{\sqrt{5}}$$

$$\Rightarrow\ \sqrt{5}\sqrt{x^2+6x+9+y^2-2y+1} = |x-2y-3|$$

On squaring both sides

$$5\left(x^2+6x+9+y^2-2y+1\right) = x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow\ 5x^2+30x+45+5y^2-10y+5-x^2-4y^2-9+4xy-12y+6x = 0$$

$$\Rightarrow\ 4x^2+y^2+36x-22y+4xy+41 = 0$$

is required equation

(iv) Given: Focus $(1,2)$, Vertex $(3,2)$

Focus and vertex implies that axis of parabola is parallel to *x*-axis and opening to left side. Therefore eq. of parabola with vertex $(3,2)$

$$(y-2)^2 = -4a(x-3) \text{ .... (i)}$$

Now $a =$ Distance between focus and vertex

$$= \sqrt{(3-1)^2+(2-2)^2} = \sqrt{4+0} = 2$$

Putting in (i)

$$(y-2)^2 = -4(2)(x-3) \quad \Rightarrow\ y^2-4y+4 = -8x+24$$

$$\Rightarrow\ y^2-4y+4+8x-20 = 0 \quad \Rightarrow\ y^2-4y+8x-20 = 0 \text{ is req. eq.}$$

(v) *Do yourself*

(vii) Directrix: $y = 3$ i.e. $y-3 = 0$

Vertex $(2,2)$

Since axis of parabola is parallel to y-axis (because directrix is parallel to x-axis). And opening is downward.

So equation of parabola with vertex $(h,k) = (2,2)$

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-2)^2 = -4a(y-2)$$

Now a = Distance of vertex $(2,2)$ from directrix

$$= \frac{|2-3|}{\sqrt{(0)^2+(1)^2}} = \frac{|-1|}{1} = 1$$

Putting in (i)

$$(x-2)^2 = -4(1)(y-2)$$

$$\Rightarrow x^2-4x+4 = -4y+8 \qquad \Rightarrow x^2-4x+4+4y-8 = 0$$

$$\Rightarrow x^2-4x+4y-4 = 0$$ is require equation.

(viii) Directrix: $y = 1$

Latusractum $= 4a = 8 \Rightarrow a = 2$

$\because$ Parabola is open downward

$\therefore$ Consider vertex $= (h,-1)$

And equation of parabola

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-h)^2 = -4(2)(y+1) \qquad \Rightarrow x^2-2hx+h^2 = -8y-8$$

$$\Rightarrow x^2-2hx+8y+h^2+8 = 0$$ is req. eq.

(ix) Axis of parabola: $y = 0$

Let vertex is (h,k)

$\because$ it lies on x-axis $\therefore$ $k=0$

Now equation of parabola with vertex $(h,0)$

$$(y-0)^2 = 4a(x-h)$$

$$\Rightarrow y^2 = 4a(x-h) \;\ldots\ldots\ldots\ldots \text{(i)}$$

$\because$ $(2,1)$ lies on parabola (i)

$\therefore$ $(1)^2 = 4a(2-h)$

$$\Rightarrow 1 = 4a(2-h) \;\ldots\ldots\ldots\ldots \text{(ii)}$$

Also $(11,-2)$ lies on parabola (i)

$$(-2)^2 = 4a(11-h)$$

$$\Rightarrow 4 = 4a(11-h)$$

$$\Rightarrow 1 = a(11-h) \;\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Dividing (i) & (ii)

$$\frac{1}{1} = \frac{4a(2-h)}{a(11-h)}$$

$$\Rightarrow 1 = \frac{4(2-h)}{(11-h)} \qquad \Rightarrow 11-h = 8-4h$$

$$\Rightarrow 4h-h = 8-11 \qquad \Rightarrow 3h = -3 \qquad \Rightarrow h = -1$$

Putting in (ii)

$$1 = 4a(2-(-1))$$

$$\Rightarrow 1 = 4a(3) \qquad \Rightarrow 1 = 12a \qquad \Rightarrow a = \frac{1}{12}$$

Using in (i)

$$y^2 = 4\left(\frac{1}{12}\right)(x-(-1)) \qquad \Rightarrow 3y^2 = x+1$$

$$\Rightarrow 3y^2 - x - 1 = 0$$

is the required equation.

(x) Axis parallel to y-axis, then points (0,3), (3,4) and (4,11) lie on the graph.
Equation of parabola axis parallel to y-axis with vertex (h,k)

$$(x-h)^2 = 4a(y-k) \text{ .... (i)}$$

$\because$ $(0,3)$ lies on parabola (i)

$$\therefore (0-h)^2 = 4a(3-k)$$

$$\Rightarrow h^2 = 12a - 4ak \text{ .... (ii)}$$

Also $(3,4)$ lies on parabola (i)

$$(3-h)^2 = 4a(4-k)$$

$$\Rightarrow 9 - 6h + h^2 = 16a - 4ak$$

$$\Rightarrow h^2 - 6h + 9 = 16a - 4ak \text{ .... (iii)}$$

Also $(4,11)$ lies on parabola (i)

$$(4-h)^2 = 4a(11-k)$$

$$\Rightarrow 16 - 8h + h^2 = 44a - 4ak$$

$$\Rightarrow h^2 - 8h + 16 = 44a - 4ak \text{ .... (iv)}$$

Subtracting (ii) & (iii)

$$\begin{array}{l} h^2 \qquad\qquad = 12a - 4ak \\ \underset{+}{h^2} \underset{-}{-6h} \underset{-}{+9} = \underset{+}{+16a} - 4ak \\ \hline \qquad 6h - 9 = -4a \text{ ....} (v) \end{array}$$

Now subtracting (iii) & (iv)

$$h^2-6h+9 = 16a-4ak$$
$$h^2-8h+16 = +44a-4ak$$
$$\underline{\ \ +\quad -\qquad -\quad +\ \ }$$
$$2h-7=-28a \quad ....(vi)$$

Multiplying (vi) with 3 and subtracting from (v)

$$6h-21 = -84a$$
$$+6h-9 = -4a$$
$$\underline{\ -\quad +\qquad +\ }$$
$$-12=-80a$$

$$\Rightarrow \boxed{a = \frac{12}{80} = \frac{3}{20}}$$

Putting in (v)

$$6h-9 = -40\left(\frac{3}{20}\right) \Rightarrow 6h = 9-\frac{3}{5} = \frac{42}{5} \Rightarrow \boxed{h=\frac{7}{5}}$$

Putting value of a and h in (ii), we get

$$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right)-4\left(\frac{3}{20}\right)k \quad \Rightarrow \frac{49}{25} = \frac{9}{5}-\frac{3}{5}k$$

$$\Rightarrow \frac{3}{5}k = \frac{9}{5}-\frac{49}{25} \quad \Rightarrow \frac{3}{5}k = -\frac{4}{25} \quad \Rightarrow \boxed{k=-\frac{4}{15}}$$

Now putting the value of a, h and k in (i), we get

$$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right) \quad \Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}\left(y+\frac{4}{15}\right)$$

$$\Rightarrow x^2-\frac{14}{5}x+\frac{49}{25} = \frac{3}{5}y+\frac{4}{25} \quad \Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{49}{25}-\frac{4}{25} = 0$$

$$\Rightarrow x^2-\frac{14}{5}x-\frac{3}{5}y+\frac{9}{5} = 0$$

$$\Rightarrow 5x^2-14x-3y+9 = 0$$

is the required equation.

Question # 3

Find an equation of the parabola having its focus at the origin and dirctrix parallel to the
(i) $x-axis$ (ii) $y-axis$

Solution

(i) When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$$y = h \quad \text{(parallel to x-axis)}$$

i.e. $y-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$$|PF| = \perp\text{ar distance of } P(x,y) \text{ from directrix}$$

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \dfrac{|y-h|}{\sqrt{(0)^2+(1)^2}}$

$\Rightarrow \sqrt{x^2+y^2} = \dfrac{|y-h|}{1}$

On squaring

$\Rightarrow x^2+y^2 = y^2-2hy+h^2 \quad \Rightarrow x^2+y^2-y^2+2hy-h^2 = 0$

$\Rightarrow x^2+2hy-h^2 = 0$ is req. equation.

ii) When directrix is parallel to y-axis.

When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$x = h$ (parallel to y-axis)

i.e. $x-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \dfrac{|x-h|}{\sqrt{(1)^2+(0)^2}}$

$\Rightarrow \sqrt{x^2+y^2} = \dfrac{|x-h|}{1}$

On squaring

$\Rightarrow x^2+y^2 = x^2-2hx+h^2 \quad \Rightarrow x^2+y^2-x^2+2hx-h^2 = 0$

$\Rightarrow y^2+2hx-h^2 = 0$ is req. equation

Question # 4

Show that an equation of parabola with focus at $(a\cos\alpha, a\sin\alpha)$ and directrix $x\cos\alpha + y\sin\alpha + a = 0$ is $(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha)$

Solution Focus: $F(a\cos\alpha, a\sin\alpha)$

Directrix: $x\cos\alpha + y\sin\alpha + a = 0$

Let $P(x,y)$ be any point on parabola then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$\Rightarrow \sqrt{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2} = \dfrac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$

On squaring

$(x-a\cos\alpha)^2+(y-\sin\alpha)^2 = \dfrac{|x\cos\alpha + y\sin\alpha + a|^2}{1}$

$\Rightarrow x^2-2ax\cos\alpha+a^2\cos^2\alpha+y^2-2ay\sin\alpha+a^2\sin^2\alpha$

$= x^2\cos^2\alpha+y^2\sin^2\alpha+a^2+2ax\cos\alpha+2ay\sin\alpha+2xy\sin\alpha\cos\alpha$

$$\Rightarrow\ x^2 - x^2\cos\alpha + y^2 - y^2\sin\alpha + a^2\left(\cos^2\alpha + \sin^2\alpha\right)$$
$$= a^2 + 2ax\cos\alpha + 2ay\sin\alpha + 2xy\sin\alpha\cos\alpha + 2ax\cos\alpha + 2ay\sin\alpha$$
$$\Rightarrow\ x^2\left(1-\cos^2\alpha\right) + y^2\left(1-\sin^2\alpha\right) + a^2(1) - a^2 - 2xy\sin\alpha\cos\alpha$$
$$= 4ax\cos\alpha + 4ay\sin\alpha$$
$$\Rightarrow\ x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha = 4a\left(x\cos\alpha + y\sin\alpha\right)$$
$$\Rightarrow\ \left(x\sin\alpha - y\cos\alpha\right)^2 = 4a\left(x\cos\alpha + y\sin\alpha\right)$$

is equation of parabola which is given.

Question # 5

Show that the ordinate at any point P of the parabola is a mean propositional between the length of the latusrectum and the abscissa of P .

Solution

Consider equation of parabola

$$y^2 = 4ax$$
$$\Rightarrow\ y\cdot y = 4a\cdot x$$
$$\Rightarrow\ \frac{4a}{y} = \frac{y}{x}$$
$$\Rightarrow\ \frac{\textit{latus ractum}}{\textit{ordinate}} = \frac{\textit{ordinate}}{\textit{abscissa}}$$

$\Rightarrow$ ordinate is mean proportional between latus rectum and abscissa.

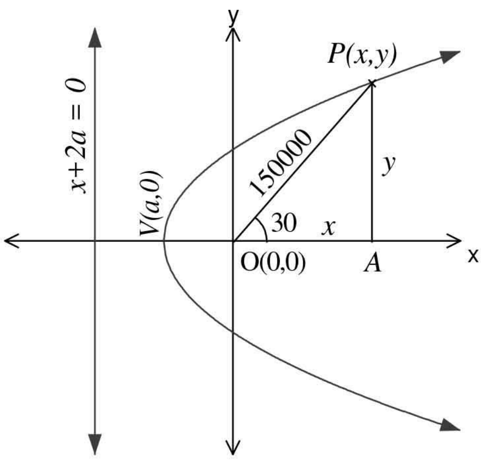
Question # 6

A comet has a parabolic orbit with the earth at the focus. When the comet is $150,000km$ from the earth, the line joining the comet and the earth makes an angle of 30 with the asix of the parabola. How close will the comet come to the earth

Solution Suppose earth be at focus which is origin and $V(-a,0)$ be vertex of parabola.

Then directrix of parabola;

$$x = -2a$$
$$\Rightarrow\ x + 2a = 0$$



Let comet be at a point $P(x,y)$ then by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow\ \sqrt{(x-0)^2 + (y-0)^2} = \frac{|x+2a|}{\sqrt{(1)^2 + (0)^2}}$$
$$\Rightarrow\ \sqrt{x^2 + y^2} = |x+2a|$$

On squaring

$$x^2 + y^2 = (x+2a)^2 \quad \text{..........} \ (i)$$

Also by Pythagoras theorem in $\triangle$ ABC

$$|OA|^2 + |AP|^2 = |OP|^2$$

$$\Rightarrow\ x^2 + y^2 = (150000)^2 \ \ldots\ldots\ldots\ldots\ (ii)$$

Comparing (i) and (ii)

$$(x+2a)^2 = (150000)^2$$

$$\Rightarrow\ x+2a = \pm 150000 \ \ldots\ldots\ldots\ldots\ (iii)$$

Now from right triangle OAP

$$\cos 30^\circ = \frac{|OA|}{|OP|} \qquad \Rightarrow\ \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

$$\Rightarrow\ x = \frac{\sqrt{3}}{2}(150000)$$

Using in (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

$$\Rightarrow\ 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000) \qquad \Rightarrow\ 2a = \pm 150000 - \sqrt{3}(75000)$$

$$\Rightarrow\ 2a = 75000\left(\pm 2 - \sqrt{3}\right) \qquad \Rightarrow\ a = 37500\left(\pm 2 - \sqrt{3}\right)$$

Since a is shortest distance and can't be negative

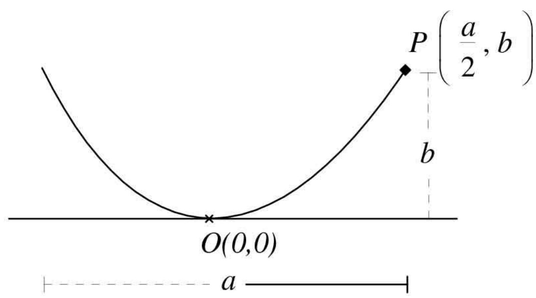
Therefore $\quad a = 37500\left(2 - \sqrt{3}\right) Km$

Question # 7

Find an equation of the parabola formed by the cables of a suspension bridge whose span is a m and the vertical height of the supporting towards is b m.

Solution Consider equation of parabola with vertex $O(0,0)$

$$x^2 = 4a'y \ \ldots\ldots\ldots\ (i)$$



Since $P\left(\frac{a}{2}, b\right)$ lies on parabola

$$\left(\frac{a}{2}\right)^2 = 4a'(b) \quad \Rightarrow\ a' = \frac{a^2}{16b}$$

Putting in (i)

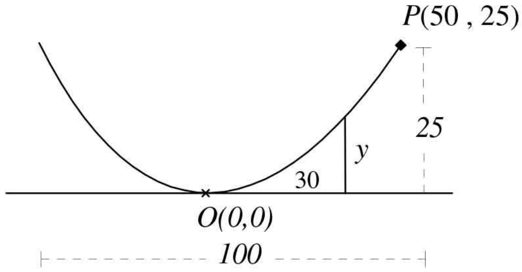
$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

$$\Rightarrow\ x^2 = \frac{a^2}{4b}y \quad \text{is required equation.}$$

Question # 8

A parabolic arch has a 100m base and height 25m. Find the height of the arch at the point 30m from the centre of the base.

Solution Suppose equation of parabola with vertex $(0,0)$



$$x^2 = 4ay \quad \ldots\ldots\ldots\ldots\ (i)$$

From figure, we see that $P(50,25)$ lies on parabola.

$$(50)^2 = 4a(25)$$

$$\Rightarrow 2500 = 100a \qquad \Rightarrow a = 25$$

Putting in (i)

$$x^2 = 4(25)y \quad \Rightarrow x^2 = 100y$$

When $x = 30$

$$(30)^2 = 100y$$

$$\Rightarrow y = \frac{900}{100} \quad \Rightarrow y = 9$$

Hence the required height = 9m

Question # 9

Show that tangent at any point P of a parabola makes equal angles with the line PF and the line through P parallel to the axis of the parabola, F being focus. (These angles are called respectively angle of incidence and angle of reflection).

Solution

Suppose the parabola

$$y^2 = 4ax \quad \ldots\ldots\ldots\ (i)$$

Let $P(x_1, y_1)$ be any point on parabola, then

$$y_1^2 = 4ax_1 \quad \ldots\ldots\ldots (ii)$$

Now differentiating (i) w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}4ax \quad \Rightarrow 2y\frac{dy}{dx} = 4a \quad \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent at $(x_1, y_1) = m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1}$

Now slope of $PS = m_2 = \frac{y_1 - 0}{x_1 - a}$

$$\Rightarrow m_2 = \frac{y_1}{x_1 - a}$$

Now slope of line parallel to axis of parabola $= m_3 = 0$

(because axis of parabola is along x-axis)

Let θ_1 be angle between tangent and line parallel to axis of parabola, then

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)} = \frac{2a/y_1}{1}$$

$$\Rightarrow \tan\theta_1 = \frac{2a}{y_1} \text{ ………..(iii)}$$

Let θ_2 be angle between tangent and PS, then

$$\tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)} = \frac{\frac{y_1^2 - 2a(x_1 - a)}{y_1(x_1 - a)}}{\frac{x_1 - a + 2a}{x_1 - a}} = \frac{y_1^2 - 2ax_1 + 2a^2}{y_1(x_1 + a)}$$

$$= \frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 + a)} \qquad \text{from (ii)}$$

$$= \frac{2ax_1 + 2a^2}{y_1(x_1 + a)} = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$$

$$\Rightarrow \tan\theta_2 = \frac{2a}{y_1} \text{ ……….(iv)}$$

From (iii) and (iv)

$$\tan\theta_1 = \tan\theta_2 \qquad \Rightarrow \theta_1 = \theta_2$$

as required