EXERCISE 6.9

1. By a rotation of axes, eliminate the xy-term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2 - 4xy + y^2 - 6 = 0$

Solution. $4x^2 - 4xy + y^2 - 6 = 0$ ... (I)

Here $a = 4$, $b = 1$, $2h = -4$ the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-1} = -\frac{4}{3} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{4}{3}$$

$$6\tan\theta = 4\tan^2\theta - 4 \Rightarrow 4\tan^2\theta - 6\tan\theta - 4 = 0$$

$$2\tan^2\theta - 3\tan q - 2 = 0$$

$$\tan\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$= 2, -\frac{1}{2} \Rightarrow \tan\theta = 2 \text{ (as } \theta \text{ is in the first quadrant)}$$

Now $\tan\theta = 2 = \frac{2}{1} \Rightarrow$ base $= 1$, $\perp = 2$, hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = \frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y \\ y &= X\sin\theta + Y\cos\theta = \frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y \end{aligned}\right\} \quad \dots (2)$$

Substituting these expressions for x and y into (1), we get

$$4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)^2 - 4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)\left(\frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y\right)$$
$$+\left(\frac{2}{\sqrt{5}}X + \frac{2}{\sqrt{5}}Y\right)^2 - 6 = 0$$

$$4\left(\frac{1}{5}X^2 - \frac{4}{5}XY + \frac{4}{5}Y^2\right) - 4\left(\frac{2}{5}X^2 - \frac{3}{5}XY + \frac{2}{5}Y^2\right)$$
$$+\left(\frac{4}{5}X^2 - \frac{4}{5}XY + \frac{1}{5}Y^2\right) - 6 = 0$$

$$\left(\frac{4}{5} - \frac{8}{5} + \frac{4}{5}\right)X^2 + \left(-\frac{16}{5} + \frac{12}{5} + \frac{4}{5}\right)XY + \left(\frac{16}{5} + \frac{8}{5} + \frac{1}{5}\right)Y^2 - 6 = 0$$

$$25Y^2 - 30 = 0 \Rightarrow Y^2 = \frac{6}{5} \Rightarrow Y = \pm\sqrt{\frac{6}{5}}$$

represents a pair of lines. To find their equations in xy - plane, we have

From (2), we have

$$X - 2Y = \sqrt{5}\,x \quad (3)$$

$$2X + Y = \sqrt{5}\,y \quad (4)$$

Multiplying (3) by 2, we get

$$2X - 4Y = 2\sqrt{5}\,x \quad (5)$$

Subtracting equation (5) from (6), we get

$$5Y = \sqrt{5}\ y - 2\sqrt{5}\ x \quad \Rightarrow \quad Y = \frac{\sqrt{5}}{5}(y - 2x) = -\frac{1}{\sqrt{5}}(2x - y)$$

$$\pm\sqrt{\frac{6}{5}} = -\frac{1}{\sqrt{5}}(2x - y) \quad \Rightarrow \quad \pm\sqrt{6} = -(2x - y)$$

$$2x - y \pm \sqrt{6} = 0 \quad \Rightarrow \quad 2x - y + \sqrt{6} = 0\ ,\ 2x - y - \sqrt{6} = 0$$

(ii) Identify : $x^2 - 2xy + y^2 - 8x - 8y = 0$

Solution. $x^2 - 2xy + y^2 - 8x - 8y = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = -2$ the angle θ though which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-2}{1 - 1} = \infty \quad \Rightarrow \quad 2\theta = 90^\circ \quad \Rightarrow \quad \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{aligned}\right] \quad ... (2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 8\left(\frac{X - Y}{\sqrt{2}}\right) - 8\left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) - \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2X.Y + Y^2)$$

$$- \frac{8}{\sqrt{2}}(X - Y) - \frac{8}{\sqrt{2}}(X + Y) = 0$$

$$X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 8\sqrt{2}\ X$$

$$+ 8\sqrt{2}\ Y - 8\sqrt{2}\ X - 8\sqrt{2}\ Y = 0$$

$$4Y^2 - 16\sqrt{2}\ X = 0 \quad \Rightarrow \quad Y^2 = 4\sqrt{2}\ X \qquad (3)$$

which represents a parabola. In xy-plane, we have

From 2i), we have

$$X - Y = \sqrt{2}\ x \qquad (4)$$

and $$X + Y = \sqrt{2}\ y \qquad (5)$$

Adding (3) and (4), we get

$$2X = \sqrt{2}\,(x + y) \quad \Rightarrow \quad X = \frac{1}{\sqrt{2}}(x + y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x + y) - Y = \sqrt{2}\,x \quad \Rightarrow \quad Y = -\sqrt{2}\,x + \frac{1}{\sqrt{2}}(x + y)$$

$$= \frac{-2x + x + y}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y - x)$$

Thus $X = \frac{1}{\sqrt{2}}(x + y)$ and $Y = \frac{1}{\sqrt{2}}(y - x)$

Elements of the parabola are

Focus of (3) is $Y = 0$, $X = \sqrt{2}$

i.e., $\frac{1}{\sqrt{2}}(x + y) = \sqrt{2}$ and $\frac{1}{\sqrt{2}}(y - x) = 0$

$x + y = 2$ and $y - x = 0$

Adding: $x + y = 2$

$\underline{-x + y = 0}$

$2y = 2 \Rightarrow y = 1$

Put $y = 1$ in $x + y = 2$, we get

$x + 1 = 2 \Rightarrow x = 1$

$(1, 1)$ is the focus of (1)

Vertex of (3) is $X = 0$, $Y = 0$

i.e., $\frac{1}{\sqrt{2}}(x + y) = 0 \Rightarrow x + y = 0$

and $\frac{1}{\sqrt{2}}(y - x) = 0 \Rightarrow -x + y = 0$

Solving, we get $x = 0$, $y = 0$

Vertex: $(0, 0)$ is the vertex of (1).

Axis: $Y = 0$ i.e., $\frac{1}{\sqrt{2}}(y - x) = 0 \Rightarrow x - y = 0$

Equation of directrix of (3) is

$$X = -\sqrt{2} \Rightarrow \frac{x + y}{\sqrt{2}} = -\sqrt{2} \Rightarrow \frac{x + y}{\sqrt{2}} + \sqrt{2} = 0$$

$x + y + 2 = 0$ is the directrix in xy-coodinate system.

(iii) Identify: $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$

Solution. $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = 2$ the angle θ through which axes be rotated to given by

$$\tan 2q = \frac{2h}{a-b} = \frac{2}{1-1} = \frac{2}{0} \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right\} \quad ...(2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)$$

$$-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) + \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2XY + Y^2)$$

$$+ 2(X - Y) - 2(X + Y) + 2 = 0$$

$$X^2 - 2XY + Y^2 + 2X^2 - 2Y^2 + X^2 + 2XY + Y^2 + 4X - 4Y - 4X$$

$$- 4Y + 4 = 0$$

$$4X^2 - 8Y + 4 = 0 \Rightarrow X^2 - 2Y + 1 = 0$$

$$\therefore X^2 = 2\left(Y - \frac{1}{2}\right) \quad ... (3)$$

Which represents a parabola.

From (ii), we have

$$X - Y = \sqrt{2}\,x \quad ... (4)$$

and $$X + Y = \sqrt{2}\,y \quad ... (5)$$

Adding (4) and (5), we get

$$2X = \sqrt{2}\,x + \sqrt{2}\,y \Rightarrow 2X = \sqrt{2}(x + y) \Rightarrow X = \frac{1}{\sqrt{2}}(x + y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x+y) - Y = \sqrt{2}\,x \implies Y = \frac{1}{\sqrt{2}}(x+y) - \sqrt{2}\,x$$

$$= \frac{x+y-2x}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y-x)$$

Thus $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(y-x)$

Elements of parabola are

Focus of (3) is $X = 0,\ Y - \frac{1}{2} = \frac{1}{2} \implies Y = 1$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$ and $\frac{1}{\sqrt{2}}(y-x) = 1$

i.e., $x + y = 0$ and $y - x = \sqrt{2}$

Adding: $x + y = 0$

$-x + y = \sqrt{2}$

$2y = \sqrt{2} \implies y = \frac{1}{\sqrt{2}}$

Put $y = \frac{1}{\sqrt{2}}$ in $x + y = 0 \implies x = -y = -\frac{1}{\sqrt{2}}$

Focus: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the focus of (1)

Vertex of (3) is $X = 0,\ Y - \frac{1}{2} = 0 \implies Y = \frac{1}{2}$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0 \implies x + y = 0$

and $\frac{1}{\sqrt{2}}(y-x) = \frac{1}{2} \implies y - x = \frac{1}{\sqrt{2}}$

Solving, we get $x = -\frac{1}{2\sqrt{2}},\ y = \frac{1}{2\sqrt{2}}$

Vertex $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ is the vertex of (1)

Axis $X = 0$ i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$

$\implies x + y = 0$

Equation of directrix of (3) is

$$Y - \frac{1}{2} = -\frac{1}{2} \Rightarrow \frac{y-x}{\sqrt{2}} = 0 \Rightarrow y - x = 0 \Rightarrow x - y = 0$$

is the directrix in xy–coordinate system.

(iv) $x^2 + xy + y^2 - 4 = 0$

Solution. $x^2 + xy + y^2 - 4 = 0$ ... (1)

Here $a = 1,\ b = 1,\ 2h = 1$ the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{1}{1-1} = \frac{1}{0} = \infty \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X.\frac{1}{\sqrt{2}} - Y.\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X.\frac{1}{\sqrt{2}} + Y.\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right] \quad \text{(ii)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 4 = 0$$

$$\left(\frac{X^2 - 2XY + Y^2}{2}\right) + \left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right) - 4 = 0$$

$$X^2 - 2XY + Y^2 + X^2 - Y^2 + X^2 + 2XY + Y^2 - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{X^2}{8/3} + \frac{Y^2}{8} = 1 \qquad \text{... (3)}$$

Which represents an ellipse.

From (2), we have

$$X - Y = \sqrt{2}\,x \qquad \text{... (4)}$$

$$X + Y = \sqrt{2}\,y \qquad \text{... (5)}$$

Adding (4) and (5) $2X = \sqrt{2}\,x + \sqrt{2}\,y \Rightarrow X = \frac{1}{\sqrt{2}}(x + y)$

Subtracting (iv) and (v):

$$-2Y = \sqrt{2}\,x - \sqrt{2}\,y \Rightarrow Y = \frac{1}{\sqrt{2}}(y - x)$$

Elements of ellipse are

Centre of (3), is $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x+y) = 0 \Rightarrow x+y = 0$$

and $\frac{1}{\sqrt{2}}(y-x) = 0 \Rightarrow -x+y = 0 \Rightarrow x = 0 , y = 0$

Hence $C(0, 0)$ is the centre of (1)

Vertices of (3) are: $X = 0$, $Y = \pm 2\sqrt{2}$

$$X = 0 \Rightarrow \frac{1}{\sqrt{2}}(x+y) = 0 \Rightarrow x+y = 0$$

and $Y = \pm\sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}(y-x) = \pm 2\sqrt{2} \Rightarrow -x+y = \pm 4$

and

$\Rightarrow x+y = 0$	$x+y = 0$
$-x+y = 4$	$-x+y = -4$
Adding: $2y = 4 \Rightarrow y = 2$	Adding: $2y = -4 \Rightarrow y = -2$
$\Rightarrow x = -y = -2$	$x = -y = -(-2) = 2$
$(-2, 2)$	$(2, -2)$

$(-2, 2)$, $(2, -2)$, as vertices of (1)

Equation of major axis: $X = 0 \Rightarrow x+y = 0$

Equation of minor axis: $Y = 0 \Rightarrow x-y = 0$

Ecentricity: $e = \frac{\sqrt{a^2-b^2}}{a} = \frac{\sqrt{8-\frac{8}{3}}}{2\sqrt{2}} = \frac{4}{2\sqrt{6}} = \frac{2}{\sqrt{6}}$

Foci of (3) are $X = 0$, $Y = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$, $-\frac{1}{\sqrt{2}}(x-y) = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$

$\Rightarrow x+y = 0,$	$x+y = 0$
$-x+y = \frac{2\sqrt{8}}{\sqrt{3}}$	$-x+y = \frac{-2\sqrt{8}}{\sqrt{3}}$
Adding: $2y = \frac{2\sqrt{8}}{\sqrt{3}} \Rightarrow y = \frac{2\sqrt{2}}{\sqrt{3}}$	Adding: $2y = -\frac{2\sqrt{8}}{3} \Rightarrow y = \frac{-2\sqrt{2}}{3}$

$x + y = 0 \Rightarrow x = -y = -\frac{2\sqrt{2}}{3}$

$x + y = 0$

$\Rightarrow \quad x = -y = -\left(\frac{-2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}}{3}$

Hence $\left(\frac{-2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right)$ and $\left(\frac{2\sqrt{2}}{3}, \frac{-2\sqrt{2}}{3}\right)$ are the foci of (1).

(v) $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$

Solution. $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0 \quad \ldots (1)$

Here $a = 7,\ b = 13,\ 2h = -6\sqrt{3}$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-6\sqrt{3}}{7-13} = \frac{-6\sqrt{3}}{-6} = \sqrt{3}$$

$\Rightarrow \quad 2\theta = 60° \quad \Rightarrow \quad \theta = 30°$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 30° - Y\sin 30° = X\cdot\frac{\sqrt{3}}{2} - Y\cdot\frac{1}{2} = \frac{\sqrt{3}\,x - Y}{2} \\ y &= X\sin 30° + Y\cos 30° = X\cdot\frac{1}{2} + Y\,\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}\,Y}{2} \end{aligned}\right] \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$7\left(\frac{\sqrt{3}\,X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}\,X - Y}{2}\right)\left(\frac{X + \sqrt{3}\,Y}{2}\right) + 13\left(\frac{\sqrt{3}\,X + Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 - 2\sqrt{3}\,XY + Y^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}\,X^2 + 2XY - \sqrt{3}\,Y^2}{4}\right) + 13\left(\frac{X^2 + 2\sqrt{3}\,Y + 3Y^2}{4}\right) - 16 = 0$$

$$\frac{(21X^2 - 14\sqrt{3}\,XY + 7Y^2)}{4} - \frac{(18X^2 + 12\sqrt{3}\,XY - 18Y^2)}{4} + \frac{(13X^2 + 26\sqrt{3}\,XY + 39Y^2)}{4} - 16 = 0$$

$$21X^2 - 14\sqrt{3}\,XY + 7Y^2 - 18Y^2 - 12\sqrt{3}\,XY + 18Y^2 + 13X^2 + 26\sqrt{3}\,XY + 39Y^2 - 64 = 0$$

$16X^2 + 64Y^2 = 64$

$\Rightarrow \dfrac{X^2}{4} + \dfrac{Y^2}{1} = 1$ ... (3)

Which represents an ellipse.

From (ii), we have

$\sqrt{3}\ X - Y = 2x$ ... (4)

$X + \sqrt{3}\ Y = 2y$ ... (5)

Multiplying (iv) by $\sqrt{3}$, we get

$3X - \sqrt{3}\ Y = 2\sqrt{3}\ x$ ... (6)

Adding (5) and (6), we get

$4X = 2y + 2\sqrt{3}\ x \quad \Rightarrow \quad X = \frac{1}{2}(\sqrt{3}\ x + y)$

Multiplying (5) by $\sqrt{3}$, we get

$\sqrt{3}\ X + 3Y = 2\sqrt{3}\ Y$ ... (7)

Subtracting (7) from (4), we get

$-4Y = 2\sqrt{3}\ y - 2x \quad \Rightarrow \quad Y = \frac{1}{2}\left(x - \sqrt{3}\ y\right)$

Thus $X = \frac{1}{2}(\sqrt{3}\ x + y)$ and $Y = \frac{1}{2}\left(x - \sqrt{3}\ y\right)$

Elements of ellipse are

Centre of (3) is $X = 0,\ Y = 0$

$X = 0 \quad \left(\frac{1}{2}(\sqrt{3}\ x + y) = 0 \quad \left(\sqrt{3}\ x + y = 0 \right.\right.$

$Y = 0 \quad \left(\frac{1}{2}(x - \sqrt{3}\ y) = 0 \quad \left(x - \sqrt{3}\ y = 0 \right.\right.$

Solving these equations, we get $x = 0,\ y = 0$

Hence, $C\ (0, 0)$ centre of (1).

Vertices of (3) are $X = \pm a = \pm 2$ and $Y = 0$

$X = \pm 2 \quad \Rightarrow \quad \frac{1}{2}(\sqrt{3}\ x + y) = \pm 2 \quad \Rightarrow \quad \sqrt{3}\ x + y = \pm 4$

$Y = 0 \quad \Rightarrow \quad \frac{1}{2}(x - \sqrt{3}\ y) = 0 \quad \Rightarrow \quad x - \sqrt{3}\ y = 0$

$\sqrt{3}\ x + y = 4$ ... (4)

$x - \sqrt{3}\ y = 0$ ... (5)

Multiplying (4) by $\sqrt{3}$ and adding these equations, we get

$\sqrt{3}\ x + y = -4$ ... (6)

$x - \sqrt{3}\ y = 0$ ... (7)

Multiplying (6) by $\sqrt{3}$ and adding these equations, we get

$4x = 4\sqrt{3} \Rightarrow x = \sqrt{3}$

(5) $\Rightarrow \sqrt{3}\,y = x = \sqrt{3} \Rightarrow y = 1$

$4x = -4\sqrt{3} \Rightarrow x = -\sqrt{3}$

(7) $\Rightarrow \sqrt{3}\,y = x = -\sqrt{3} \Rightarrow y = -1$

$(\sqrt{3}, 1)$, $(-\sqrt{3}, -1)$, as vertices of (1)

Eccentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{4-1}}{2} = \dfrac{\sqrt{3}}{2}$

Foci of (3) are: $X = \pm\sqrt{3}$, $Y = 0$

$$X = \pm\sqrt{3} \Rightarrow \frac{1}{2}(\sqrt{3}\,x + y) = \pm\sqrt{3} \Rightarrow \sqrt{3}\,x + y = \pm 2\sqrt{3}$$

$$Y = 0 \Rightarrow \frac{1}{2}(\sqrt{3}\,y - x) = 0 \Rightarrow -x + \sqrt{3}\,y = 0$$

$\sqrt{3}\,x + y = 2\sqrt{3}$ ... (8)

$-x + \sqrt{3}\,y = 0$ ... (9)

Multiplying (9) by $\sqrt{3}$ and adding these equations, we get

$4y = 2\sqrt{3} \Rightarrow y = \dfrac{\sqrt{3}}{2}$

(9) $\Rightarrow$ x \r(3) y = \r(3) . \f(\r(3),2) = \f(3,2)

$(\frac{3}{2}, \frac{\sqrt{3}}{2})$

$\sqrt{3}\,x + y = -2\sqrt{3}$ ... (10)

$-x + \sqrt{3}\,y = 0$ ... (11)

Multiplying (11) by $\sqrt{3}$ and adding these equations, we get

$4y = -2\sqrt{3} \Rightarrow y = \dfrac{-\sqrt{3}}{2}$

(11) $\Rightarrow$ x = \r(3) y $= \sqrt{3}\left(\dfrac{-\sqrt{3}}{2}\right) = \dfrac{-3}{2}$

$(\frac{-3}{2}, \frac{-\sqrt{3}}{2})$

Hence $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{-3}{2}, \frac{-\sqrt{3}}{2}\right)$, as foci of (1).

Equation of major axis: $Y = 0 \Rightarrow \frac{1}{2}\left(\sqrt{3}\,y - x\right) = 0 \Rightarrow x - \sqrt{3}y = 0$

Equation of minor axis: $X = 0 \Rightarrow \frac{1}{2}\left(\sqrt{3}\,x + y\right) = 0 \Rightarrow \sqrt{3}x + y = 0.$

(vi) Identify. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$ ... (1)

Here $a = 4$, $b = 7$, $2h = -4$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-4}{4-7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\frac{2\tan q}{1-\tan^2 q} = \frac{4}{3} \implies 6\tan\theta = 4 - 4\tan^2\theta$$

$$4\tan^2\theta + 6\tan\theta - 4 = 0 \implies 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$= \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2} \implies \tan\theta = \frac{1}{2}$$

Now $\tan\theta = \frac{1}{2} \implies$ base $= 2$, $\perp = 1$, so hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X \cdot \frac{2}{\sqrt{5}} - Y \cdot \frac{1}{\sqrt{5}} = \frac{2X - Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\sin\theta = X \cdot \frac{1}{\sqrt{5}} + Y \cdot \frac{2}{\sqrt{5}} = \frac{X + 2Y}{\sqrt{5}} \end{aligned}\right\} \ldots (2)$$

Substituting these expressions for x, and y into (i), we get

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2$$
$$+ 12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\left(\frac{4X^2 - 4XY + Y^2}{5}\right) - 4\left(\frac{2X^2 + 3XY - 2Y^2}{5}\right) + 7\left(\frac{X^2 + 4XY + 4Y^2}{5}\right)$$
$$+ \frac{24X - 12Y}{\sqrt{5}} + \frac{6X + 12Y}{\sqrt{5}} - 9 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2$$
$$+ \sqrt{5}\,(24X - 12Y) + \sqrt{5}\,(6X + 12Y) - 45 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2 + 24\sqrt{5}\,X$$
$$- 12\sqrt{5}\,Y + 6\sqrt{5}\,X + 12\sqrt{5}\,Y - 45 = 0$$

$$15X^2 + 40Y^2 + 30\sqrt{5}\,X - 45 = 0 \implies 3X^2 + 8Y^2 + 6\sqrt{5}\,X - 9 = 0$$

$$3(X^2 + 2\sqrt{5}\,X) + 8Y^2 = 9 \implies 3(X^2 + 2\sqrt{5}\,X + (\sqrt{5})^2) + 8Y^2 = 9 + 15$$

$$3(X + \sqrt{5})^2 + 8Y^2 = 24 \implies \frac{(X + \sqrt{5})^2}{8} + \frac{Y^2}{3} = 1 \qquad (3)$$

which represents an ellipse.

From (2), we have

$$2X - Y = \sqrt{5}\,x \qquad \dots (4)$$

$$X + 2Y = \sqrt{5}\,y \qquad \dots (5)$$

Multiplying (5) by 2 and subtracting from (4), we get

$$5Y = 2\sqrt{5}\,y - \sqrt{5}\,x \Rightarrow Y = \frac{1}{\sqrt{5}}(-x + 2Y)$$

Put $Y = \frac{1}{\sqrt{5}}(-x + 2y)$ in (5), we get

$$X + \frac{2}{\sqrt{5}}(-x + 2y) = \sqrt{5}\,y \Rightarrow X = \sqrt{5}\,y - \frac{2}{\sqrt{5}}(-x + 2y)$$

$$= \sqrt{5}\,y + \frac{2}{\sqrt{5}}x - \frac{4}{\sqrt{5}}y$$

$$= \frac{1}{\sqrt{5}}(2x + y)$$

Thus $X = \frac{1}{\sqrt{5}}(2x + y)$ and $Y = \frac{1}{\sqrt{5}}(-x + 2y)$.

For centre of (3) $X + \sqrt{5} = 0, Y = 0 \Rightarrow X = -\sqrt{5}, Y = 0$

$$X = -\sqrt{5} \Rightarrow \frac{1}{\sqrt{5}}(2x + y) = -\sqrt{5} \Rightarrow 2x + y = -5 \qquad (7)$$

$$Y = 0 \Rightarrow \frac{1}{\sqrt{5}}(-x + 2y) = 0 \Rightarrow -x + 2y = 0 \qquad (8)$$

Multiplying equation (8) by 2, we get

$$-2x + 4y = 0 \qquad (9)$$

Adding equation (7) and (9), we get

$$5y = -5 \Rightarrow y = -1$$

Equation. (8) $\Rightarrow x = 2y = 2(-1) = -2$

Hence $C(-2, -1)$ is the centre of (1)

Vertices of (3) are $X + \sqrt{5} = \pm\sqrt{8}$, $Y = 0$

$$X + \sqrt{5} = \sqrt{8}, Y = 0$$

$$X + \sqrt{5} = \sqrt{8} \Rightarrow \frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = \sqrt{8}$$

$$2x + y + 5 = \sqrt{40}$$

$$2x + y = -5 + \sqrt{40} \qquad \dots (10)$$

$$Y = 0 \quad \Rightarrow \quad -x + 2y = 0 \qquad \dots (11)$$

Multiplying (11) by 2

$$-2x + 4y = 0 \qquad \dots (12)$$

Adding equation (10) and equation (12), we get

$$5y = -5 + \sqrt{40} \quad \Rightarrow \quad y = -1 + \sqrt{\frac{8}{5}}$$

$$(12) \Rightarrow \quad x = 2y = 2\left(-1 + \sqrt{\frac{8}{5}}\right) = - + \sqrt{\frac{32}{5}}$$

Similarly, solving $X + \sqrt{5} = -\sqrt{8}$, and $Y = 0$, we get

$$x = -2 - \sqrt{\frac{32}{5}}\ , \ y = -1 - \sqrt{\frac{8}{5}}$$

$$\therefore \quad \left(-2 + \sqrt{\frac{32}{5}}\ , -1 + \sqrt{\frac{8}{5}}\right), \left(-2 - \sqrt{\frac{32}{2}}\ , -1 - \sqrt{\frac{8}{5}}\right)$$

are the vertices of (1)

Eccentricity:- $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{8-3}}{\sqrt{8}} = \dfrac{\sqrt{5}}{\sqrt{8}} = \sqrt{\dfrac{5}{8}}$

Foci of (3) are $X + \sqrt{5} = \pm\sqrt{5}\ ,\ Y = 0$

$$X + \sqrt{5} = \sqrt{5} \quad \text{and} \quad Y = 0$$

$$X + \sqrt{5} = \sqrt{5} \quad \Rightarrow \quad X = 0$$

$$\frac{2x + y}{\sqrt{5}} = 0 \quad \Rightarrow \quad 2x + y = 0 \qquad \dots (13)$$

$$Y = 0 \quad \Rightarrow \quad x + 2y = 0 \qquad \dots (14)$$

Multiplying equation (14) by 2, we get

$$-2x + 4y = 0 \qquad \dots (15)$$

Adding (13) and (15), we get

$$5y = 0 \quad \Rightarrow \quad y = 0$$

Equation (14) $\Rightarrow \quad x = 2y = 2(0) = 0$

Similarly, solving $X + \sqrt{5} = -\sqrt{5}$ and $Y = 0$.

We get $x = -4,\ y = -2$

Thus $(0, 0), (-4, -2)$ are the foci of (1).