

(IV) Circle

Equations of tangent in different forms

(i) Point form:

Equation of tangent to the circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) Slope form:

Equation of tangent in terms of slope 'm' is $y = mx \pm a\sqrt{1+m^2}$
($\therefore c^2 = a^2(1+m^2)$)

Equations of Normal

(i) Parabola $y^2 = 4ax$ is at (x_1, y_1)

$$y - y_1 = \frac{-y_1}{2a} (x - x_1)$$

(ii) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(iii) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

EXERCISE 6.7

Q.1: Find equations of tangent and normal to each of the following at the indicated point.

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Solution:

Equation of tangent at $(at^2, 2at)$ is

$$yy_1 = 2a(x + x_1)$$

$$y(2at) = 2a(x + at^2)$$

$$2ayt = 2ax + 2a^2t^2$$

$$2ayt = 2a(x + at^2)$$

$$yt = x + at^2$$

And equation of normal at $(at^2, 2at)$ is

$$y - y_1 = \frac{-y_1}{2a} (x - x_1)$$

$$y - 2at = \frac{-2at}{2a} (x - at^2)$$

$$y - 2at = -tx + at^3$$

$$tx + y - 2at - at^3 = 0$$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$

Solution:

Equation of tangent $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$$\frac{x(a \cos \theta)}{a^2} + \frac{y(b \sin \theta)}{b^2} = 1$$

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

and equation of normal

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\frac{a^2x}{a \cos \theta} - \frac{b^2y}{b \sin \theta} = a^2 - b^2$$

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

(iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$

Solution:

Equation of tangent

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\frac{x a \sec \theta}{a^2} - \frac{y b \tan \theta}{b^2} = 1$$

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

And equation of normal

$$\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

$$\frac{xa^2}{a \sec \theta} + \frac{yb^2}{b \tan \theta} = a^2 + b^2$$

$$\frac{xa}{\sec \theta} + \frac{yb}{\tan \theta} = a^2 + b^2$$

OR $x a \cos \theta + y b \cot \theta = a^2 + b^2$

Q.2: Write equation of the tangent to the given conic at the indicated point

(i) $3x^2 = -16y$ at the points whose ordinate is -3

Solution:

$$3x^2 = -16y \quad \dots\dots (1)$$

Put $y = -3$ in

$$3x^2 = -16(-3)$$

$$3x^2 = 48 \quad \Rightarrow \quad x^2 = 16$$

$$\Rightarrow x = \pm 4$$

Hence points are

$(4, -3)$ & $(-4, -3)$

Now diff. (1) w.r.t 'x'

$$6x = -16 \frac{dy}{dx}$$

$$\frac{6x}{-16} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-3}{8} x$$

$$m = \text{Slope} = \frac{dy}{dx} \Big|_{(4, -3)} = \frac{-3}{8} (4) = \frac{-3}{2}$$

$$\text{Also } m = \frac{dy}{dx} \Big|_{(-4, -3)} = \frac{-3}{8} (4) = \frac{3}{2}$$

\therefore Equation of tangent at $(4, -3)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 3 = \frac{-3}{2} (x - 4)$$

$$2y + 6 = -3x + 12$$

$$3x + 2y = 6$$

$$3x + 2y - 6 = 0$$

Equation of tangent at $(-4, -3)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 3 = \frac{+3}{2} (x + 4)$$

$$2y + 6 = 3x + 12$$

$$3x - 2y = -6$$

$$3x - 2y + 6 = 0$$

(ii) $3x^2 - 7y^2 = 20$ at points where $y = -1$.

Solution:

$$3x^2 - 7y^2 = 20 \quad \dots\dots (1)$$

Put $y = -1$ in (1)

$$3x^2 - 7(-1)^2 = 20$$

$$3x^2 = 20 + 7$$

$$3x^2 = 27 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

Thus the required points on the conic are $(3, -1)$ & $(-3, -1)$

Now diff (1) w.r.t. 'x' we have

$$6x - 14y \frac{dy}{dx} = 0$$

$$14 \frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{14y} = \frac{3x}{7y}$$

$$\text{Now } m = \text{Slope} = \left. \frac{dy}{dx} \right|_{(3, -1)} = \frac{9}{-7} \text{ Also } m = \left. \frac{dy}{dx} \right|_{(-3, -1)} = \frac{9}{7}$$

Therefore equation of tangent at (3, -1) is

$$y - y_1 = m(x - x_1)$$

$$y + 1 = \frac{-9}{7} (x - 3)$$

$$7y + 7 = -9x + 27$$

$$9x + 7y = 20$$

$$9x + 7y - 20 = 0 \quad \text{Ans}$$

Equation of tangent at (-3, -1)

$$y - y_1 = m(x - x_1)$$

$$y + 1 = \frac{9}{7} (x + 3)$$

$$7y + 7 = 9x + 27$$

$$9x - 7y = -20$$

$$9x - 7y + 20 = 0 \quad \text{Ans}$$

(iii) $3x^2 - 7y^2 + 2x - y - 48 = 0$, at point where $x = 4$

Solution:

$$3x^2 - 7y^2 + 2x - y - 48 = 0 \quad \dots (1)$$

Put $x = 4$ in (1)

$$3(4)^2 - 7y^2 + 2(4) - y - 48 = 0$$

$$48 - 7y^2 + 8 - y - 48 = 0$$

$$\Rightarrow -7y^2 - y + 8 = 0 \quad \Rightarrow 7y^2 + y - 8 = 0$$

$$7y^2 + 8y - 7y - 8 = 0$$

$$y(7y + 8) - 1(7y + 8) = 0$$

$$(7y + 8)(y - 1) = 0$$

Either

$$7y + 8 = 0, \quad y - 1 = 0$$

$$y = \frac{-8}{7}, \quad y = 1$$

Therefore, required points on the conic are $(4, \frac{-8}{7})$ & $(4, 1)$

$$\text{Now diff. (1) w.r.t. 'x'} \quad 6x - 14y \frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$

$$(-14y - 1) \frac{dy}{dx} = -6x - 2$$

$$\frac{dy}{dx} = \frac{6x+2}{14y+1}$$

$$m = \frac{dy}{dx} \Big|_{(4,1)} = \frac{6(4)+2}{14(1)+1} = \frac{26}{15} \quad \text{Also } m = \frac{dy}{dx} \Big|_{(4, -\frac{8}{7})} = \frac{6(4)+2}{14\left(\frac{-8}{7}\right)+1} = \frac{26}{-15}$$

Equation of tangent at (4, 1) is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{26}{15} (x - 4)$$

$$15y - 15 = 26x - 104$$

$$26x - 15y - 89 = 0 \quad \text{Ans}$$

Equation of tangent at $(4, -\frac{8}{7})$ is

$$y - y_1 = m(x - x_1)$$

$$y + \frac{8}{7} = \frac{-26}{15} (x - 4)$$

$$105y - 120 = -182x + 728$$

$$182x + 105y - 608 = 0 \quad \text{Ans}$$

Q.3: Find equations of the tangents to each of the following through the given point

(i) $x^2 + y^2 = 25$, through (7, -1)

Solution:

$$x^2 + y^2 = 25 \Rightarrow r = 5$$

We know that condition of tangency for the circle is

$$c^2 = r^2 (1 + m^2)$$

$$c^2 = 25 (1 + m^2)$$

$$\Rightarrow c = \pm 5 \sqrt{1 + m^2}$$

Let the required equation of tangent be

$$y = mx + c \quad \dots (1) \quad \text{Putting value of } C \text{ in (1)}$$

$$y = mx \pm 5 \sqrt{1 + m^2} \quad \dots (2)$$

Since tangent line passes through point (7, -1), therefore

$$-1 = 7m \pm 5 \sqrt{1 + m^2}$$

$$\pm 5 \sqrt{1 + m^2} = 7m + 1 \quad \text{Squaring}$$

$$25(1 + m^2) = (7m + 1)^2$$

$$25 + 25m^2 = 49m^2 + 1 + 14m$$

$$-24m^2 - 14m + 24 = 0$$

$$12m^2 + 7m - 12 = 0$$

$$12m^2 + 16m - 9m - 12 = 0$$

$$4m(3m + 4) - 3(3m + 4) = 0$$

$$(3m + 4)(4m - 3) = 0$$

$$m = \frac{-4}{3} \quad m = \frac{3}{4}$$

with $m = \frac{-4}{3}$ (2) becomes

$$y = -\frac{4}{3}x \pm 5\sqrt{1 + \frac{16}{9}}$$

$$= -\frac{4}{3}x \pm 5\frac{5}{3}$$

$$3y = -4x \pm 25$$

$$4x + 3y \pm 25 = 0$$

(ii) $y^2 = 12x$ through (1, 4)

Solution:

$$y^2 = 12x$$

As standard form is

$$y^2 = 4ax$$

$$4a = 12 \Rightarrow a = 3$$

Let $y = mx + c$ (1) be the required equation of tangent. For Parabola we know that condition of tangency is $c = \frac{a}{m} = \frac{3}{m}$ put in (1)

$$y = mx + \frac{3}{m} \text{ (2)}$$

Since tangent line passes through point (1, 4)

So (2) becomes

$$4 = m + \frac{3}{m} \Rightarrow 4m = m^2 + 3$$

$$m^2 - 4m + 3 = 0$$

$$(m - 1)(m - 3) = 0$$

$$m = 1, \quad m = 3 \quad \text{Put in (1)}$$

$$y = x + 3 \quad \& \quad y = 3x + \frac{3}{m}$$

$$x - y + 3 = 0 \quad y = 3x + \frac{3}{3}$$

$$y = 3x + 1$$

$$3x - y + 1 = 0 \quad \text{Ans}$$

(iii) $x^2 - 2y^2 = 2$ through (1, -2)

Solution:

$$x^2 - 2y^2 = 2$$

$$\frac{x^2}{2} - \frac{y^2}{1} = 1$$

$$\Rightarrow a^2 = 2, \quad b^2 = 1$$

with $m = \frac{3}{4}$ (2) becomes

$$y = \frac{3x}{4} \pm 5\sqrt{1 + \frac{9}{16}}$$

$$= \frac{3x}{4} \pm \frac{25}{4}$$

$$4y = 3x \pm 25$$

$$3x - 4y \pm 25 = 0$$

For hyperbola, we know that condition of tangent is

$$c^2 = a^2 m^2 - b^2$$

$$\Rightarrow c^2 = 2m^2 - 1 \Rightarrow c = \pm \sqrt{2m^2 - 1}$$

Let $y = mx + c$ be tangent to the given hyperbola then $y = mx \pm \sqrt{2m^2 - 1}$ (1)

Since (1) passes through (1, -2) (1) becomes

$$-2 = m \pm \sqrt{2m^2 - 1}$$

$$-2 - m = \pm \sqrt{2m^2 - 1} \quad \text{Squaring}$$

$$4 + m^2 + 4m = 2m^2 - 1$$

$$2m^2 - 1 - m^2 - 4m - 4 = 0$$

$$m^2 - 4m - 5 = 0$$

$$\Rightarrow (m - 5)(m + 1) = 0$$

$$\Rightarrow m = 5, \quad m = -1$$

Putting values of m in (1) we get

$$y = 5x \pm \sqrt{2(25) - 1}, \quad y = -x \pm \sqrt{2 - 1}$$

$$y = 5x \pm \sqrt{49}, \quad = -x \pm 1$$

$$y = 5x \pm 7, \quad y + x \pm 1 = 0$$

$$5x - y \pm 7 = 0 \quad \text{Ans}$$

Q.4: Find equations of normal to the Parabola $y^2 = 8x$, which are parallel to the line $2x + 3y = 10$.

Solution:

$$y^2 = 8x \quad \dots\dots (1)$$

$$2x + 3y = 10 \quad \dots\dots\dots (2)$$

Diff. (1) w.r.t. 'x'

$$2y \frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

$$m_1 = \frac{dy}{dx} = \frac{4}{y}$$

$$m_1 = \text{Slope of normal} = \frac{-y}{4}$$

$m_2 =$ Slope of line

$$= \frac{-\text{coeff of } x}{\text{coeff of } y}$$

$$= -\frac{2}{3}$$

Since normal and given line are Parallel

$$m_1 = m_2$$

$$\frac{-y}{4} = -\frac{2}{3} \Rightarrow y = \frac{8}{3} \quad \text{Put in (1)}$$

$$\left(\frac{8}{3}\right)^2 = 8x$$

$$\frac{64}{9 \times 8} = x \quad \Rightarrow \quad x = \frac{8}{9}$$

Required point $(\frac{8}{9}, \frac{8}{3})$

with $y = \frac{8}{3}$, m_1 become

$$m_1 = -\frac{8}{3} \times \frac{1}{4} = -\frac{2}{3}$$

Required equation of normal at $(\frac{8}{9}, \frac{8}{3})$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{8}{3} = \frac{-2}{3} (x - \frac{8}{9})$$

$$3y - 8 = -2 \left(\frac{9x - 8}{9} \right)$$

$$27y - 72 = -18x + 16$$

$$18x + 27y - 88 = 0$$

Q.5: Find equations of tangents to the ellipse $\frac{x^2}{4} + y^2 = 1$, which are parallel to the line $2x - 4y + 5 = 0$.

Solution:

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$2x - 4y + 5 = 0$$

$$\Rightarrow a^2 = 4, \quad b^2 = 1 \quad m = \frac{-\text{coeff of } x}{\text{coeff of } y} = \frac{-2}{-4} = \frac{1}{2}$$

We know that condition of tangent for ellipse is

$$c^2 = a^2 m^2 + b^2$$

$$c^2 = 4m^2 + 1$$

$$c = \pm \sqrt{4m^2 + 1}$$

Since tangent is parallel to line $2x - 4y + 5 = 0$

$$\therefore \text{Slope is also } m = \frac{1}{2}$$

$$c = \pm \sqrt{4 \left(\frac{1}{4} \right) + 1} = \pm \sqrt{2}$$

Let the equation of required tangent by

$$y = mx + c$$

$$y = \frac{1}{2}x \pm \sqrt{2}$$

$$2y = x \pm 2\sqrt{2}$$

$$x - 2y \pm 2\sqrt{2} = 0 \quad \text{Ans}$$

Q.6: Find equations of the tangents to the conics $9x^2 - 4y^2 = 36$ Parallel to $5x - 2y + 7 = 0$.

Solution:

$$9x^2 - 4y^2 = 36$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \quad (\text{Dividing by 36})$$

$$\Rightarrow a^2 = 4, \quad b^2 = 9$$

$$5x - 2y + 7 = 0$$

$$m = \text{slope of line} = \frac{5}{2}$$

For hyperbola, we know that

$$c^2 = a^2m^2 - b^2$$

$$c^2 = 4m^2 - 9$$

Since tangent and given line are parallel so their slopes are same. Thus $m = \frac{5}{2}$

$$c^2 = 4\left(\frac{25}{4}\right) - 9 \quad c^2 = 16 \quad \Rightarrow \quad c = \pm 4$$

Let $y = mx + c$ be the required equation of the tangent then $y = \frac{5}{2}x \pm 4$

$$2y = 5x \pm 8$$

$$5x - 2y \pm 8 = 0 \quad \text{Ans.}$$

Q.7: Find equations of common tangents to the given conics.

$$(i) \quad x^2 = 80y \quad \& \quad x^2 + y^2 = 81$$

Solution:

$$x^2 = 80y \quad \dots (1) \quad x^2 + y^2 = 81 \quad \dots (2)$$

Let $y = mx + c$ (3) be the required common tangent. Let a be radius of circle then (2) becomes $a^2 = 81$ Put in (1)

$$x^2 = 80(mx + c)$$

$$x^2 - 80mx - 80c = 0$$

For equal roots, we know that $\text{Disc} = 0$

$$b^2 - 4ac = 0$$

$$(-80m)^2 - 4(1)(-80c) = 0$$

$$80(80m^2 + 4c) = 0$$

$$80m^2 + 4c = 0 \quad c = -20m^2$$

$$\text{Condition of tangency for circle is } c^2 = a^2(1 + m^2) \quad (4)$$

$$(-20m^2)^2 = 81(1 + m^2)$$

$$400m^4 = 81 + 81m^2$$

$$400m^4 - 81m^2 - 81 = 0$$

By Quadratic Formula

$$m^2 = \frac{-(-81) \pm \sqrt{(-81)^2 - 4(400)(-81)}}{2(400)}$$

$$= \frac{81 \pm \sqrt{136161}}{800} = \frac{9}{16}$$

$$m = \pm \frac{3}{4}$$

$$\therefore c = -20\left(\frac{9}{16}\right) = -\frac{45}{4}$$

Putting values of m & c in $y = mx + c$

$$y = \pm \frac{3}{4}x - \frac{45}{4}$$

$$4y = \pm 3x - 45$$

$$\pm 3x - 4y - 45 = 0 \quad \text{Ans.}$$

$$(ii) \quad y^2 = 16x \quad \& \quad x^2 = 2y$$

Solution:

$$y^2 = 16x \quad \dots (1) \quad x^2 = 2y \quad \dots (2)$$

$$y^2 = 4ax$$

$$4a = 16$$

$$\boxed{a = 4}$$

We know that condition of tangency for Parabola is $c = \frac{a}{m}$

$$c = \frac{4}{m}$$

Let $y = mx + c \dots\dots (3)$ be required tangent

then $y = mx + \frac{4}{m}$ Putting value of y in (2)

$$x^2 = 2\left(mx + \frac{4}{m}\right) \Rightarrow mx^2 = 2m^2x + 8$$

$$mx^2 - 2m^2x - 8 = 0$$

For equal roots, we know that Disc = 0

$$\text{i.e; } b^2 - 4ac = 0$$

$$(-2m^2)^2 - 4(m)(-8) = 0$$

$$4m^4 + 32m = 0$$

$$4m(m^3 + 8) = 0$$

$$m = 0, \quad m^3 = -8, \quad m = -2$$

Equation of tangent is

$$y = mx + c$$

$$y = -2x + \frac{4}{-2}$$

$$y = -2x - 2$$

$$\boxed{2x + y + 2 = 0} \quad \text{Ans.}$$

Q.8: Find the points of intersection of the given conics.

$$(i) \quad \frac{x^2}{18} + \frac{y^2}{8} = 1 \quad \& \quad \frac{x^2}{3} - \frac{y^2}{3} = 1$$

Solution:

$$\frac{x^2}{18} + \frac{y^2}{8} = 1 \quad \& \quad \frac{x^2}{3} - \frac{y^2}{3} = 1$$

$$8x^2 + 18y^2 = 144 \quad x^2 - y^2 = 3 \quad \dots\dots (2)$$

$$4x^2 + 9y^2 = 72 \quad \dots (1) \quad (\text{Dividing by 2})$$

Multiplying Eq. (2) by 9 & add in (1)

$$9x^2 - 9y^2 = 27$$

$$\frac{4x^2 + 9y^2 = 72}{13x^2 = 99}$$

$$13x^2 = 99$$

$$x^2 = \frac{99}{13} \quad \Rightarrow \quad x = \pm \sqrt{\frac{99}{13}}$$

Put in (2)

$$\frac{99}{13} - y^2 = 3$$

$$\frac{99}{13} - 3 = y^2$$

$$\frac{99 - 39}{13} = y^2$$

$$y^2 = \frac{60}{13} \quad \Rightarrow \quad y = \pm \sqrt{\frac{60}{13}}$$

Points of intersection are $\left(\pm \sqrt{\frac{99}{13}}, \pm \sqrt{\frac{60}{13}} \right)$ Ans.

(ii) $x^2 + y^2 = 8$ & $x^2 - y^2 = 1$

Solution:

$$x^2 + y^2 = 8 \dots (1) \quad x^2 - y^2 = 1 \dots (2)$$

Adding (1) & (2)

$$x^2 + y^2 = 8$$

$$x^2 - y^2 = 1$$

$$2x^2 = 9 \Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \pm \frac{3}{\sqrt{2}}$$

Put in (1) $\frac{9}{2} + y^2 = 8$

$$y^2 = 8 - \frac{9}{2}$$

$$y^2 = \frac{16-9}{2} = \frac{7}{2}$$

$$y = \pm \sqrt{\frac{7}{2}}$$

Hence points of intersection are $\left(\pm \frac{3}{\sqrt{2}}, \pm \sqrt{\frac{7}{2}} \right)$ Ans

(iii) $3x^2 - 4y^2 = 12$ & $3y^2 - 2x^2 = 7$

Solution:

$$3x^2 - 4y^2 = 12 \dots\dots\dots (1)$$

$$3y^2 - 2x^2 = 7 \dots\dots\dots (2)$$

Multiplying equation (1) by (2) & (2) by 3 and adding

$$6x^2 - 8y^2 = 24$$

$$-6x^2 + 9y^2 = 21$$

$$y^2 = 45 \Rightarrow y = \pm \sqrt{45}$$

Put in (2)

$$-2x^2 + 3(45) = 7$$

$$-2x^2 + 135 = 7$$

$$135 - 7 = 2x^2$$

$$128 = 2x^2$$

$$x^2 = 64 \Rightarrow x = \pm 8$$

Hence points of intersection are

$$(\pm 8, \pm \sqrt{45}) \text{ Ans.}$$

(iv) $3x^2 + 5y^2 = 60$ and $9x^2 + y^2 = 124$

Solution:

$$3x^2 + 5y^2 = 60 \quad \dots\dots (1) \quad \quad 9x^2 + y^2 = 124 \quad \dots\dots (2)$$

Multiplying (1) by (3) & Subtracting from (2)

$$9x^2 + y^2 = 124$$

$$\underline{-9x^2 + 15y^2 = -180}$$

$$-14y^2 = -56$$

$$y^2 = 4 \quad \Rightarrow \quad y = \pm 2$$

Put in (1)

$$9x^2 + 4 = 124$$

$$9x^2 = 120$$

$$x^2 = \frac{120}{9} = \frac{40}{3} \quad x = \pm \sqrt{\frac{40}{3}}$$

Hence points of intersection are $\left(\pm \sqrt{\frac{40}{3}} \pm 2 \right)$