

EXERCISE 6.6

Q.1: Find an equation of the hyperbola with given data. Sketch the graph of each.

(i) Center $(0, 0)$ Focus $(6, 0)$, Vertex $(4, 0)$

Solution:

$$\text{Given } (ae, 0) = (6, 0) \quad , \quad (a, 0) = (4, 0)$$

$$(c, 0) = (6, 0) \quad \quad a = 4$$

$$c = 6$$

For hyperbola

$$c^2 = a^2 + b^2$$

$$(6)^2 = (4)^2 + b^2$$

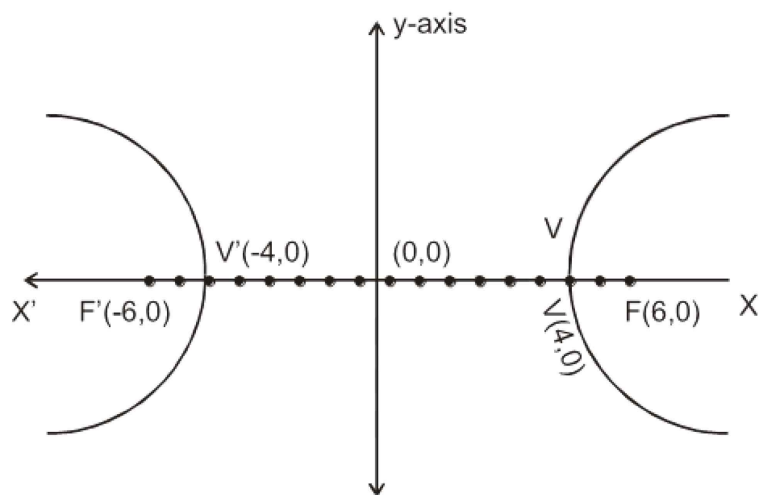
$$36 - 16 = b^2$$

$$20 = b^2$$

Required equation of hyperbola is

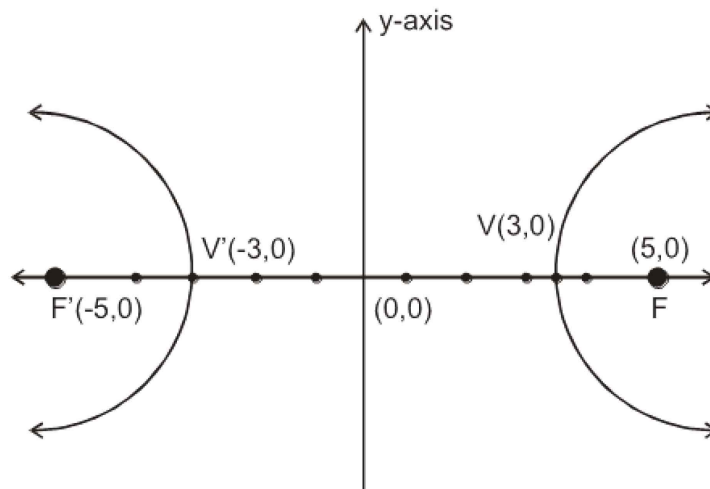
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$



(ii) Foci $(\pm 5, 0)$, vertex $(3, 0)$

Solution:



Foci $(\pm 5, 0)$ Vertex $(3, 0)$

$$(\pm c, 0) = (\pm 5, 0) \quad (a, 0) = (3, 0)$$

$$\boxed{c = 5}$$

$$\boxed{a = 3}$$

For hyperbola

$$c^2 = a^2 + b^2$$

$$(5)^2 = (3)^2 + b^2$$

$$25 = 9 + b^2$$

$$16 = 9 + b^2$$

$$16 = b^2$$

Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{Ans.}$$

(iii) Foci $(2 \pm 5\sqrt{2}, -7)$, Length of the transverse axis 10.

Solution:

$$(\pm ae, 0) = (2 \pm 5\sqrt{2}, -7)$$

$$\text{Foci are } (2 + 5\sqrt{2}, -7) \text{ \& } F' (2 - 5\sqrt{2}, -7)$$

Also given $2a = 10$

$$a = 5$$

Center = mid point of foci

$$= \left(\frac{2 + 5\sqrt{2} + 2 - 5\sqrt{2}}{2}, \frac{-7 - 7}{2} \right)$$

$$\text{Center} = (2, -7)$$

We know that

$$c = \text{distance between center and focus}$$

$$= \sqrt{(2 - 2 - 5\sqrt{2})^2 + (-7 + 7)^2} = \sqrt{50}$$

For hyperbola

$$c^2 = a^2 + b^2$$

$$(\sqrt{50})^2 = (5)^2 + b^2$$

$$50 - 25 = b^2$$

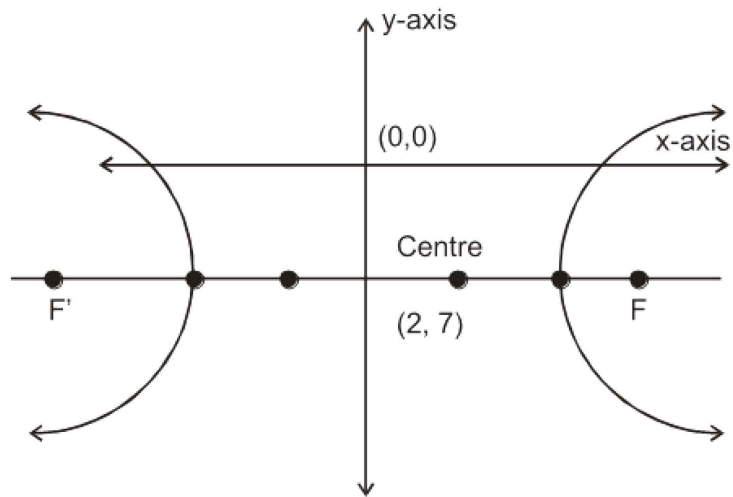
$$25 = b^2$$

$$b^2 = 25$$

with center $(2, -7)$ equation of hyperbola is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x - 2)^2}{25} - \frac{(y + 7)^2}{25} = 1 \quad \text{Ans}$$



(iv) Foci $(0, \pm 6)$, $e = 2$ (Lahore Board 2011)

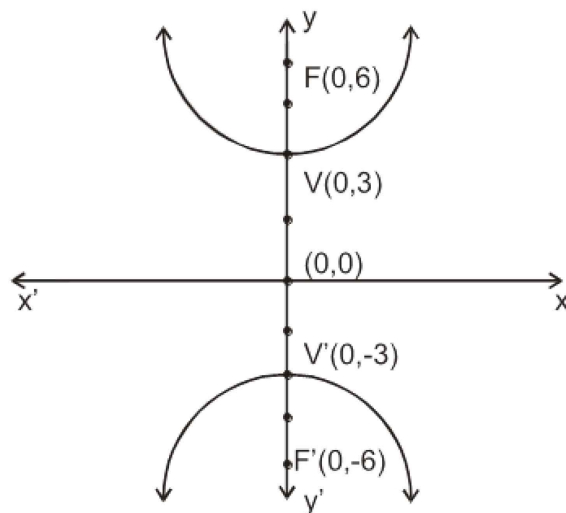
Solution:

$$\text{Foci } (0, \pm c) = (0, \pm 6)$$

$$c = 6 , e = 2$$

$$ae = c$$

$$a = \frac{c}{e} = \frac{6}{2} = \boxed{3 = a}$$



For hyperbola

$$c^2 = a^2 + b^2$$

$$(6)^2 = (3)^2 + b^2$$

$$36 - 9 = b^2$$

$$b^2 = 27$$

Equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{x^2}{27} = 1$$

(v) Foci $(0, \pm 9)$ directrix $y = \pm 4$

Solution:

$$\text{Foci } (0, \pm ae) = (0, \pm 9) \quad , \quad y = \pm \frac{a}{e}$$

$$ae = 9 \quad , \quad \pm 4 = \pm \frac{a}{e}$$

$$\Rightarrow c = 9 \quad , \quad e = \frac{a}{4}$$

$$\Rightarrow a \left(\frac{a}{4} \right) = 9$$

$$a^2 = 36$$

For hyperbola

$$c^2 = a^2 + b^2$$

$$(9)^2 = 36 + b^2$$

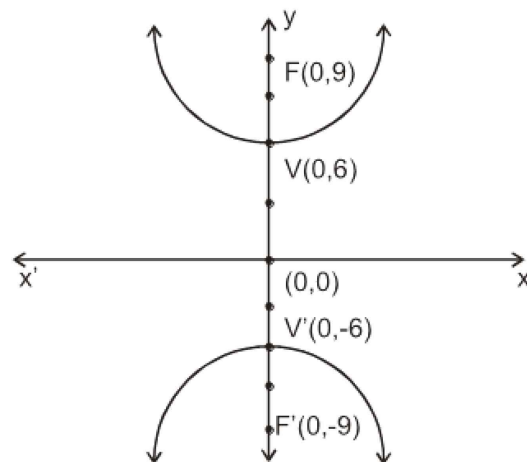
$$81 - 36 = b^2$$

$$b^2 = 45$$

Required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{36} - \frac{x^2}{45} = 1$$



(vi) Centre (2, 2) horizontal transverse axis of length 6 & eccentricity $e = 2$.

Solution:

$$\text{Centre } (2, 2) \text{ \& } 2a = 6 \Rightarrow \boxed{a = 3} \text{ \& } e = 2$$

$$c = ae = 3(2) = 6$$

For hyperbola

$$c^2 = a^2 + b^2$$

$$(6)^2 = (3)^2 + b^2$$

$$36 - 9 = b^2$$

$$b^2 = 27$$

with centre (2, 2) required equation of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$$

$$\text{foci are } (\pm ae, 0) = (\pm 6, 0)$$

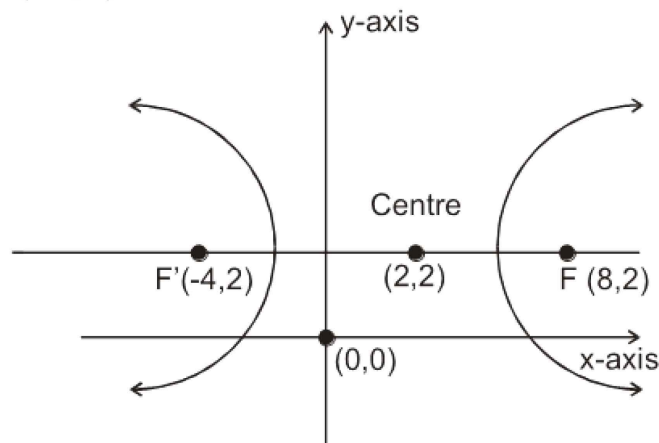
$$(x-2, y-2) = (\pm 6, 0)$$

$$x-2 = \pm 6 \quad y-2 = 0$$

$$x = \pm 6 + 2 \quad y = 2$$

$$x = 8, -4 \quad y = 2$$

$$\therefore F(8, 2) \text{ \& } F'(-4, 2)$$



(vii) Vertices (2, ±3) and (0, 5) lies on the curve.

Solution:

Center of the hyperbola = mid point of vertices

$$= \left(\frac{2+0}{2}, \frac{3+5}{2} \right)$$

$$\text{Center} = (1, 4)$$

$$\text{Since } V = (2, 3) \quad \& \quad V' = (2, -3)$$

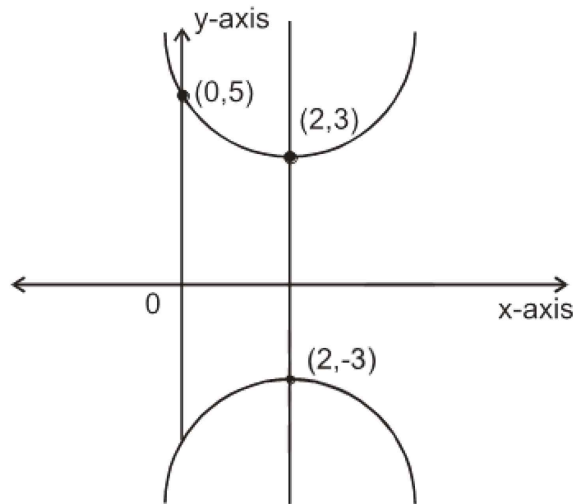
$$2a = |VV'| = \sqrt{(2-2)^2 + (3+3)^2} = 6$$

$$\boxed{a = 3}$$

With center (2, 0) equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-0)^2}{9} - \frac{(x-2)^2}{b^2} = 1 \quad (1)$$



Since (0, 5) lies on (1)

$$\frac{25}{9} - \frac{4}{b^2} = 1$$

$$\frac{25}{9} - 1 = \frac{4}{b^2}$$

$$\frac{16}{9} = \frac{4}{b^2}$$

$$b^2 = \frac{9}{4} \quad (1) \text{ becomes } \frac{y^2}{9} - \frac{(x-2)^2}{\frac{9}{4}} = 1 \quad \text{Ans.}$$

(viii) Foci (5, -2) (5, 4) & One vertex (5, 3)

Solution:

Center = mid point of foci

$$= \left(\frac{5+5}{2}, \frac{-2+4}{2} \right)$$

Center = (5, 1)

$$\begin{aligned}
 c &= \text{distance between center \& focus} \\
 &= \sqrt{(5-5)^2 + (1-4)^2} \\
 &= \sqrt{9} \\
 &= 3
 \end{aligned}$$

$$\boxed{c = 3}$$

$$\begin{aligned}
 a &= \text{distance between vertex \& center} \\
 &= \sqrt{(5-5)^2 + (3-1)^2} \\
 &= \sqrt{4} \\
 &= 2
 \end{aligned}$$

$$\boxed{a = 2}$$

For hyperbola

$$\begin{aligned}
 c^2 &= a^2 + b^2 \\
 (3)^2 &= (2)^2 + b^2 \\
 9 - 4 &= b^2
 \end{aligned}$$

$$\boxed{b^2 = 5}$$

with center (5, 1) required equation of hyperbola is

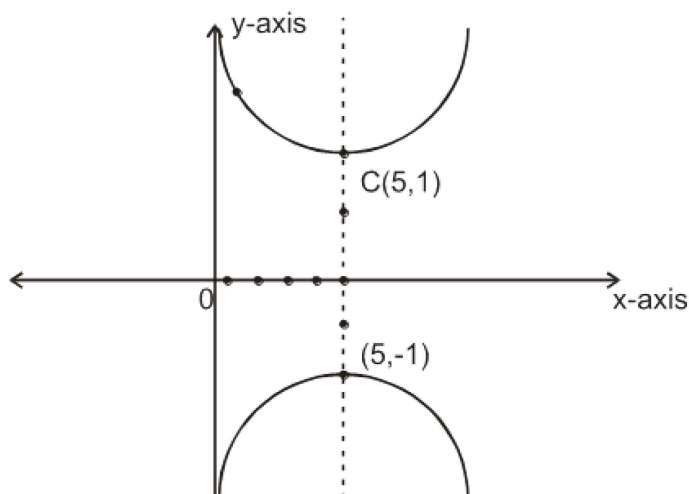
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1$$

The second vertex is

$$(5, 1 - 2)$$

$$(5, -1)$$



Q.2: Find the center, foci, eccentricity, vertices and equations of directrices of each of the following. (i) $x^2 - y^2 = 9$

Solution:

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

$$\Rightarrow a^2 = 9, \quad b^2 = 9$$

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + 9 = 18$$

$$c^2 = 18 \Rightarrow c = 3\sqrt{2}$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{18}{9} = 2$$

$$\boxed{e = \sqrt{2}}$$

$$\text{foci} = (\pm c, 0) = (\pm 3\sqrt{2}, 0)$$

$$\text{Vertices} = (\pm a, 0) = (\pm 3, 0)$$

$$\text{Directrix } x = \pm \frac{a}{e}$$

$$\text{Directrix } x = \pm \frac{3}{\sqrt{2}}$$

Center is (0, 0)

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$a^2 = 4 \quad b^2 = 9$$

$$\therefore c^2 = a^2 + b^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$$

$$\therefore ae = c \Rightarrow e = \frac{c}{a}$$

$$\text{eccentricity} \quad \boxed{e = \frac{\sqrt{13}}{2}}$$

$$\text{foci} = (\pm c, 0) = (\pm \sqrt{13}, 0)$$

$$\text{Vertices} = (\pm a, 0) = (\pm 2, 0)$$

$$\text{Directrix } x = \pm \frac{a}{e}$$

$$x = \pm \frac{2}{\sqrt{13}}$$

$$x = \pm \frac{4}{\sqrt{13}}$$

Center is (0, 0)

$$(iii) \quad \frac{y^2}{16} - \frac{x^2}{9} = 1$$

Solution:

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$$a^2 = 16, \quad b^2 = 9$$

$$\therefore c^2 = a^2 + b^2$$

$$= 16 + 9$$

$$c^2 = 25$$

$$\text{eccentricity } ae = c \Rightarrow e = \frac{c}{a} \Rightarrow \frac{5}{4}$$

$$\boxed{e = \frac{5}{4}}$$

$$\text{foci} = (0, \pm c)$$

$$= (0, \pm 5)$$

$$\text{Vertices} = (0, \pm a) = (0, \pm 4)$$

$$\text{Directrix } y = \pm \frac{a}{e}$$

$$\text{Directrix } y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$$

Center is (0, 0)

$$(iv) \quad \frac{y^2}{4} - x^2 = 1$$

Solution:

$$\frac{y^2}{4} - \frac{x^2}{1} = 1$$

Center = (0, 0)

$$a^2 = 4, \quad b^2 = 1$$

$$c^2 = a^2 + b^2$$

$$c^2 = 4 + 1$$

$$c^2 = 5$$

$$\text{eccentricity } ae = c \Rightarrow e = \frac{c}{a} = \boxed{\frac{\sqrt{5}}{2} = e}$$

$$\text{foci} = (0, \pm c) = (0, \pm \sqrt{5})$$

$$\text{Vertices} = (0, \pm a) = (0, \pm 2)$$

$$\text{Directrix } y = \pm \frac{a}{e}$$

$$\text{Directrix } y = \pm \frac{2}{\frac{\sqrt{5}}{2}}$$

$$= \pm \frac{4}{\sqrt{5}}$$

$$(v) \quad \frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$$

Solution:

$$\text{Let } x-1 = X, \quad y-1 = Y$$

$$\frac{X^2}{2} - \frac{Y^2}{9} = 1$$

$$a^2 = 2, \quad b^2 = 9 \quad c^2 = a^2 + b^2 = 2 + 9 = 11$$

$$ae = c, \quad e = \frac{c}{a} = \frac{\sqrt{11}}{\sqrt{2}}$$

$$\text{Eccentricity } e = \sqrt{\frac{11}{2}}$$

$$\text{Coordinates of foci } (\pm c, 0)$$

$$(X, Y) = (\pm \sqrt{11}, 0)$$

$$(x-1, y-1) = (\pm \sqrt{11}, 0)$$

$$x-1 = \pm \sqrt{11} \quad y-1 = 0$$

$$x = 1 \pm \sqrt{11} \quad y = 1$$

$$\text{foci } (1 \pm \sqrt{11}, 1)$$

$$\text{For center Put } X = 0, \quad Y = 0$$

$$x-1 = 0, \quad y-1 = 0$$

$$\Rightarrow x = 1, \quad y = 1$$

$$\boxed{\text{cenere} = (1, 1)}$$

$$\begin{aligned}
 \text{Vertices } (X, Y) &= (\pm\sqrt{2}, 0) \\
 (x-1, y-1) &= (\pm\sqrt{2}, 0) \\
 x-1 &= \pm\sqrt{2} = y-1 = 0 \\
 x &= 1 \pm \sqrt{2} \quad y = 1
 \end{aligned}$$

$$\text{Vertices are } \boxed{(1 + \sqrt{2}, 1), (1 - \sqrt{2}, 1)}$$

Equation of directrix

$$\begin{aligned}
 X &= \pm \frac{a}{e} \\
 x-1 &= \pm \frac{\sqrt{2}}{\frac{\sqrt{11}}{\sqrt{2}}} \\
 x-1 &= \pm \frac{2}{\sqrt{11}} \\
 \boxed{x &= 1 \pm \frac{2}{\sqrt{11}}}
 \end{aligned}$$

$$(vi) \quad \frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Solution:

$$\text{Let } y+2 = Y, \quad x-2 = X$$

Given equation becomes

$$\frac{Y^2}{9} - \frac{X^2}{16} = 1$$

$$a^2 = 9, \quad b^2 = 16 \quad c^2 = a^2 + b^2 = 9 + 16 = 25$$

$$\Rightarrow c = \pm 5$$

$$ae = c$$

$$\text{Eccentricity } \Rightarrow e = \frac{c}{a} = \frac{5}{3}$$

$$\text{Foci } (X, Y) = (0, \pm c)$$

$$(x-2, y+2) = (0, \pm 5)$$

$$x-2 = 0, \quad y+2 = \pm 5$$

$$x = 2, \quad y = 2 \pm 5$$

$$y = -7, 3$$

$$\text{Foci } (2, -7), (2, 3)$$

For the center put $X = 0$ $Y = 0$

$$\begin{aligned} x - 2 &= 0 & , & & y + 2 &= 0 \\ x &= 2 & , & & y &= -2 \end{aligned}$$

Center $(2, -2)$

$$\begin{aligned} \text{Vertices } (X, Y) &= (0, \pm a) \\ (x - 2, y + 2) &= (0, \pm 3) \\ x - 2 &= 0 & y + 2 &= \pm 3 \\ x &= 2 & y &= -2 \pm 3 \\ & & y &= -5, 1 \end{aligned}$$

Vertices

$$(2, -5) \text{ \& } (2, 1)$$

Equations of directrices are

$$\begin{aligned} Y &= \pm \frac{a}{e} \\ y + 2 &= \pm \frac{3}{\frac{5}{3}} \\ y &= -2 \pm \frac{9}{5} \quad \text{Ans.} \end{aligned}$$

(vii) $9x^2 - 12x - y^2 - 2y + 2 = 0$

Solution:

$$\begin{aligned} 9x^2 - 12x - y^2 - 2y &= -2 \\ 9(x^2 - \frac{12}{9}x) - 1(y^2 + 2y) &= -2 \\ 9(x^2 - \frac{4}{3}x + \frac{4}{9}) - 1(y^2 + 2y + 1) &= -2 + 4 - 1 \\ 9(x - \frac{2}{3})^2 - 1(y + 1)^2 &= 1 \\ \frac{(x - \frac{2}{3})^2}{\frac{1}{9}} - \frac{(y + 1)^2}{1} &= 1 \quad (1) \end{aligned}$$

Let $x - \frac{2}{3} = X$, $y + 1 = Y$ Equation (1) becomes

$$\frac{X^2}{\frac{1}{9}} - \frac{Y^2}{1} = 1 \quad \text{which represents hyperbola}$$

$$a^2 = \frac{1}{9} \quad , \quad b^2 = 1$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{\frac{1}{9} + 1}{\frac{1}{9}} = \frac{\frac{10}{9}}{\frac{1}{9}} = 10$$

$$e = \sqrt{10} \quad c^2 = \frac{10}{9} \Rightarrow c = \frac{\sqrt{10}}{3}$$

Coordinates of foci (X, Y) = ($\pm C$, 0)

$$(x - \frac{2}{3}, y + 1) = (\pm \frac{\sqrt{10}}{3}, 0)$$

$$x - \frac{2}{3} = \pm \frac{\sqrt{10}}{3} \quad y + 1 = 0$$

$$x = \frac{2}{3} \pm \frac{\sqrt{10}}{3} \quad y = -1$$

$$x = \frac{2 \pm \sqrt{10}}{3} \quad , \quad y = -1$$

Foci ($\frac{2 \pm \sqrt{10}}{3}$, -1)

For center put X = 0, Y = 0

$$x - \frac{2}{3} = 0 \quad , \quad y + 1 = 0$$

$$x = \frac{2}{3} \quad y = -1 \quad \text{Center } (\frac{2}{3}, -1)$$

Vertices ($\pm a$, 0)

$$(X + Y) = (\pm \frac{1}{3}, 0)$$

$$(x - \frac{2}{3}, y + 1) = (\pm \frac{1}{3}, 0)$$

$$x - \frac{2}{3} = \pm \frac{1}{3} \quad , \quad y + 1 = 0$$

$$x = \frac{2}{3} \pm \frac{1}{3} \quad y = -1$$

$$\text{Vertices } (1, -1) \quad , \quad (\frac{1}{3}, -1)$$

Equation of directrices are

$$X = \pm \frac{a}{e}$$

$$x - \frac{2}{3} = \pm \frac{1}{3} \div \sqrt{10}$$

$$x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$$

(viii) $4y^2 + 12y - x^2 + 4x + 1 = 0$

Solution:

$$4(y^2 + 3y) - 1(x^2 - 4x) = -1$$

$$4(y^2 + 3y + \frac{9}{4}) - 1(x^2 - 4x + 4) = -1 + 9 - 4$$

$$4(y + \frac{3}{2})^2 - 1(x - 2)^2 = 4$$

$$\frac{\left(y + \frac{3}{2}\right)^2}{1} - \frac{(x-2)^2}{4} = 1 \quad (1)$$

Let $y + \frac{3}{2} = Y$, $x - 2 = X$

Equation (1) becomes

$$\frac{Y^2}{1} - \frac{X^2}{4} = 1 \quad (\text{Represents a hyperbola})$$

$$a^2 = 1, \quad b^2 = 4$$

$$\therefore c^2 = a^2 + b^2 = 1 + 4 = 5 \Rightarrow c = \sqrt{5}$$

Eccentricity $ae = c \Rightarrow e = \frac{c}{a} = \frac{\sqrt{5}}{1} = \boxed{\sqrt{5} = e}$

Foci $(X, Y) = (0, \pm c)$

$$(x - 2, y + \frac{3}{2}) = (0, \pm \sqrt{5})$$

$$x - 2 = 0 \quad y + \frac{3}{2} = \pm \sqrt{5}$$

$$x = 2 \quad y = \frac{-3}{2} \pm \sqrt{5}$$

Foci $(2, \frac{-3}{2} \pm \sqrt{5})$

For center Put $X = 0, Y = 0$

$$x - 2 = 0, \quad y + \frac{3}{2} = 0$$

$$x = 2 \quad y = \frac{-3}{2}$$

Center $(2, \frac{-3}{2})$

Vertices $(X, Y) = (0, \pm a)$

$$(x-2, y+\frac{3}{2}) = (0, \pm 1)$$

$$x-2 = 0, \quad y + \frac{3}{2} = \pm 1$$

$$x = 2, \quad y = \frac{-3}{2} \pm 1$$

$$y = \frac{-1}{2}, \frac{-5}{2}$$

Vertices are $(2, \frac{-1}{2})$, $(2, \frac{-5}{2})$

Equation of directrices are

$$Y = \pm \frac{a}{e}$$

$$y + \frac{3}{2} = \pm \frac{1}{\sqrt{5}}$$

$$y = \frac{-3}{2} \pm \frac{1}{\sqrt{5}} \quad \text{Ans}$$

(ix) $x^2 - y^2 + 8x - 2y - 10 = 0$

Solution:

$$x^2 + 8x - y^2 - 2y = 10$$

$$(x^2 + 8x + 16) - (y^2 + 2y + 1) = 10 + 16 - 1$$

$$(x+4)^2 - (y+1)^2 = 25$$

$$\frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1 \quad (i)$$

Let $x+4 = X$, $y+1 = Y$

(i) Becomes $\frac{X^2}{25} - \frac{Y^2}{25} = 1$ (which represents hyperbola)

$$a^2 = 25, \quad b^2 = 25 \quad c^2 = a^2 + b^2 = 50 \Rightarrow c = \sqrt{50}$$

$$ae = c$$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{50}}{5} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

$$\text{Foci} = (\pm c, 0)$$

$$(X, Y) = (\pm 5\sqrt{2}, 0)$$

$$(x+4, y+1) = (\pm 5\sqrt{2}, 0)$$

$$x+4 = \pm 5\sqrt{2}, \quad y+1 = 0$$

$$x = -4 \pm 5\sqrt{2} \quad y = -1 \quad \text{foci } (-4 \pm 5\sqrt{2}, -1)$$

For center put $X = 0$, $Y = 0$

$$x + 4 = 0, \quad y + 1 = 0 \Rightarrow x = -4, \quad y = -1$$

Center $(-4, -1)$

Vertices $(X, Y) = (\pm a, 0)$

$$(x + 4, y + 1) = (\pm 5, 0)$$

$$x + 4 = \pm 5 \quad y + 1 = 0$$

$$x = -4 \pm 5 \quad y = -1$$

$$= -9, 1$$

Vertices $(-9, -1)$ & $(1, -1)$

Equations of directrices

$$X = \pm \frac{a}{e}$$

$$x + 4 = \pm \frac{5}{\sqrt{2}}$$

$$x = -4 \pm \frac{5}{\sqrt{2}} \quad \text{Ans.}$$

$$(x) \quad 9x^2 - y^2 - 36x - 6y + 18 = 0$$

Solution:

$$9x^2 - 36x - y^2 - 6y = -18$$

$$9(x^2 - 4x) - 1(y^2 + 6y) = -18$$

$$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) = -18 + 36 - 9$$

$$9(x - 2)^2 - 1(y + 3)^2 = 9$$

$$\frac{(x - 2)^2}{1} - \frac{(y + 3)^2}{1} = 1 \quad (i)$$

Let $x - 2 = X$, $y + 3 = Y$ Equation (i) becomes

$$\frac{X^2}{1} - \frac{Y^2}{9} = 1$$

$$a^2 = 1, \quad b^2 = 9$$

$$\therefore c^2 = a^2 + b^2 \Rightarrow c^2 = 1 + 9 = 10 \quad \boxed{c^2 = 10}$$

$$\therefore ae = c$$

$$\text{Eccentricity} \quad \therefore e = \frac{c}{a} = \frac{\sqrt{10}}{1} \quad \boxed{e = \sqrt{10}}$$

$$\text{Foci} \quad (\pm c, 0) = (\pm \sqrt{10}, 0)$$

$$(X, Y) = (\pm \sqrt{10}, 0)$$

$$(x - 2, y + 3) = (\pm \sqrt{10}, 0)$$

$$x - 2 = \pm\sqrt{10} \quad , \quad y + 3 = 0$$

$$x = 2 \pm \sqrt{10} \quad , \quad y = -3$$

Foci $(2 \pm \sqrt{10}, -3)$

For center Put $X = 0$, $Y = 0$

$$x - 2 = 0 \quad y + 3 = 0$$

$$x = 2 \quad , \quad y = -3 \quad \boxed{\text{center } (2, -3)}$$

Vertices $(X, Y) = (\pm a, 0)$

$$(x - 2, y + 3) = (\pm 1, 0)$$

$$x - 2 = \pm 1 \quad y + 3 = 0$$

$$x = 2 \pm 1 \quad y = -3$$

$$x = 3, 1 \quad , \quad y = -3 \quad \boxed{\text{Vertices } (3, -3), (1, -3)}$$

Equations of directrices

$$X = \pm \frac{a}{e}$$

$$x - 2 = \pm \frac{1}{\sqrt{10}}$$

$$x = 2 \pm \frac{1}{\sqrt{10}} \quad \text{Ans}$$

Q.3: Let $0 < a < c$ and $F(-c, 0)$, $F'(c, 0)$ be two fixed points. Show that the set of points $P(x, y)$ such that $|PF| - |PF'| = \pm 2a$ is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Solution:

Given $F(-c, 0)$, $F'(c, 0)$ and $p(x, y)$

$$\text{Also } |PF| - |PF'| = 2a$$

$$\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$

$$\sqrt{(x + c)^2 + y^2} = 2a + \sqrt{(x - c)^2 + y^2}$$

Squaring on both sides

$$(\sqrt{(x + c)^2 + y^2})^2 = (2a + \sqrt{(x - c)^2 + y^2})^2$$

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 + 4a\sqrt{(x - c)^2 + y^2}$$

$$4cx - 4a^2 = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$

$$4(cx - a^2) = 4a^2\sqrt{x^2 + c^2 - 2cx + y^2}$$

Again Squaring

$$c^2x^2 + a^4 - 2cxa^2 = a^2(x^2 + c^2 - 2cx + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 - a^2x^2 - a^2c^2 + 2a^2cx - a^2y^2 = 0$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

Dividing throughout by $c^2 - a^2$

$$\frac{x^2(c^2 - a^2)}{c^2 - a^2} - \frac{a^2y^2}{c^2 - a^2} = \frac{a^2(c^2 - a^2)}{c^2 - a^2}$$

$$x^2 - \frac{a^2y^2}{c^2 - a^2} = a^2$$

Dividing throughout by a^2 $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$ Hence proved.

Q.4: Find an equation of the hyperbola with foci $(-5, -5)$ & $(5, 5)$ vertices $(-3\sqrt{2}, -3\sqrt{2})$ and $(3\sqrt{2}, 3\sqrt{2})$.

Solution:

Since $2a =$ distance between two vertices

$$2a = \sqrt{(3\sqrt{2} + 3\sqrt{2})^2 + (3\sqrt{2} + 3\sqrt{2})^2}$$

$$2a = \sqrt{(6\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{72 + 72} = \sqrt{144} = 12$$

$$2a = 12 \quad \boxed{a = 6}$$

Also given $F(-5, -5)$ & $F'(5, 5)$. Let $P(x, y)$ be any point on the hyperbola such that $|PF| - |PF'| = 2a$

$$\sqrt{(x+5)^2 + (y+5)^2} - \sqrt{(x-5)^2 + (y-5)^2} = 12$$

Squaring $\sqrt{(x+5)^2 + (y+5)^2} = 12 + \sqrt{(x-5)^2 + (y-5)^2}$

$$x^2 + 25 + 10x + y^2 + 25 + 10y = x^2 + 25 - 10x + y^2 + 25 - 10y + 144 + 24\sqrt{(x-5)^2 + (y-5)^2}$$

$$20x + 20y - 144 = 24\sqrt{(x-5)^2 + (y-5)^2}$$

$$4(5x + 5y - 36) = 4[6\sqrt{(x-5)^2 + (y-5)^2}]$$

Again squaring on both sides

$$(5x + 5y - 36)^2 = 36(x^2 + 25 - 10x + y^2 + 25 - 10y)$$

$$25x^2 + 5y^2 + 1296 + 50xy - 360y - 360x = 36x^2 + 900 - 360x + 36y^2 + 900 - 360y$$

$$25x^2 + 25y^2 + 1296 + 50xy - 360y - 360x - 36x^2 - 900 + 360x - 36y^2 - 900 + 360y = 0$$

$$11x^2 - 50xy - 11y^2 + 504 = 0 \quad \text{Ans}$$

Q.5: For any point on hyperbola the difference of its distances from the points $(2, 2)$ & $(10, 2)$ is 6. Find an equation of hyperbola.

Solution:

Given $2a = 6$ $F = (2, 2)$ & $F' = (10, 2)$

Let $P(x, y)$ be any point on the hyperbola, such that

$$|PF| - |PF'| = 2a$$

$$\begin{aligned}\sqrt{(x-2)^2 + (y-2)^2} - \sqrt{(x-10)^2 + (y-2)^2} &= 6 \\ \sqrt{(x-2)^2 + (y-2)^2} &= 6 + \sqrt{(x-10)^2 + (y-2)^2}\end{aligned}$$

Squaring

$$\begin{aligned}x^2 + 4 - 4x + y^2 + 4 - 4y &= 36 + x^2 + 100 - 20x + y^2 + 4 - 4y + 12 \\ \sqrt{(x-10)^2 + (y-2)^2} \\ 16x - 132 &= 12 \sqrt{(x-10)^2 + (y-2)^2} \\ 4(4x - 33) &= 12 \sqrt{(x-10)^2 + (y-2)^2} \\ 4x - 33 &= 3 \sqrt{(x-10)^2 + (y-2)^2}\end{aligned}$$

Again squaring

$$\begin{aligned}16x^2 + 1089 - 264x &= 9(x^2 + 100 - 20x + y^2 + 4 - 4y) \\ 16x^2 + 1089 - 264x - 9x^2 + 900 + 180x - 9y^2 - 36 + 36y &= 0 \\ 7x^2 - 9y^2 - 84x + 36y + 153 &= 0 \quad \text{Ans}\end{aligned}$$

Q.6: Two listening posts hear the sound of an enemy gun. The difference in time is one second. If the listening posts are 1400 feet apart, write an equation of the hyperbola passing through the position of the enemy gun.

Solution:

Since the difference in time is one second, so let the listening posts F & F' near the second of the enemy gun after t, t-1 seconds respectively. If P is position of enemy then
 $|PF| - |PF'| = 2a \quad \dots\dots (1)$

Since posts are 1400 feet apart, so

$$2c = 1400 \Rightarrow c = 700$$

since distance = (Velocity) (time)

So $|PF| = (1080)t$ & $|PF'| = (1080)(t-1)$

Putting these values in (1)

$$(1080)t - (1080)(t-1) = 2a$$

$$1080t - 1080t + 1080 = 2a$$

$$a = \frac{1080}{2} = 540$$

For hyperbola, we know that $c^2 = a^2 + b^2$

$$(700)^2 = (540)^2 + b^2$$

$$b^2 = 490000 - 291600$$

$$b^2 = 198400$$

Required equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x^2}{291600} - \frac{y^2}{198400} = 1 \quad \text{Ans}$$

Tangents to Conics

(I) Parabola

Equations of Tangent in different forms

(i) Point form:

The equation of the tangent to the parabola

$$y^2 = 4ax \text{ at the point } (x, y) \text{ is } yy_1 = 2a(x + x_1)$$

(ii) Slope form:

The equation of the tangent to the parabola

$$y^2 = 4ax \text{ in the term of slope 'm' is}$$

$$y = mx + \frac{a}{m} \quad (\because c = \frac{a}{m})$$

Note:

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1 , y^2 by yy_1 , x by $\frac{1}{2}(x + x_1)$, y by $\frac{1}{2}(y + y_1)$ and xy by $\frac{xy_1 + yx_1}{2}$

(II) Ellipse

Equations of tangents in different forms.

(i) Point form:

Equations of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope m is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad (\because c = \pm \sqrt{a^2m^2 + b^2})$$

(III) Hyperbola

Equations of tangent in different forms

(i) Point form:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(ii) Slope form:

The equation of tangent to hyperbola in terms of 'm' is

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad (c = \pm \sqrt{a^2m^2 - b^2})$$

(IV) Circle

Equations of tangent in different forms

(i) Point form:

Equation of tangent to the circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) Slope form:

Equation of tangent in terms of slope 'm' is $y = mx \pm a\sqrt{1+m^2}$
($\therefore c^2 = a^2(1+m^2)$)

Equations of Normal

(i) Parabola $y^2 = 4ax$ is at (x_1, y_1)

$$y - y_1 = \frac{-y_1}{2a} (x - x_1)$$

(ii) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(iii) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

EXERCISE 6.7

Q.1: Find equations of tangent and normal to each of the following at the indicated point.

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Solution:

Equation of tangent at $(at^2, 2at)$ is

$$yy_1 = 2a(x + x_1)$$

$$y(2at) = 2a(x + at^2)$$

$$2ayt = 2ax + 2a^2t^2$$

$$2ayt = 2a(x + at^2)$$

$$yt = x + at^2$$

And equation of normal at $(at^2, 2at)$ is

$$y - y_1 = \frac{-y_1}{2a} (x - x_1)$$

$$y - 2at = \frac{-2at}{2a} (x - at^2)$$